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On Some Difference Sequence Spaces Defined by Orlicz Function and Ideal Convergence in 2-Normed Space

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Abstract: In the present work, we introduce the difference sequence spaces W_0^I (||. , . ||, M, λ, a, Δ), W^I (||. , . ||, M, λ, a, Δ) and W_∞^I (||. , . ||, M, λ, a, Δ) in 2-normed space using Orlicz function and ideal convergence. We will examine some of their topological properties.

Keywords: *Difference sequence space*, *Orlicz function*, *Paranormed space*, *Ideal convergence*, 2-*normed space*.

1. Introduction

In functional analysis and related areas of mathematics, a sequence space is a special case of function space if the domain is restricted to the set of natural numbers ℕ. It is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, the set ω of all functions from the set of natural numbers ℕ to the field K of real or complex numbers can be turned into a vector space. A sequence space is defined as a linear subspace of ω. Let ℓ_∞, c_0 and c be the linear spaces of bounded, null and convergent sequences with complex terms respectively and the norm is given by $||x|| = \sup_k |x_k|$, where $k \in \mathbb{N}$.

Before proceeding to the main results, we recall some definitions and notations that are used in this paper.

Definition 1.1: An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is convex, continuous and non-decreasing with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.(Krasnosel'skiî and Rutickiî, [11])

Definition 1.2: An Orlicz function M is said to satisfy Δ_2-condition for all values of x if there exists a constant $L > 0$ such that $M(2x) \leq LM(x)$ for all $x \geq 0$. It is equivalent to the condition

$M(Kt) \leq Q\, KM(t)$, $\forall t$ and $K > 1$.

The function $M(t) = t^p$, $1 < p < \infty$ and $t \geq 0$ is an Orlicz function which does not satisfy the Δ_2-condition but the function $M(t) = \alpha\, |t|^p$, $1 < p < \infty$ and $t \geq 0$ is an Orlicz function which satisfies the Δ_2-condition since $M(2t) = \alpha\, 2^p\, |t|^p = 2^p M(t)$. (Krasnosel'skiî and Rutickiî, [11])

Definition 1.3: Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct the scalar-valued sequence space ℓ_M of scalars (x_k) such that

$$\ell_M = \left\{ \bar{x} = (x_k) \in \omega \ : \ \sum_{k=1}^{\infty} .M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M endowed with a norm

$$||\bar{x}||_M = \inf\left\{ \rho > 0: \ \sum_{k=1}^{\infty} .M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

forms a Banach space which is called an Orlicz sequence space.

Orlicz sequence space ℓ_M plays an important role in functional analysis and is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p : 1 \le p < \infty$.

For more details about Orlicz function and its subsequent use, we refer a few: Bhardwaj and Bala [2], Dutta et al. [3], Ghimire and Pahari[6] , Kamthan and Gupta [8], Krasnosel'skiî and Rutickiî [11], Lindenstrauss and Tzafriri [12] , Maddox [13], Pahari [17], Parashar and Choudhary [18], and many others.

Definition 1.4: A paranormed space (X, G) is a linear space X with zero element θ together with a function $G : X \to \mathbb{R}_+$ (called a paranorm on X) which satisfies the following properties:

*PN*1: $G(\theta) = 0$;

*PN*2: $G(x) = G(-x)$ for all $x \in X$;

*PN*3: $G(x + y) \le G(x) + G(y)$ for all $x, y \in X$; and

*PN*4: Scalar multiplication is continuous.

Note that the continuity of scalar multiplication is equivalent to

(i) if $G(x_n) \to 0$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $G(\alpha_n x_n) \to 0$ as $n \to \infty$ and

(ii) if $\alpha_n \to 0$ as $n \to \infty$ and x be any element in X, then $G(\alpha_n x) \to 0$, (see, Wilansky[25]).

A paranorm is called total if $G(x) = 0$ implies $x = \theta$.

