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Some Characteristics of the Farey Sequences with Ford Circles

Bal Bahadur Tamang[1], Shes Raj Kumar Sharma[2], Sita Laxmi Lawaju[3], Ajaya Singh[4]

[1]*Department of Mathematics, Mahendra Ratna Multiple Campus, Ilam, Tribhuvan University, Nepal*
[2]*Dept of Mathematics, Bhairahawa Multiple Campus, Tribhuvan University, Nepal*
[3,4]*Central Department of Mathematics, Tribhuvan University, Kiryipur, Kathmandu, Nepal*
Email: [1]*bal.tamang@mrmc.tu.edu.np*, [2]*shesrajbhattarai@gmail.com,*
[3]lawajusita@gmail.com, [4]singh.ajaya1@gmail.com

Corresponding Author: Bal Bahadur Tamang

Abstract: *Farey sequence is a pattern of rational numbers that approximates irrational numbers. In this paper, we use the Farey sequence to describe the Ford Circles. Its results and applications are equally fascinating as its pattern. Hurwitz Theorem is the main outcome of the approximation of irrational by rational numbers. Also, we examine the relationship between the Ford circle and the Farey sequence for rational values between* 0 *and* 1.

Keywords: *Farey sequence, Irrational number, Mediant, Approximation, Ford circle, Hurwitz Theorem.*

1.Introduction

Two-integer fractions with a non-zero denominator can be used to represent rational numbers. If not, it is an irrational number. The Farey sequence helps to classify rational and irrational numbers. A brief history of the Farey sequences after John Farey, who initially proposed that F_n can be created from F_{n-1}can be found in the work of Hardy and Wright [11]. The Farey sequence [7]was created by John Farey and is an outstanding method for producing appropriate fractions in the range$[0,1]$. The Farey sequence $F_n$of order n is a collection of proper, irreducible, positive fractions with denominators less than or equal to n and arranged in ascending order of their values. The Farey sequence is related to various studies, the major of which concentrate on fraction theory [11, 13].The Farey sequence connection with rational numbers between $0=\left(\frac{0}{1}\right)$and $1=\left(\frac{1}{1}\right)$. A collection of all reduced rational numbers with non-exceeding $n$denominators, known as the Farey sequence of order n, and is arranged in size order and collection of all integers of order 2in the following:

$$\left\{\dots,\frac{-3}{1},\frac{-5}{2},\frac{-2}{1},\frac{-3}{2},\frac{-1}{1},\frac{-1}{2},\frac{0}{1},\frac{1}{2},\frac{1}{1},\frac{3}{2},\frac{2}{1},\frac{5}{2},\frac{3}{1},\dots\right\}$$

While Mr. Flitcon's concluded that implementation the correct number of elements in $F_{99}$with the exception of 0 and 1, it did not give a formula or even a list of those elements. Charles Haros was tasked with creating a mathematical table to convert between fractions and decimals after the new French government passed legislation going to require that all of France switch to the metric system in place of imperial measurements in 1791, it was Charles Haros' responsibility to develop a mathematical table to translate between fractions and decimals. The French Revolution provided an unlikely motivation for the first publication of F_{99}in [8]. Charles Haros[1] provided some major sequence-related results in conjunction with a description of how the F_{99}Farey sequence was built using the mediant property and noted that if two numbers $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are Farey neighbors, then $|b_1a_2 - a_1b_2| = 1$, and formed F_{99} using the mediant property. The two fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are Farey neighbors and appear adjacent to one another in some Farey sequence if they have the properties that $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ and $b_1a_2 - a_1b_2 = 1$,then the median [7] of these two fractions is given by$\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1+a_2}{b_1+b_2}$. The denominator of each term in F_n, though, cannot be greater than n.The collection of irreducible rational number$\frac{a_1}{b_1}$, where $0 \le a_1 < b_1 \le n$, $(a_1, b_1) = 1$are ordered in ascending order to form the Farey sequence $F_n$for every positive integer n. Except for F_1, every F_n has odd numbers of terms, and the middle term is always $\frac{1}{2}$. Introduce the mediant fraction $\frac{p}{q} \oplus \frac{r}{s} = \frac{P+r}{q+s}$ between terms $\frac{p}{q}$and $\frac{r}{s}$, where $q + s \ge n$[2, 4, 10].When the fractions of $F_1$are improperly added together, the outcome is $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, which is a new fraction that lies between the first two. Adding the first two fractions of F_2 gives the mediant $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$, and the last two fractions of F_2 give the mediant $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, separating them in F_3. The Farey sequence can be found by repeatedly calculating the median between each pair of fractions in the preceding sequence. We can write $\frac{0}{1}$ and $\frac{1}{1}$in the first row F_1. Since $1 + 1 \le 2$, we insert $\frac{0+1}{1+1}$ between $\frac{0}{1}$ and $\frac{1}{1}$, to get $\frac{0}{1}$, $\frac{1}{2}$, $\frac{1}{1}$ for the second row F_2. Using this process, we can construct theFarey sequences table in the first eight rows of the following table as follows:

