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Modified Generalized Exponential Distribution

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Abstract: *In this article, a new distribution having three called Modified Generalized Exponential Distribution is proposed. Important statistical properties of the proposed model like survival function, hazard rate function, the probability density function (PDF), the cumulative distribution function (CDF), quantile function, skewness, and kurtosis are discussed here. Least Square Estimation (LSE), Cramer-Von Mises (CVM) and Maximum Likelihood estimators (MLE) methods are used for estimation of parameters using R programming software. A data set is discussed and performed the goodness-of-fit to assess the applications of the proposed distribution. Various methods of model comparison and model validation are also used. The proposed model Modified Generalized Exponential Distribution is more applicable as compared to some existing probability model.*

Keywords: *Exponential distribution, Estimation, Hazard function, Cramer-von Mises, Maximum likelihood.*

1. Introduction

Probability models are very useful in reliability analysis of different fields of biological science, applied statistics as well as engineering. During the modeling of the data, probability models available so far may not produce better fit in modeling reliability data. Due to this reason, researchers have been adjusting traditional probability models and describing the acceptance of those models in practice. This can be done by adding one or more extra parameters to the baseline distribution. Addition of extra parameters generates new probability models. These modified models usually provide a better fit to the data than the traditional models.

In the last decades, exponential model is frequently used as baseline model to create new probability models. In literature, we can find a lot of modifications of the exponential distributions. Some of the modified distributions are generalized exponential (GE) [Gupta & Kundu, 2007], generalized inverted exponential distribution [Abouammoh & Alshingiti, 2009], gamma EE [Ristic & Balakrishnan, 2012] and exponential extension (EE) model [Kumar, 2010] etc.
These lifetime models may have bathtub-shaped Hazard rate function (hrf).In real life we can find many data that have bathtub-shaped hrf. In literature we can also find many modifications of Weibull distribution. The two parameter Weibull distribution is given as

$$\overline{F}(y,\lambda,\beta) = \exp[-(\lambda, y)]^{\beta} \quad (1)$$

Above distribution does not have bathtub hrf. This distribution is modified to generate several distributions that possess bathtub hrf. One of the modifications of Weibull distribution is exponentiated Weibull distribution [Mudholkar & Srivastava (1993)]. Taking appropriate limits on beta integrated distribution [Lai et al. (2016)] to get new lifetime distributions as

$$\overline{F}(y) = \exp[ay^{b}.\exp(\lambda y)] \tag{2}$$

Here, generalized exponential distribution is modified to introduce new probability model called modified generalized exponential distribution. The CDF of generalized exponential distribution is given as,

$$F(x,\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha} \tag{3}$$

The CDF and PDF of the Modified Generalized exponential (MGE) model can be given as,

$$F(x;\alpha,\beta,\lambda) = \left[1 - \exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha} \quad ; \ \alpha > 0, \beta > 0, \lambda > 0, x > 0$$

$$f(x;\alpha,\beta,\lambda) = \alpha\lambda\left(1+\beta x\right)\exp\left(\beta x - \lambda x e^{\beta x}\right)\left[1 - \exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha-1}$$

2. Model Analysis

Modified Generalized Exponential (MGE) distribution:

Cumulative distribution function of *Modified Generalized Exponential* distribution is defined by

$$F(x;\alpha,\beta,\lambda) = \left[1 - \exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha} \quad ; \ \alpha > 0, \beta > 0, \lambda > 0, x > 0 \tag{4}$$

And the PDF of Modified Generalized Exponential distribution can be as

$$f(x;\alpha,\beta,\lambda) = \alpha\lambda\left(1+\beta x\right)\exp\left(\beta x - \lambda x e^{\beta x}\right)\left[1 - \exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha-1} \tag{5}$$

Survival function:

The Survival function of MGE model is

$$R(x;\alpha,\beta,\lambda) = 1 - \left[1 - \exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha} \quad ; x > 0 \tag{6}$$

Hazard rate function:

Hazard rate function of MGE distribution with parameters $\left(\alpha,\beta,\lambda\right)$ is

$$h\left(x\right) = \alpha\lambda(1+\beta x)e^{\beta x}e^{-\lambda x e^{\beta x}}\left[1 - e^{-\lambda x e^{\beta x}}\right]^{\alpha-1}\left[1 - \left(1 - e^{-\lambda x e^{\beta x}}\right)^{\alpha}\right]^{-1} \tag{7}$$