The concept of paranorm is closely related to linear metric space, (see, Wilansky [25]) and its studies on sequence spaces were initiated by Maddox [13] and many others. In particular, various types of paranormed sequence spaces were further investigated by several workers Bhardwaj and Bala [2], Parashar and Choudhary [18] and many others.

Next, we recall the definition of difference sequence spaces.

Definition 1.5: Kizmaz [9] defined the difference sequence spaces by

$c_o(\Delta) = \{x = (x_i): \Delta x \in c_o\}$,

$c(\Delta) = \{x = (x_i): \Delta x \in c\}$

$l_\infty(\Delta) = \{x = (x_i): \Delta x \in l_\infty\}$

where, $\Delta x = (\Delta x_i) = (x_i - x_{i+1})$ and showed that these spaces are Banach spaces with the norm given by

$$\|x| = |x_1| + ||\Delta x||_\infty .$$

A sequence $x = (x_i)$is called Δ-convergent if the $\lim x_i$ is finite and hence exists.

Every convergent sequence is Δ-convergent but not conversely. For, consider the sequence $x_k = k + 7$ for all natural numbers i. Then, $(\Delta x)_i = (x_i - x_{i+1}) = -1$ for each $i \in \mathbb{N}$. Thus, $x = (x_i)$ is divergent but it is Δ-convergent.

Definition 1.6: For sequence $(x_k) \in S$ and for all scalars (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$, $(\alpha_x x_k) \in S$,then the sequence space S is called solid (normal).

A sequence space S is called a sequence algebra if $(x_k) . (y_k) = (x_k y_k) \in S$ whenever $(x_k), (y_k) \in S$.

Definition 1.7: Let X be a vector space with dim $(X) > 1$. A mapping $\|.,.\| : X \times X \to \mathbb{R}$ satisfying.

($2N_1$) $\|x, y\| \ge 0$ and $\|x, y\| = 0 \Leftrightarrow x, y$ are linearly dependent

($2N_2$) $\|x, y\| = \|y, x\|$

($2N_3$) $\|\alpha x, y\| = |\alpha|\,\|x, y\|$ for any real number α.

($2N_4$) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for all $x, y, z \in X$,

is called a 2-norm. The pair $(X, \|.,.\|)$ is called a 2-normed space.

The notion of 2–normed space was initiated by Gahler [5]in 1960's as an interesting linear generalization of a normed linear space.

The 2-norm is used to measure the area of parallelogram spanned by two vectors. Geometrically, a 2-norm function generalizes the concept of area function of parallelogram. For example, consider $X = \mathbb{R}^2$, being equipped with $\|\bar{x}, \bar{y}\| = |x_1 y_2 - x_2 y_1|$, where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$.

Then $(X, \|.,.\|)$ forms a 2–normed space and $\|\bar{x}, \bar{y}\|$ represents the area of the parallelogram spanned by the associated vectors $\bar{x}$ and $\bar{y}$.

Subsequently the interesting linear generalization of a normed linear was studied by Freese and Cho [4], White and Cho [24] and many others. Recently a lot of activities have been started by many researchers to study this concept in different directions which characterized 2-normed and generalized 2-normed spaces for instances: Açikgöz [1] and Savas [21] and others.

Definition 1.8: A sequence (x_n) in a 2-normed space $(X, \|.,.\|)$ is called Cauchy if $\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0$ for all $z \in X$ and convergent if there is $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, z\| = 0 \text{ for all } z \in X.$$

A complete 2-normed space is called a 2-Banach space.

Definition 1.9: Let A be a subset of $\mathbb{N}$. The natural density $\delta(A)$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \le n\}|, \text{ provided that the limit exists.}$$

A sequence $x = (x_n) \in \omega$ is said to be statistically convergent to a number $\ell \in \mathbb{R}$, if for all $\varepsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |x_n - \ell| \ge \varepsilon\} = 0$.

Definition 1.10: Let X be a non-empty set then a class I of subsets of X is said to be an ideal if

(i) $A \in I$ and $B \subset A$ implies that $B \in I$ (Hereditary property)

(ii) $A, B \in I$ implies that $A \cup B \in I$ (Additive property)

If I of X further satisfies $\{x\} \in I$ for each $x \in X$, then it is called admissible ideal.