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$
$$F_7 = \left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$$
$$F_8 = \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\}$$

Table No. 1. Iteratively construct Farey sequences up to 8 order

Table No. 1 has several fascinating characteristics, at least up until this row. Every fraction is in its reduced form, and every reduced fraction fractions $\frac{a_1}{b_1}$ such that $0 \le \frac{a_1}{b_1} \le 1$ and $b_1 \le n$appear on the n^{th} row. If successive fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are found in the n^{th} row, then $b_1a_2 - a_1b_2 = 1$and $b_1 + b_2 \le n$.Thus, Table No. 1,gives the rational number $\frac{a_1}{b_1}$ with $\gcd(a_1, b_1) = 1$ is reduced form or lowest terms.

2.Main Results

Farey sequences [5, 9] appear in a variety of scientific fields and are connected to the theory of prime numbers. Farey sequences also have connections to unsolved mathematical puzzles, such as the Riemann hypothesis [3]. The best approximations for irrational numbers can be found using the Stern-Brocot tree [7]. Recent research has connected the components of a Farey sequence to the singularities of the Inverse Chirp Z-Transform [14], which is a generalization of the Inverse Fast Fourier Transform [15].The Farey sequence can be shown using Ford circles. The Stern-Brocot tree, which was created by removing unnecessary branches, has a subtree defined by the Farey sequence F_n [7]. Although the Stern Brocot tree was developed independently, it has many similarities with the Farey sequence, which is created by the mediant characteristic of the fractions. Stern Brocot tree ranges from 0 to ∞ [1], while the Farey sequence is between 0and 1.The Farey sequence with the Ford circle is discussed in detail as follows, along with some of the essential characteristics that are important to our paper.

Theorem 2.1: Suppose $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$are consecutive fractions in any row, then $\frac{a_1+b_1}{a_2+b_2}$is the only rational fraction with the smallest denominator between these two.

We will discuss the rational Approximations with the Farey sequence. We have approximate $\frac{1}{\pi} = 0.318309886183791$ with a denominator no larger than 100. Probably $\frac{\mathbf{32}}{100} = \frac{8}{\mathbf{25}}$ is the first and simplest approximation. The mediant can be used to approximate the Farey sequence. Iterations of the mediant in the Farey sequence can be used to find a reasonable approximation for an irrational integer between 0 and 1.The first Farey sequence occurs in the interval $\left\{\frac{0}{1}, \frac{1}{1}\right\}$. The median that corresponds to this is $\frac{1}{2}$. Choosing whether $\frac{1}{\pi}$ lies in $\left\{\frac{0}{1}, \frac{1}{2,}\right\}$ or $\left\{\frac{1}{2}, \frac{1}{1}\right\}$ is the next stage. As a result of $\frac{1}{\pi} < \frac{1}{2}$, the new interval is $\left\{\frac{0}{1}, \frac{1}{2}\right\}$, and $\frac{1}{2}$ is the current approximation. With the help of this interval, a new mediant and subsequent approximation can be found. The mediant for the new interval $\left\{\frac{0}{1}, \frac{1}{2}\right\}$is $\frac{1}{2}$. So, $\frac{29}{91}$ is a rational approximation of $\frac{1}{\pi}$. It should be noted that $\frac{7}{22}$ is still a better approximation than $\frac{29}{91}$,showing that while a different approximation is found after each step, it is not always the most recent fraction that is found after each repetition that provides the best approximation. The approximation is $\frac{57}{179}$, which is more accurate than $\frac{7}{22}$.