Reverse hazard function of MGE:

Reverse hazard function of MGE model can be expressed as

$$h_{rev}(x) = \alpha\lambda(1+\beta x)e^{\beta x}e^{-\lambda x e^{\beta x}}\left[1 - e^{-\lambda x e^{\beta x}}\right]^{\alpha-1}\left[1 - e^{-\lambda x e^{\beta x}}\right]^{-\alpha} ; \ x > 0, \tag{8}$$

The various shapes of pdf and hazard rate function of MGE$\left(\alpha,\beta,\lambda\right)$ at various values of constants are displayed in Figure 1.

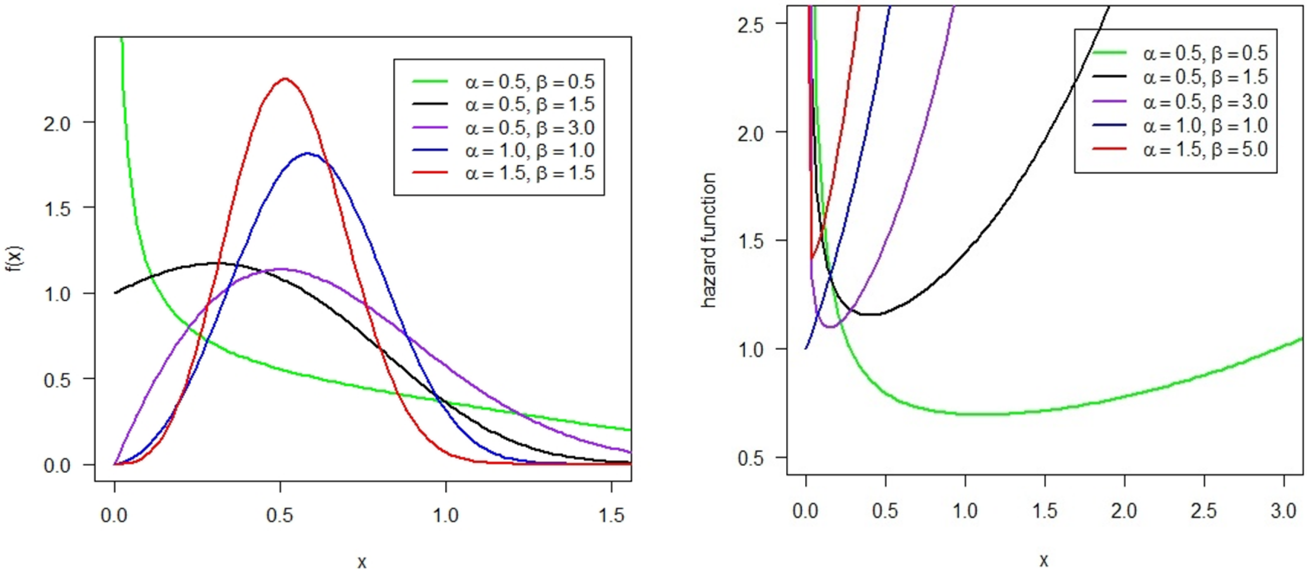

***Figure* 1:** *Graphs of PDF in left panel and HRF in right panel for fixed* λ.

Cumulative hazard function:

The cumulative hazard rate function of the MGE (α, β, λ) is defined as

$$H(x) = \int_{-\infty}^{x} h(y)\,dy = -\log\left[1 - F(x)\right]$$

$$= -\log\left[1 - \left[1 - e^{-\lambda x e^{\beta x}}\right]^{\alpha}\right]; \;\; x > 0,\; \alpha, \beta, \lambda > 0 \qquad (9)$$

Quantile function:

Let *X* be a non-negative random variable with CDF as $F_X(x)$ then quantile function can be defined by

$$\lambda x e^{\beta x} + \log\left(1 - p^{(1/\alpha)}\right) = 0 \quad ; 0 < p < 1.$$

$$\log(\lambda) + \log(x) + \beta x + \log\left[\log\left(1 - p^{(1/\alpha)}\right)\right] = 0 \quad ; 0 < p < 1.$$

Random Deviate Generation:

Random deviate generation for the MGE(α, β, λ) is displayed in expression (10) as,

$$\lambda x e^{\beta x} + \log\left(1 - u^{(1/\alpha)}\right) = 0 \quad ; 0 < u < 1. \qquad (10)$$

where *u* has the uniform *U*(0, 1) distribution.