Definition 1.11: A sequence $x = (x_n) \in \omega$ is said to be ideal convergence (I-convergence) to a number $\ell \in \mathbb{R}$ if for all $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - \ell| \ge \varepsilon|\} \in I$.

The notion of ideal convergence was first introduced by Kostyrko et al. [10] as a generalization of both usual and statistical convergence which was introduced by Fast and Steinhaus in 1951. For more details about the sequence spaces defined by ideal convergence, one may refer to Hazarika et al. [7], Mursaleen and Alotaibi[14], Mursaleen and Mohiuddine [15], Mursaleen and Sharma[16], Sahiner et al.[19], Salat et al.[20], Savas([21],[22]), Tripathy and Hazarika[23], and many others.

2. Main Results

Let $(X, \|.,.\|)$ be a 2-normed space and M be an Orlicz function. Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity with $\lambda_n + 1 \le \lambda_{n+1}$ and $\lambda_1 = 0$ and I be an admissible ideal of $\mathbb{N}$. Let ω be the space of all sequences defined over $(X, \|.,.\|)$.

Let $a = (a_k)$ be a bounded sequence of positive real numbers and $I_n = [n + 1 - \lambda_n, n]$.

By extending the work done by Savas [21], we now introduce and study the following classes of difference sequences

1. $$W_0^I(\|.,.\|, M, \lambda, a, \Delta) = \left\{x \in \omega : \forall \varepsilon > 0 \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \ge \varepsilon\right\} \in I \text{ for some } \rho > 0 \text{ and for all } z \in X\right\}$$

2. $$W^I(\|.,.\|, M, \lambda, a, \Delta) = \left\{x \in \omega : \forall \varepsilon > 0 \left\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\left\|\frac{\Delta x_k - L}{\rho}, z\right\|\right)\right]^{a_k} \ge \varepsilon\right\} \in I\right.$$

for some $\rho > 0$, $L \in X$ and $\forall z \in X\}$.

3. $W_\infty(\|\cdot,\cdot\|, M, \lambda, a, \Delta) = \{x \in \omega : \exists\, K > 0$ such that $\sup_{n\in\mathbb{N}} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \le K$

for some $\rho > 0$ and for all $z \in X$.

4. $W^{I}_\infty(\|\cdot,\cdot\|, M, \lambda, a, \Delta) = \{x \in W : \exists K > 0$ s.t. $\left\{n\in\mathbb{N} : \sup_{n\in\mathbb{N}} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \ge K\right\} \in I$

for some $\rho > 0$ and for all $z \in X\}$

Throughout this article, we shall use the following inequalities

If $0 \le a_k \le \sup a_k = H$, $D = \max(1, 2^{H-1})$ then $|\alpha_k + \beta_k|^{a_k} \le D\, \{|\alpha_k|^{a_k} + |\beta_k|^{a_k}\}$

for all $\alpha_k, \beta_k \in \mathbb{C}$. Also, $|\alpha|^{a_k} \le \max(1, |a|^H)$ for all $\alpha \in \mathbb{C}$.

In this work, we shall investigate some topological properties of the classes defined above.

Theorem 2.1: The class $W^{I}_0(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$, $W^{I}(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ and $W^{I}_\infty(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ are linear spaces.

Proof:

Let $x, y \in W^{I}_0(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ and $\alpha, \beta \in \mathbb{R}$. Then, for some $\rho_1, \rho_2 > 0$

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho_1}, z\right\|\right)\right]^{a_k} \ge \varepsilon\right\} \in I \text{ and } \left\{n\in\mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta y_k}{\rho_2}, z\right\|\right)\right]^{a_k} \ge \varepsilon\right\} \in I$$