Theorem 2.2: If $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are Farey fractions contained in F_n such that they are not connected by another Farey fraction of order n, then

$$\left|\frac{a_1}{b_1} - \frac{a_1 + a_2}{b_1 + b_2}\right| = \frac{1}{b_1(b_1 + b_2)} \leq \frac{1}{b_1(n + 1)}$$

$$and \quad \left|\frac{a_2}{b_2} - \frac{a_1 + a_2}{b_1 + b_2}\right| = \frac{1}{b_2(b_1 + b_2)} \leq \frac{1}{b_2(n + 1)}$$

Proof:

Suppose that $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are Farey fractions contained in F_n, then

$$\left|\frac{a_1}{b_1} - \frac{a_1 + a_2}{b_1 + b_2}\right| = \left|\frac{a_1 b_1 + a_1 b_2 - a_1 b_1 - a_2 b_1}{b_1(b_1 + b_2)}\right| = \frac{1}{b_1(b_1 + b_2)}$$

$$\Rightarrow \left|\frac{a_1}{b_1} - \frac{a_1 + a_2}{b_1 + b_2}\right| = \frac{1}{b_1(b_1 + b_2)}$$

Since $(b_1 + b_2) \geq n + 1 \Rightarrow \frac{1}{(b_1+b_2)} \leq \frac{1}{n+1} \Rightarrow \frac{1}{b_1(b_1+b_2)} \leq \frac{1}{b_1(n+1)}, b_1 \neq 0$

Therefore, $\left|\frac{a_1}{b_1} - \frac{a_1+a_2}{b_1+b_2}\right| = \frac{1}{b_1(b_1+b_2)} \leq \frac{1}{b_1(n+1)}$

Similarly, we have $\left|\frac{a_2}{b_2} - \frac{a_1+a_2}{b_1+b_2}\right| = \frac{1}{b_2(b_1+b_2)} \leq \frac{1}{b_2(n+1)}$

Theorem 2.3: The inequalities $\frac{1}{uv} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{v^2}\right)$ and $\frac{1}{u(u+v)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{(u+v)^2}\right)$ cannot both hold if u and v are positive integers.

Proof:

We have the inequalities

$$\frac{1}{uv} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{v^2}\right) \Leftrightarrow \frac{1}{uv} \geq \frac{1}{\sqrt{5}u^2v^2}(v^2 + u^2) \Leftrightarrow \sqrt{5}uv \geq (u^2 + v^2)$$

Also,

$$\frac{1}{u(u+v)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{(u+v)^2}\right) \Leftrightarrow \frac{1}{u(u+v)} \geq \frac{1}{\sqrt{5}u^2(u+v)^2}((u+v)^2 + u^2)$$
$$\Leftrightarrow \sqrt{5}u(u+v) \geq ((u+v)^2 + u^2)$$

Assume for contradiction that both

$\sqrt{5}uv \geq (u^2 + v^2)and\sqrt{5}u(u+v) \geq ((u+v)^2 + u^2)$ are true.

Adding these two, we have

$$\sqrt{5}u(u+2v) \geq ((u+v)^2 + 2u^2 + v^2)$$
$$\Leftrightarrow \sqrt{5}(u^2 + 2uv) \geq 3u^2 + 2uv + 2v^2 \Leftrightarrow 3u^2 + 2uv + 2v^2\sqrt{5}(u^2 + 2uv)$$
$$\Leftrightarrow (3-\sqrt{5})u^2 - 2(\sqrt{5}-1)uv + 2v^2 \leq 0 \Leftrightarrow (6-2\sqrt{5})u^2 - 4(\sqrt{5}-1)uv + 4v^2 \leq 0$$
$$\Leftrightarrow (5-\sqrt{5}+1)u^2 - 4(\sqrt{5}-1)uv + 4v^2 \leq 0 \Leftrightarrow (\sqrt{5}-1)^2u^2 - 4(\sqrt{5}-1)uv + 4v^2 \leq 0$$
$$\Leftrightarrow (2v - (\sqrt{5}-1)u)^2 \leq 0 \Leftrightarrow 2v - (\sqrt{5}-1)u = 0 \Leftrightarrow \sqrt{5} = \frac{2v+u}{u}$$

But that contradicts $\sqrt{5}$ is irrational. Hence, the inequalities

$$\frac{1}{uv} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{v^2}\right) \text{ and } \frac{1}{u(u+v)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{u^2} + \frac{1}{(u+v)^2}\right)$$

cannot both hold if u and v are positive integers.