Skewness and Kurtosis:

The Bowley's coefficient of skewness based on quartiles is,

$$S_B = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)}$$

Coefficient of kurtosis is,

$$K-Moors=\frac{Q(0.875)-Q(0.625)-Q(0.125)+Q(0.375)}{Q(0.75)-Q(0.25)}$$

3. Parameter Estimation

3.1. Maximum Likelihood Estimation

There are different methods of parameter estimation available in statistics. For estimation of parameters, we used three most applicable and useful methods of estimation. Firstly we have used maximum likelihood estimation method. Let n denotes the number of items, then summation up to n in density function of MGE to get the log- likelihood function of random sample drawn from MGE distribution is given in (5.3.1). That is, taking log and then summation up to n in density function of proposed model. Resulting log-likelihood function is given in expression (11)

$$\ell(\alpha,\beta,\lambda\mid \underline{x})=n\log\alpha+n\log\lambda+\sum_{i=1}^{n}\log(1+\beta x_i)+\beta\sum_{i=1}^{n}x_i-\lambda\sum_{i=1}^{n}x_i e^{\beta x_i}$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\left(-\lambda x_i\, e^{\beta x_i}\right)\right] \qquad (11)$$

As α, β, and λ are the unknown parameters of the distribution, so for estimation, getting differentiation with respect to these unknown parameters and resulting expressions are mentioned below as,

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\left[1-e^{-\lambda x_i e^{\beta x_i}}\right]$$

$$\frac{\partial\ell}{\partial\beta}=\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)+\sum_{i=1}^{n}x_i-\lambda\sum_{i=1}^{n}x_i^2 e^{\beta x_i}+(\alpha-1)\lambda\sum_{i=1}^{n}\left[x_i^2 e^{\beta x_i}e^{-\lambda x_i e^{\beta x_i}}\right]$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i e^{\beta x_i}+(\alpha-1)\sum_{i=1}^{n}\left[x_i e^{\beta x_i}e^{-\lambda x_i e^{\beta x_i}}\right]$$

Equating $\frac{\partial\ell}{\partial\alpha}=\frac{\partial\ell}{\partial\beta}=\frac{\partial\ell}{\partial\lambda}=0$ and solving simultaneously for α, β, and λ, we estimated ML estimators of the parameters MGE(α,β,λ) model.

Generally, this is not possible for solving non-linear equations above so with the aid of suitable computer package we can estimate them easily.

Suppose $\underline{\Theta}=(\alpha,\beta,\lambda)$ is the parameter vector of MGE(α,β,λ) with respective MLE of Θ as $\hat{\underline{\Theta}}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$. The asymptotic normality is given as, $\left(\hat{\underline{\Theta}}-\underline{\Theta}\right)\rightarrow N_3\left[0,\left(I(\underline{\Theta})\right)^{-1}\right]$ .where,

$$I(\underline{\Theta})=-\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right)\\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right)\\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right)\end{pmatrix}$$

Since $\underline{\Theta}$ is unknown so it is worthless that the MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$. Using the estimated value of parameters one can approximate the asymptotic variance. And observed fisher information matrix $O(\underline{\hat{\Theta}})$ can be used as an estimate of $I\left(\underline{\Theta}\right)$ in form of hessian matrix H as,

$$O(\underline{\hat{\Theta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \hat{\alpha}^2} & \frac{\partial^2 l}{\partial \hat{\alpha}\partial \hat{\beta}} & \frac{\partial^2 l}{\partial \hat{\alpha}\partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\beta}\partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\beta}^2} & \frac{\partial^2 l}{\partial \hat{\beta}\partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\lambda}\partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\lambda}\partial \hat{\beta}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} \end{pmatrix}_{\Big|\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\right)} = -H\left(\underline{\Theta}\right)_{\Big|\left(\underline{\Theta}=\underline{\hat{\Theta}}\right)}$$