Since M is an Orlicz function and $\|\cdot,\cdot\|$ is a 2-norm, we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\alpha\Delta x_k + \beta\Delta y_k}{(|\alpha|\rho_1 + |\beta|\rho_2)}, z\right\|\right)\right]^{a_k}$$

$$\le D\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[\frac{|\alpha|}{(|\alpha|\rho_1 + |\beta_1|\rho_2)} M\left(\left\|\frac{\Delta x_k}{\rho_1}, z\right\|\right)\right]^{a_k} + D\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[\frac{|\beta|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M\left(\left\|\frac{\Delta y_k}{\rho_2}, z\right\|\right)\right]^{a_k}$$

$$\le DL\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho_1}, z\right\|\right)\right]^{a_k} + DL\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta y_k}{\rho_2}, z\right\|\right)\right]^{a_k}$$

where, $L = \max\left[1, \left(\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}\right)^H, \left(\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2}\right)^H\right]$

Then, we can write

$$\left\{n\in\mathbb{N} : \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\alpha\Delta x_k + \beta\Delta y_k}{|\alpha|\rho_1 + |\beta|\rho_2}, z\right\|\right)\right] \ge \varepsilon\right\}$$

$$\subseteq \left\{n\in\mathbb{N} : DL\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho_1}, z\right\|\right)\right]^{a_k} \ge \frac{\varepsilon}{2}\right\} \cup \left\{n\in\mathbb{N} : DL\cdot\frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta y_k}{\rho_2}, z\right\|\right)\right]^{a_k} \ge \frac{\varepsilon}{2}\right\}$$

Clearly the two sets on the right hand side belong to I and hence the left side too. This shows that the space $W^{I}_0(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ is a linear space.

In the same way, one can show that $W^{I}(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ and $W^{I}_\infty(\|\cdot,\cdot\|, M, \lambda, a, \Delta)$ are linear spaces.

Theorem 2.2: The space $W_\infty(\|.,.\|, M, \lambda, a, \Delta)$ is a paranormed space with respect to the paranorm

$$g_n(x) = \inf \left\{\rho^{a_n/H};\ \rho > 0 \text{ such that } \left(\sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M \left\|\frac{\Delta x_k}{\rho}, z\right\|\right]^{a_k}\right)^{\frac{1}{H}} \le 1,\ \forall\, z \in X\right\}$$

Proof:

Obviously $g_n(\theta) = 0$ and $g_n(-x) = x$ easily follow, so PN_1 and PN_2 are obvious.

To proceed the further proof, for PN_3, let $x = (x_k)$ and $y = (y_k)$ in $W_\infty(\|.,.\|, M, \lambda, a, \Delta)$. Let

$$A(x) = \left\{\rho > 0 : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \le 1,\ \forall\, z \in X\right\}$$

$$A(y) = \left\{\rho > 0 : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta y_k}{\rho}, z\right\|\right)\right]^{a_k} \le 1,\ \forall\, z \in X\right\}$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. Also let $\rho = \rho_1 + \rho_2$ then we can write

$$\sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\left\|\frac{(\Delta x_k + \Delta y_k)}{\rho}, z\right\|\right) \le \frac{\rho_1}{\rho_1+\rho_2} \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\left\|\frac{(\Delta x_k)}{\rho_1}, z\right\|\right) + \frac{\rho_2}{\rho_1+\rho_2} \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\left\|\frac{(\Delta y_k)}{\rho_2}, z\right\|\right)$$

Thus, $\sup_n \left(\frac{1}{\lambda_n}\right) \sum_{k\in I_n} M\left(\left\|\frac{(\Delta x_k + \Delta y_k)}{\rho_1+\rho_2}, z\right\|\right)^{a_k} \le 1$

and
$$\begin{aligned} g_n(x+y) &\le \inf\{(\rho_1+\rho_2)^{a_n/H} : \rho_1 \in A(x),\ \rho_2 \in A(y)\} \\ &\le \inf\{\rho_1^{a_n/H} : \rho_1 \in A(x)\} + \inf\{\rho_2^{a_n/H} : \rho_2 \in A(y)\} \\ &= g_n(x) + g_n(y) \end{aligned}$$

Next we prove PN_4 i.e., the scalar multiplication is continuous.Let $\alpha^m \to \alpha$ where $\alpha, \alpha^m \in \mathbb{C}$ and let $g_n(x^m - x) \to 0$ as $m \to \infty$. We have to show that $g_n(\sigma^m x^m - \sigma_x) \to 0$ as $m \to \infty$.