Theorem 2.4: (Hurwitz's Theorem[12])

Given that there are an infinite number of rational integers $\frac{u}{v}$that satisfy the condition

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{\sqrt{5}v^2} \qquad (1)$$

whereα is an irrational number.

Proof

Suppose that $n > 0$and the Farey sequence of order n contains the fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$in succession, such that $\frac{a_1}{b_1} < \alpha < \frac{a_2}{b_2}$. We assert that one of the three fractions $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$,and $\frac{a_1+a_2}{b_1+b_2}$ can operate as $\frac{u}{v}$ in (1). Assume that this is false,either $\alpha < \frac{a_1+a_2}{b_1+b_2}$ or $\alpha > \frac{a_1+a_2}{b_1+b_2}$

Case -I. If $\alpha < \frac{a_1+a_2}{b_1+b_2}$

Assume that $\alpha - \frac{a_1}{b_1} \geq \frac{1}{\sqrt{5}b_1^2}$, $\frac{a_1+a_2}{b_1+b_2} - \alpha \geq \frac{1}{(b_1+b_2)^2\sqrt{5}}$, $\frac{a_2}{b_2} - \alpha \geq \frac{1}{\sqrt{5}b_2^2}$

Adding the first and last inequalities

$$\alpha - \frac{a_1}{b_1} + \frac{a_2}{b_2} - \alpha \geq \frac{1}{\sqrt{5}b_1^2} + \frac{1}{\sqrt{5}b_2^2} \Rightarrow \frac{a_2}{b_2} - \frac{a_1}{b_1} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right)$$

$$\Rightarrow \frac{a_2 b_1 - a_1 b_2}{b_1 b_2} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right)$$

$$\Rightarrow \frac{1}{b_1 b_2} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right) \tag{2}$$

Adding first and second inequalities

$$\alpha - \frac{a_1}{b_1} + \frac{a_1 + a_2}{b_1 + b_2} - \alpha \geq \frac{1}{\sqrt{5}b_1^2} + \frac{1}{(b_1 + b_2)^2\sqrt{5}}$$

$$\Rightarrow \frac{-a_1 b_1 - a_1 b_2 + a_1 b_1 + a_2 b_1}{b_1(b_1 + b_2)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{(b_1 + b_2)^2}\right)$$

$$\Rightarrow \frac{a_2 b_1 - a_1 b_2}{b_1(b_1 + b_2)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{(b_1 + b_2)^2}\right)$$

$$\Rightarrow \frac{1}{b_1(b_1 + b_2)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{(b_1 + b_2)^2}\right) \tag{3}$$

These inequalities (2) and (3) contradict theorem 3. Therefore, at least one of $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_1+a_2}{b_1+b_2}$ will operate as $\frac{u}{v}$ in inequalities (1).

Case-II. If $\alpha > \frac{a_1+a_2}{b_1+b_2}$

Assume that $\alpha - \frac{a_1}{b_1} \geq \frac{1}{\sqrt{5}b_1^2},\ \alpha - \frac{a_1+a_2}{b_1+b_2} \geq \frac{1}{(b_1+b_2)^2\sqrt{5}},\ \frac{a_2}{b_2} - \alpha \geq \frac{1}{\sqrt{5}b_2^2}$

Adding the first and last inequalities

$$\alpha - \frac{a_1}{b_1} + \frac{a_2}{b_2} - \alpha \geq \frac{1}{\sqrt{5}b_1^2} + \frac{1}{\sqrt{5}b_2^2} \Rightarrow \frac{1}{b_1 b_2} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b_1^2} + \frac{1}{b_2^2}\right) \tag{4}$$

Adding the second and last inequalities

$$\alpha - \frac{a_1 + a_2}{b_1 + b_2} + \frac{a_2}{b_2} - \alpha \geq \frac{1}{(b_1 + b_2)^2\sqrt{5}} + \frac{1}{\sqrt{5}b_2^2}$$

$$\Rightarrow \frac{a_2}{b_2} - \frac{a_1 + a_2}{b_1 + b_2} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{(b_1 + b_2)^2} + \frac{1}{b_2^2}\right)$$

$$\Rightarrow \frac{1}{b_2(b_1 + b_2)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{(b_1 + b_2)^2} + \frac{1}{b_2^2}\right) \tag{5}$$

These inequalities (4) and (5) contradict theorem 3. Therefore, at least one of $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_1+a_2}{b_1+b_2}$ will operate as $\frac{u}{v}$ in inequalities (1).