We can use Newton-Raphson technique for optimizing the likelihood can produce the observed information matrix. The variance covariance matrix is,

$$-\left[-H\left(\underline{\Theta}\right)_{\Big|\left(\underline{\Theta}=\underline{\hat{\Theta}}\right)}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) \\ \operatorname{cov}(\hat{\beta},\hat{\alpha}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\beta},\hat{\lambda}) \\ \operatorname{cov}(\hat{\lambda},\hat{\alpha}) & \operatorname{cov}(\hat{\lambda},\hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix} \tag{12}$$

Using asymptotic normality for MLEs, *100(1-b) %* approximated CI of α, β, and λ of MGE $\left(\alpha,\beta,\lambda\right)$ can be constructed as,

$$\hat{\alpha} \pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\alpha})}, \hat{\beta} \pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\beta})}, \text{ and } \hat{\lambda} \pm Z_{b/2}\sqrt{\operatorname{var}(\hat{\lambda})},$$

where $Z_{b/2}$ be upper percentile for Standard normal variate.

3.2. Least-Square Estimation

Consider $F\left(X_{(i)}\right)$ is the CDF of the variables $X_{(1)} < X_{(2)} < \dots < X_{(n)}$. Here $\{X_1, X_2, \dots, X_n\}$ is a random sample having size n with a distribution function F (.). LSE of the unknown parameters α, β, and λ, of MGE $\left(\alpha,\beta,\lambda\right)$ distribution can be obtained by minimizing (13) with respect to unknown parameters α, β, and λ.

$$A\left(X;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n}\left[F(X_{(i)}) - \frac{i}{n+1}\right]^2 \tag{13}$$

Substituting the distribution function of MGE in (13), we get

$$A\left(X;\alpha,\beta,\lambda\right) = \sum_{i=1}^{n}\left[\left\{1-e^{-\lambda x e^{\beta x_{(i)}}}\right\}^{\alpha} - \frac{i}{n+1}\right]^2 \tag{14}$$

Following expression of partial derivatives is obtained by differentiating (14) with respect to parameters.

$$\frac{\partial A}{\partial \alpha} = 2\sum_{i=1}^{n}\left[1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right]^{\alpha} \log\left[1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right]\left[\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha} - \frac{i}{n+1}\right]$$

$$\frac{\partial A}{\partial \beta}=2\sum_{i=1}^{n}\left[\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{i}{n+1}\right]\left[\alpha\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha-1}\right]$$

$$\frac{\partial A}{\partial \lambda}=2\alpha\sum_{i=1}^{n}x_i e^{\beta x_i}e^{-\lambda x_i e^{\beta x_{(i)}}}\left[\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{i}{n+1}\right]\left[1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right]^{\alpha-1}$$

By minimizing the function below we can find weighted least square estimates

$$A(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

Given that weights w_i as $w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$

Then weighted LSE can be determined by differentiating for minimization of relation (15) with respect α, β, and λ as,

$$A(X;\alpha,\beta,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left\{1-e^{-\lambda x e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{i}{n+1}\right]^2 \qquad (15)$$

3.3 Cramer-Von-Mises estimation

We can minimize the function (16) to obtain the Cramer-Von-Mises estimators of the parameters α, β and λ. That is,

$$Z(X;\alpha,\beta,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\mid\alpha,\beta,\lambda)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left\{1-e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{2i-1}{2n}\right]^2 \qquad (16)$$

Differentiating (16) with respect to α, β, and λ we get,

$$\frac{\partial Z}{\partial \alpha}=2\sum_{i=1}^{n}\left[1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right]^{\alpha}\log\left[1-e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\right]\left[\left\{1-e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{2i-1}{2n}\right]$$

$$\frac{\partial Z}{\partial \beta}=2\sum_{i=1}^{n}\left[\alpha\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha-1}\right]\left[\left\{1-e^{-\lambda x_i e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{2i-1}{2n}\right]$$

$$\frac{\partial Z}{\partial \lambda}=2\alpha\sum_{i=1}^{n}x_{(i)}e^{\beta x_{(i)}}e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\left[1-e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\right]^{\alpha-1}\left[\left\{1-e^{-\lambda x_{(i)}e^{\beta x_{(i)}}}\right\}^{\alpha}-\frac{2i-1}{2n}\right]$$

By simultaneous solving non-linear equations $\frac{\partial Z}{\partial \alpha}=0,\frac{\partial Z}{\partial \beta}=0$ and $\frac{\partial Z}{\partial \lambda}=0$

we can obtain CVM estimators.