Let
$$A(x^m) = \left\{\rho_m > 0 : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta x_k^m}{\rho_m}, z\right\|\right)\right]^{a_k} \le 1,\ \forall\, z \in X\right\}$$

$$A(x^m - x) = \left\{\rho_m^1 > 0 : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[M\left(\left\|\frac{\Delta(x_k^m - x_k)}{\rho_m^1}, z\right\|\right)\right]^{a_k} \le 1,\ \forall\, z \in X\right\}$$

If $\rho_m \in A, (x^m)$ and $\rho_m^1 \in A(x^m - x)$ then

$$M\left(\left\|\frac{\Delta(\alpha^m x_k^m - \alpha x_k)}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|}, z\right\|\right) \le M\left(\left\|\frac{\Delta(\alpha_m x_m^k - \sigma x_k^m)}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|}\right\| + \left\|\frac{\Delta(\alpha^m x_k^m - \alpha x_k)}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|}, z\right\|\right)$$

$$\le \frac{|\alpha^m - \alpha|\,\rho_m}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|} M\left(\left\|\frac{\Delta x_k^m}{\rho_m}, z\right\|\right) + \frac{|\alpha|\,\rho_m^1}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|} M\left(\left\|\frac{\Delta(x_k^m - x_k)}{\rho_m^1}, z\right\|\right)$$

This follows that

$$\left[M\left(\left\|\frac{\Delta(\alpha^m x_k^m - \alpha x_k)}{\rho_m|\alpha^m - \alpha| + \rho_m^1|\alpha|}, z\right\|\right)\right]^{a_k} \le 1$$

Consequently, we have

$$g_n(\alpha^m x^m - \alpha x) \le \inf \{(\rho_m|\alpha^m - \alpha| + \rho_m^1 |\alpha|\}^{a_n/H} : \rho_m \in A(x^m), \rho_m^1 \in A(x^m - x)\}$$

$$\le (|\alpha^m - \alpha|)^{P_n/H} \inf \{\rho_m^{a_n/H} : \rho_m \in A(x^m)\} + (|\alpha|)^{a_n/H} \inf \{(\rho_m^1)^{a_n/H} : \rho_m^1 \in A(x_m - x)\}$$

$$\le \max |\alpha^m - \alpha|^{a_n/H} g_n(x^m) + \max |\alpha|^{a_n/H} g_n(x^m - x) \to 0 \text{ as } m \to \infty.$$

Hence, the scalar multiplication is continuous.

Theorem 3: Let M, M_1, M_2 be Orlicz functions. Then we have

(i) $W_0^I(\|.,.\|, M, \lambda, a,\Delta) \subseteq W_0^I(\|.,.\|, M\circ M_1, a,\Delta)$ provided that (a_k) is such that $H_0 = \inf a_k > 0$.

(ii) $W_0^I(\|.,.\|, M_1, \lambda, a,\Delta) \cap W_0^I(\|.,.\|, M_2, \lambda, a,\Delta) \subseteq W_0^I(\|.,.\|, M_1 + M_2, \lambda, a,\Delta)$

(iii) $W_0^I(\|.,.\|, M_1,\lambda, a,\Delta) \subset W^I(\|.,.\|, M_1,\lambda, a,\Delta) \subset W_\infty^I(\|.,.\|, M,\lambda, a, \Delta)$

Proof:

Let $\varepsilon > 0$ be given. Let us choose $\varepsilon_0 > 0$ such that $\max \{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$.

Using continuity of M, we can choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M(t) < \varepsilon_0$.