The existence of $\frac{u}{v}$ fulfills the requirements that we have established the inequalities (1). Our choice of n will affect this $\frac{u}{v}$. In actuality, $\frac{u}{v}$ is either $\frac{a_1}{b_1}, \frac{a_2}{b_2}$or $\frac{a_1+a_2}{b_1+b_2}$, where $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are successive fractions in the Farey sequence of order n and $\frac{a_1}{b_1} < \alpha < \frac{a_2}{b_2}$. Using Theorem 2, we have

$$\left|\alpha - \frac{u}{v}\right| < \left|\frac{a_2}{b_2} - \frac{a_1}{b_1}\right| \leq \left|\frac{a_2}{b_2} - \frac{a_1 + a_2}{b_1 + b_2}\right| + \left|\frac{a_1 + a_2}{b_1 + b_2} - \frac{a_1}{b_1}\right| \leq \frac{1}{b_2(n+1)} + \frac{1}{b_1(n+1)} \leq \frac{2}{n+1}$$

$$\Rightarrow \left|\alpha - \frac{u}{v}\right| \leq \frac{2}{n+1}$$

We need proof that the number of satisfying the inequalities (1), $\frac{u}{v}$ is infinite. Assuming thatinequalities (1) holds to any $\frac{u_1}{v_2}$, then $\left|\alpha-\frac{u_1}{v_2}\right|>0$ and choose

$$n>\frac{2}{\left|\alpha-\frac{u_1}{v_2}\right|}\Rightarrow\left|\alpha-\frac{u_1}{v_2}\right|>\frac{2}{n}\Rightarrow\frac{2}{n}<\left|\alpha-\frac{u_1}{v_2}\right|\Rightarrow\frac{2}{n+1}<\frac{2}{n}<\left|\alpha-\frac{u_1}{v_2}\right|$$

$$\Rightarrow\frac{2}{n+1}<\left|\alpha-\frac{u_1}{v_2}\right|$$

Then, the Farey sequence of order n produces $\frac{u}{v}$ that satisfiesinequalities (1) and is such that

$$\left|\alpha-\frac{u}{v}\right|\leq\frac{2}{n+1}<\left|\alpha-\frac{u_1}{v_2}\right|$$

It follows that there are infinitely many rational integer$\frac{u}{v}$ that satisfy inequality (1).

Theorem 2.5: If a large value is used to replace $\sqrt{5}$, then Theorem 2.4 does not hold.

Proof:

In order to display, we need to verify that $\sqrt{5}$ cannot be changed by a large value.

Let us consider $\alpha=\frac{(1+\sqrt{5})}{2}$. Then,

$$(x-\alpha)\left(x-\frac{1-\sqrt{5}}{2}\right)=\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right)=x^2-x-1.$$

For integers $u, v$with $v>0$, we have

$$\left|\frac{u}{v}-\alpha\right|\left|\frac{u}{v}-\alpha+\sqrt{5}\right|=\left|\left(\frac{u}{v}-\alpha\right)\left(\frac{u}{v}-\frac{(1+\sqrt{5})}{2}+\sqrt{5}\right)\right|$$

$$=\left|\left(\frac{u}{v}-\alpha\right)\left(\frac{u}{v}-\frac{(1-\sqrt{5})}{2}\right)\right|=\left|\left(\frac{u}{v}-\frac{(1+\sqrt{5})}{2}\right)\left(\frac{u}{v}-\frac{(1-\sqrt{5})}{2}\right)\right|$$

$$=\left|\frac{u^2}{v^2}-\frac{u}{v}-1\right|=\left|\frac{u^2}{v^2}-\frac{u}{v}-1\right|=\frac{1}{v^2}|u^2-uv-v^2| \qquad (6)$$