4. Application to Real Data Set

Here, a real data set is taken for checking the suitability as well as applicability of the MGE model. The dataset is breaking stress of 100 observations of carbon fibers (in Gba) [Nichols & Padgett (2006)], from a bootstrap control chart for Weibull percentiles, "Quality and Reliability Engineering International", 22, pp. 141-151.

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 1.87, 3.15, 4.90,3.75, 3.65 ,2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22,3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56,3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92,1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71,2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38,1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80,1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.09.

We have depicted the graph of parameter profile versus log-likelihood function in Figure 2. From these graph we can say that estimated ML can be calculated uniquely.

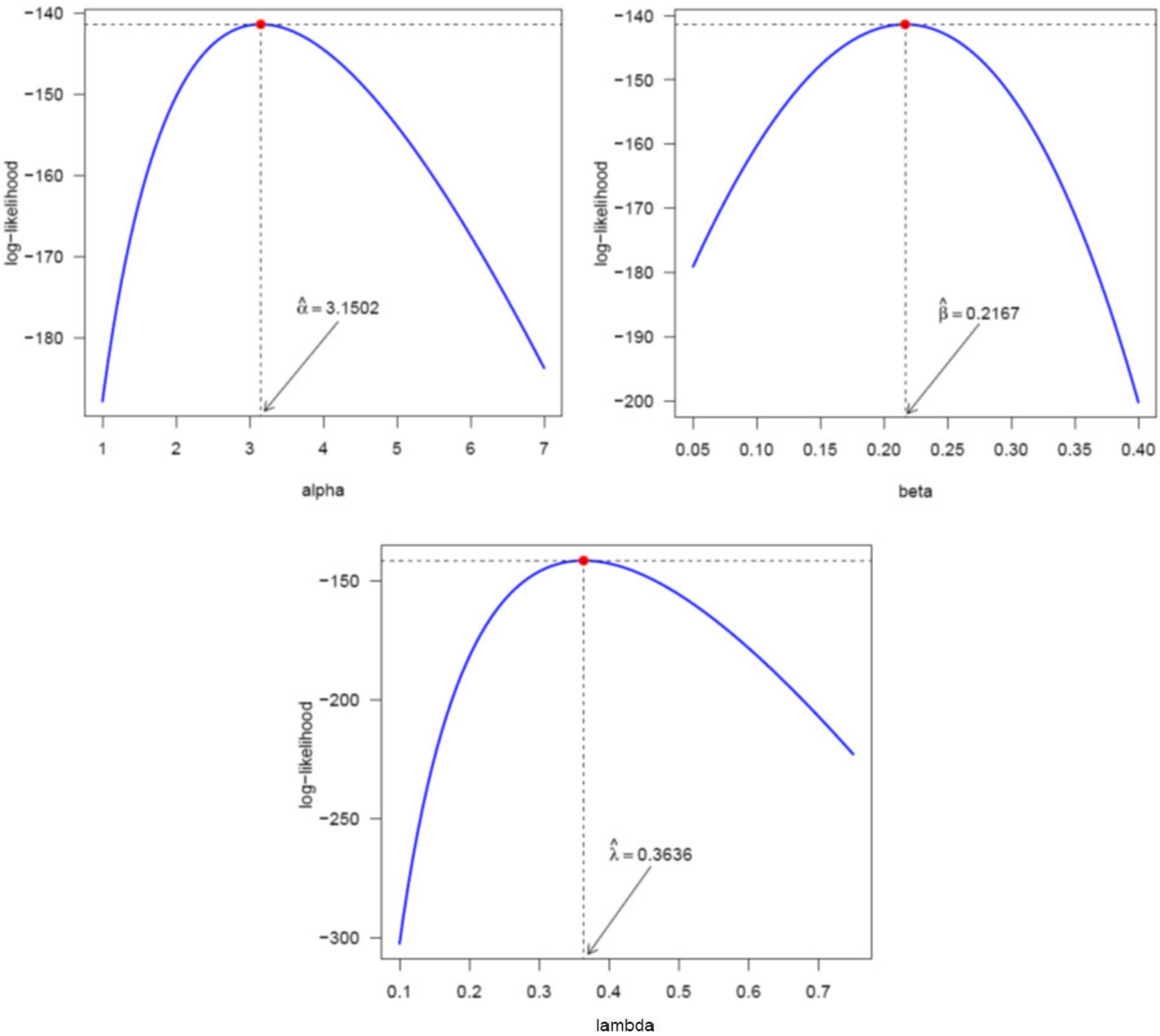

Figure 2: *Profile log-likelihood function of the parameters α, β, and λ of the MGE model.*