Let $x_k \in W_0(\|.,.\|, M_1, \lambda, a,\Delta)$. Then from definition,

$$A(\delta) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M_1\left(\left\| \frac{D_k}{\rho}, z \right\| \right) \right]^{a_k} \ge \delta^H \right\} \in I$$

Hence, if $n \in A(\delta)$ then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right]^{a_k} < \delta^H \Rightarrow \sum_{k \in I_n} \left[M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right]^{a_k} < \lambda_n \delta^H$$

$$\Rightarrow \left[M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right]^{a_k} < \delta^H, \ \forall k \in I_n$$

$$\Rightarrow M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) < \delta, \ \forall k \in I_n.$$

Using continuity of M, we have

$$M\left(M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right) < m(\delta) < \varepsilon_0, \ \forall k \in I_n$$

This implies that

$$\sum_{k \in I_n} \left[M\left(M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right) \right]^{a_k} < \lambda_n \max \{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \lambda\varepsilon.$$

Thus, $\displaystyle\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right) \right]^{a_k} < \varepsilon.$

This shows that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(M_1\left(\left\| \frac{\Delta x_k}{\rho}, z \right\| \right) \right) \right]^{a_k} \ge \varepsilon \right\} \subset A(\delta)$$

and hence belong to I. This completes the proof.

(ii) Let $(x_k) \in W_0^I(\|\cdot,\cdot\|, M_1, \lambda, a,\Delta) \cap W_0^I(\|\cdot,\cdot\|, M_2, \lambda, a,\Delta)$. We can write

$$\frac{1}{\lambda_n}\left[M_1 + M_2\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \leq D\cdot\frac{1}{\lambda_n}\left[M_1\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} + D\cdot\frac{1}{\lambda_n}\left[M_2\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k}$$

This consequently gives that

$$W_0^I(\|\cdot,\cdot\|, M_1, \lambda, a,\Delta) \cap W_0^I(\|\cdot,\cdot\|, M_2, \lambda, a,\Delta) \subseteq W_0^I(\|\cdot,\cdot\|, M_1 + M_2, \lambda, a,\Delta).$$

(iii) The inclusion $W_0^I(\|\cdot,\cdot\|, M_1, \lambda, a,\Delta) \subset W^I(\|\cdot,\cdot\|, M_1, \lambda, a,\Delta)$ is obvious.

One can easily show that $W^I(\|\cdot,\cdot\|, M_1, \lambda, a,\Delta) \subset W_\infty^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$

This completes the proof.

Theorem 4: The sequence spaces $W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ and $W_\infty^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ are solid.

Proof:

We proof $W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ is solid. Let $(x_k) \in W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ and (α_k) be a sequence of scalars having the property that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ and $F = \max_k \{1, |\alpha_k|^H\}$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{\alpha_k \Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{F}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{\Delta x_k}{\rho}, z\right\|\right)\right]^{a_k} \geq \varepsilon\right\} \in I;$$

Hence, $(\alpha_k x_k) \in W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$. Thus the space $W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ is solid.

Theorem: 5 The spaces $W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ and $W^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ are sequence algebra.

Proof:

Let $(x_k), (y_k) \in W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$. Then, for some $\rho_1, \rho_2 > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{\Delta x_k}{\rho_1}, z\right\|\right)\right]^{a_k} \geq \varepsilon\right\} \in I \quad \text{and} \quad \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{\Delta x_k}{\rho_2}, z\right\|\right)\right]^{a_k} \geq \varepsilon\right\} \in I.$$

Choose $\rho = \rho_1 + \rho_2$. Then, we can easily show that

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\left\|\frac{\Delta x_k . \Delta y_k}{\rho}, z\right\|\right)\right]^{a_k} \geq \varepsilon\right\} \in I.$$

It follows that $x_k y_k \in W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$. This shows that $W_0^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$ is a sequence algebra. For the space $W^I(\|\cdot,\cdot\|, M, \lambda, a,\Delta)$, we can prove similarly.

Conclusion

In this paper, we have examined and explore some of the results that characterize the linear topological structures in 2- normed difference sequence space by endowing it with a suitable natural paranorm. In fact, these results can be used for further generalization to investigate other properties of sequences whose values in 2- normed space.

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