As a result, of the irrationality of $\alpha$and $\sqrt{5}-\alpha$, the expression on the left in (6) is not zero. Therefore, the integer $|u^2-uv-v^2|$ is non-negative. It gives in $|u^2-uv-v^2|\geq 1$. Thus, expression (6) can be written as

$$\left|\frac{u}{v}-\alpha\right|\left|\frac{u}{v}-\alpha+\sqrt{5}\right|\geq\frac{1}{v^2}\Rightarrow\frac{1}{v^2}\leq\left|\frac{u}{v}-\alpha\right|\left|\frac{u}{v}-\alpha+\sqrt{5}\right| \qquad (7)$$

Assume that the infinite sequence $\frac{u_i}{v_i}$ and $v_i>0$ contains k positive real numbers and a rational number such that

$$\left|\frac{u_i}{v_i}-\alpha\right|<\frac{1}{kv_i^2} \qquad (8)$$

$$\Rightarrow-\frac{1}{kv_i^2}<\frac{u_i}{v_i}-\alpha<\frac{1}{kv_i^2}\Rightarrow-\frac{1}{kv_i}<u_i-\alpha v_i<\frac{1}{kv_i}$$

$$\Rightarrow\alpha v_i-\frac{1}{kv_i}<u_i<\frac{1}{kv_i}+\alpha v_i$$

This means that for each value of v_i, there are a finite number of u_i.As a result, we have, as $i\to\infty,\ v_i\to\infty$. According to the triangle inequality and the expressions (7) and (8), we have

$$\frac{1}{v_i^2}\leq\left|\frac{u_i}{v_i}-\alpha\right|\left|\frac{u_i}{v_i}-\alpha+\sqrt{5}\right|<\frac{1}{kv_i^2}\left(\frac{1}{kv_i^2}+\sqrt{5}\right)$$

$$\Rightarrow 1<\frac{1}{k}\left(\frac{1}{kv_i^2}+\sqrt{5}\right)\Rightarrow k<\left(\frac{1}{kv_i^2}+\sqrt{5}\right)$$

$$\Rightarrow k \leq \lim_{n\to\infty}\left(\frac{1}{kv_i^2}+\sqrt{5}\right)=\sqrt{5}$$

$$\Rightarrow k \leq \sqrt{5}$$

Therefore, if a large value is used to replace $\sqrt{5}$, then Theorem 4 does not hold. Hence $\sqrt{5}$ is the most important role in the Hurwitz Theorem.

Consider the centers of the circle are $\left(\frac{a}{b},\frac{1}{2b^2}\right)$ and $\left(\frac{a_1}{b_1},\frac{1}{2b_1^2}\right)$. By using the Pythagorean Theorem, one may calculate the length between the centers of two circles. Then

$$\sqrt{\left(\frac{a}{b}-\frac{a_1}{b_1}\right)^2+\left(\frac{1}{2b^2}-\frac{1}{2b_1^2}\right)^2}=\sqrt{\frac{1}{b^2b_1^2}+\frac{1}{4b^4}-\frac{1}{2b^2b_1^2}+\frac{1}{2b_1^4}}=\sqrt{\left(\frac{1}{2b^2}+\frac{1}{2b_1^2}\right)^2}=\frac{1}{2b^2}+\frac{1}{2b_1^2}$$

This shows that the distance between two circle centers is determined by adding their radii together. It follows that the circles are tangent. Ford proposed the idea in the article Fractions[6], where it was given the name of the Ford circles.

The Ford circle $C(u,v)$ is a circle with a center at $\left(\frac{u}{v},\frac{1}{2v^2}\right)$ and a radius of $\frac{1}{2v^2}$ for every rational number $\frac{u}{v}$ in lowest terms. Thus, $C(u,v)$ is the circle with radius $\frac{1}{2v^2}$ that is tangent to the x-axis at $x=\frac{u}{v}$.