Here, we have used optim () function of R language (R Core Team, 2020) and (Ming Hui, 2019), The MLEs of MGE model are calculated by maximizing the likelihood function. We got the Log-Likelihood value as l = -141.3544. Table 1 gives MLE's with their corresponding standard errors for α, β, and λ

Table 1: MLE and SE for α, β, & λ

Parameter	MLE	SE
alpha	3.1502	1.10699
beta	0,2167	0.09044
lambda	0.3636	0.16083

Graphs of P-P plot and Q-Q plot in Figure 3. From these plotted graphs it is clear that proposed model MGE fits the real data set more precisely.

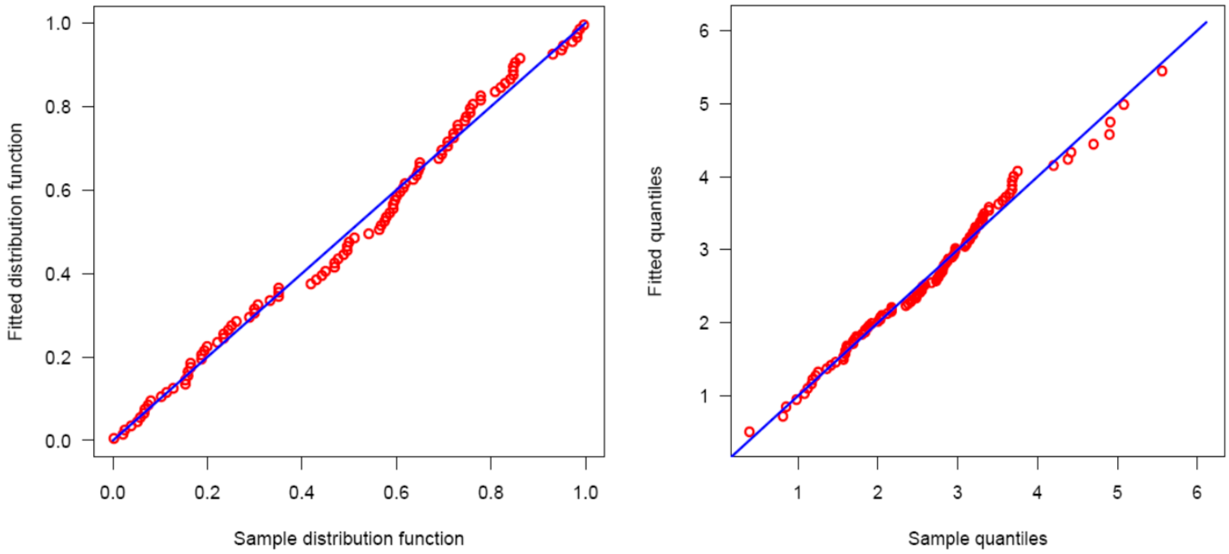

***Figure* 3:** *The P-P plot in left panel and Q-Q plot in right panel of the MGE distribution.*

Table 5.2 contains the estimated parameter values using all three methods of estimation MLE, LSE and CVE. Table also contains the values of different information criteria values like negative log-likelihood, AIC, BIC, CAIC, and HQIC. As we know that lesser the value of information criteria, better the fit of model. Here, MLE estimation method has least values among all in all the cases so MLE gives better estimates of parameter.

***Table* 2:** *Estimated parameters, log-likelihood, AIC, BIC, CAIC, and HQIC*

Methods	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	LL	AIC	BIC	CAIC	HQIC
MLE	3.1502	0.2167	0.3636	-141.3544	288.7088	296.5244	288.9588	291.8719
LSE	1.3085	0.5878	0.0729	-147.8597	301.7194	309.5349	301.9694	304.8825
CVE	1.3158	0.5995	0.0710	-148.5176	303.0352	310.8507	303.2852	306.1983

We have also calculated the KS, W and A^2 statistics & corresponding p-values on same set of real data sets using the estimated parameters using all three methods of estimation and are displayed in table 3.

***Table* 3:** *The KS, W and A^2 statistic with a p-value*

Method	KS(p-value)	W(p-value)	A^2(p-value)
MLE	0.0637(0.8118)	0.0697(0.7542)	0.413(0.8354)
LSE	0.0502(0.9623)	0.044(0.9126)	0.8363(0.4553)
CVE	0.0521(0.9487)	0.0433(0.9161)	0.8861(0.4226)

The Graph for Q-Q plot for estimation methods MLE, LSE and CVM and Histogram & the fitted density function of MGE distribution are shown in figure 4.

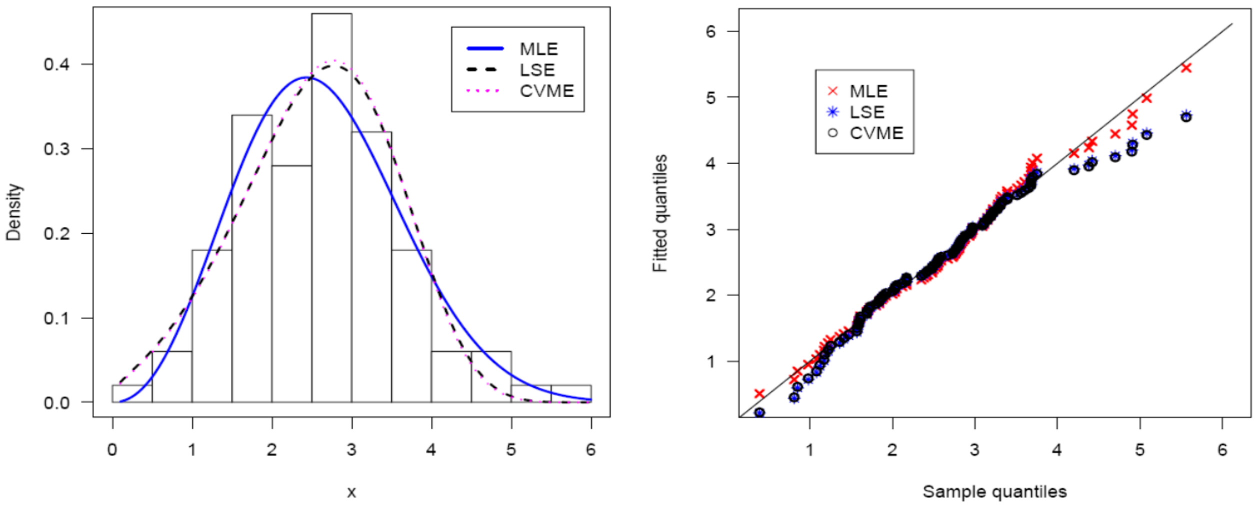

***Figure* 4:** *The Q-Q plot in right panel & Histogram and the density function of fitted distributions in left panel of different methods of estimation.*

5. Model Comparison

Here we have presented the applicability of MGE model distribution on same real dataset taking some important probability models used by previous researchers. For this, we have selected the six other distributions. These are *Generalized Exponential Extension (GEE), Generalized Gompertz*, Weibull Extension (WE) distribution, *Flexible Weibull (FW) distribution, Generalized Exponential (GE) distribution* and *Gompertz distribution (GZ).*

We have illustrated the AIC, BIC, CAIC, and HQIC for the evaluation of the applicability of MGE distribution tabulated in Table 4.

***Table* 4:** *Log-likelihood, AIC, BIC, CAIC, and HQIC*

10	LL	AIC	BIC	CAIC	HQIC
MGE	-141.3544	288.7088	296.5244	288.9588	291.8719
GEE	-141.3708	288.7416	296.5571	288.9916	291.9047
GGZ	-141.3899	288.7799	296.5954	289.0299	291.9430
WE	-141.5577	289.1153	296.9309	289.3653	292.2784
FW	-143.2775	290.5551	296.7654	290.6788	292.6638
GE	-146.1823	296.3646	301.5749	296.4883	298.4733
GZ	-149.1250	302.2500	307.4604	302.3737	304.3588

We have also calculated the values of KS, AD, and CVME values and corresponding p values when parameters are estimated using three methods MLE, LSE and CVME. It is noted that in most of the cases MGE has minimum value and higher p-values than the competing distributions.