The Ford circles generate a maximal arrangement of circles, in which each circle is above but tangent to the x-axis at a rational integer, with disjoint interiors, and no more circles of such a type can be added. They are applied in the circle approach of Ramanujan and Hardy [6] and provide a geometric perspective on continued fractions. They also provide a natural way to understand the Diophantine approximation of real numbers by rational. We build Ford circles for each Farey sequence in the lowest terms $\frac{u}{v}$ and give a circle with radii$\frac{1}{2v^2}$ above but tangent to the x-axis at $\frac{u}{v}$. Observe that there are infinitely many Ford circle tangent points occurring in every small interval of the x-axis in fig. no. 1.

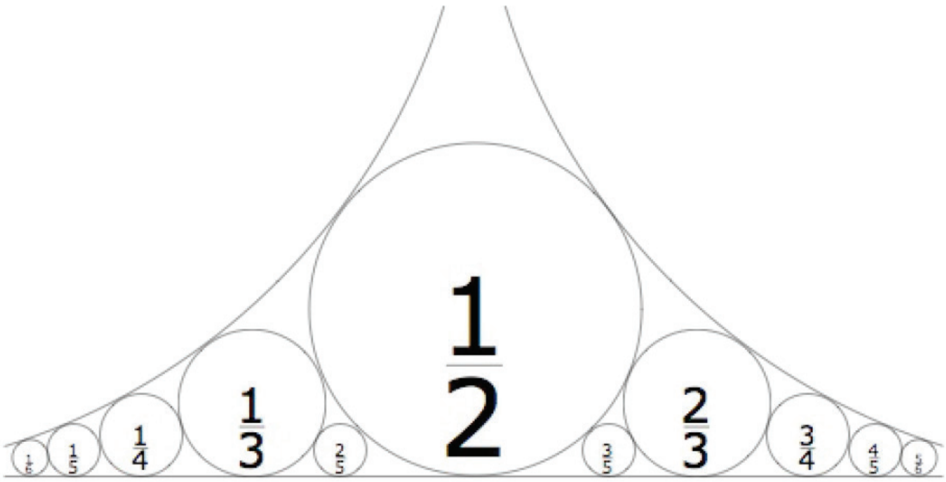

Fig. No. 1. Infinitely many Ford circle tangents at the x-axis

Consider a Farey sequence F_n in which terms $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$are three consecutive terms. Then the circles $C(a_1,b_1)andC(a_2,b_2)$are tangent at $u_1=\left(\left(\frac{a_2}{b_2}-\frac{b_1}{b_2(b_2^2+b_1^2)}\right),\frac{1}{(b_2^2+b_1^2)}\right)$.

Similarly, the circles $C(a_2,b_2)$ and $C(a_3,b_3)$ are tangent at $v_1=\left(\left(\frac{a_2}{b_2}+\frac{b_3}{b_2(b_2^2+b_3^2)}\right),\frac{1}{(b_2^2+b_3^2)}\right)$.

Furthermore, u_1 and v_1 are located on semicircles with diameters of $\frac{a_2}{b_2}-\frac{a_1}{b_1}$and $\frac{a_3}{b_3}-\frac{a_2}{b_2}$, respectively [4], which is the relation between Ford circles and the Farey sequence.Assume that Ford circles with tangents are $C(a_1,b_1)$and $C(a_2,b_2)$.

The largest Ford circle $C(a_1+a_2,b_1+b_2)$, is connected to the mediant [4], which connects the Largest Ford circle between tangent Ford circles. Also, the Farey sequence $F_n$and the set of Ford circles C_n have a one-to-one correspondence. The circles with radius $\frac{1}{2b^2}$ that are tangent to the x-axis at the fraction $\frac{a}{b}\in F_n$ are defined as Ford circles of order n and denotedC_n. So, the corresponding Ford circles are tangent if the Farey fractions $\frac{a}{b}$ and $\frac{a_1}{b_2}$ are adjacent.

3.Conclusions

There are many fascinating patterns in mathematics, and much more Mathematics involves Farey sequences. The Farey sequence can be found in a variety of mathematical structures, including Ford circles, and the Stern-Brocot tree. Additionally, they can be used to approximate irrational numbers rationally. Ford circles are infinitely many, with one for each rational number. We aim to demonstrate both the mathematical aspect of the Farey sequence and its application to the Ford Circle, rational approximation, and the Stern-Brocot tree. Specially, the use of the Farey sequence in the rational approximation of real numbers by the Hurwitz Theorem, and the Ford circle to approximate rational numbers.

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