***Table 5**: The goodness-of-fit statistics and their corresponding p-value*

Model	KS(p-value)	W(p-value)	A^2(p-value)
MGE	0.0637(0.8118)	0.0697(0.7542)	0.4130(0.8354)
GEE	0.0654(0.7862)	0.0723(0.7385)	0.4202(0.8281)
GGZ	0.0637(0.8114)	0.0708(0.7475)	0.4198(0.8286)
WE	0.0607(0.8542)	0.0635(0.7932)	0.4212(0.8268)
FW	0.0778(0.5805)	0.1103(0.5375)	0.6239(0.6253)
GE	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
GZ	0.0962(0.3129)	0.2280(0.2193)	1.7537(0.1261)

Here, we displayed graph of goodness-of-fit of MGE model and the considered models that are shown in Figure 5. It exhibits that the proposed model MGE fits data better to real data compared to other distribution taken in considerations.

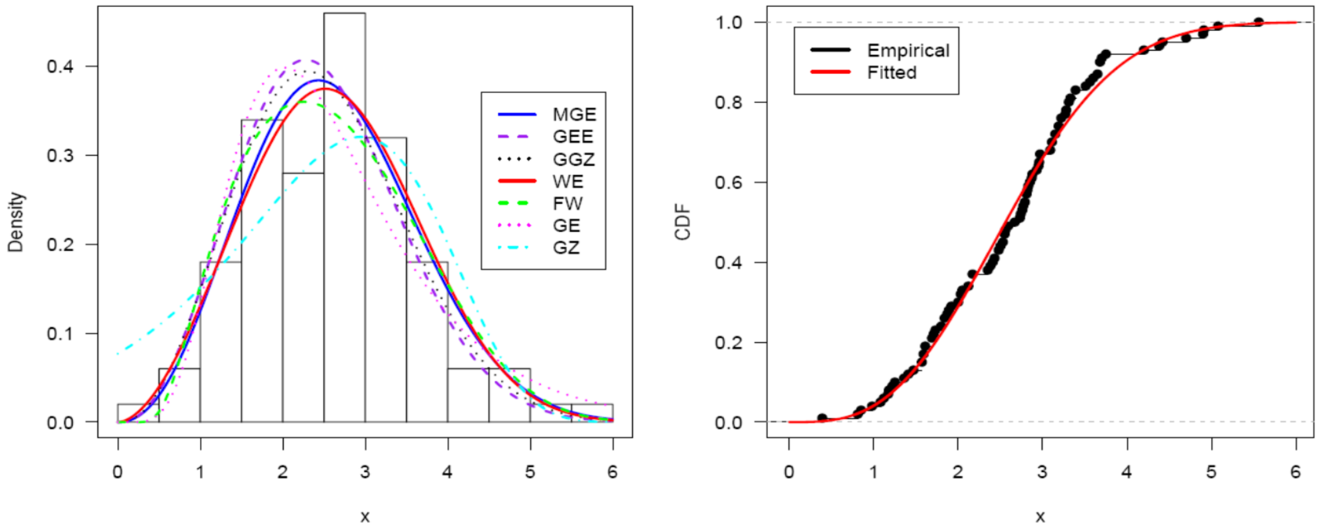

***Figure 5**: Histogram and fitted density function in left and graph for Empirical distribution function with estimated distribution function in right panel of MGE*

Models for Comparison:

(i). **Generalized Exponential Extension (GEE) Distribution:**

The PDF of GEE given by [Lemonte, 2013] is

$$f_{GEE}(x)=\alpha\beta\lambda\,(1+\lambda x)^{\alpha-1}\,exp\left\{1-(1+\lambda x)^{\alpha}\right\}\left[1-exp\left\{1-(1+\lambda x)^{\alpha}\right\}\right]^{\beta-1}\;;x>0$$

(ii). **Generalized Gompertz distribution**:

The PDF of Gompertz distribution [El-Gohary et al., 2013] with parameters α, λ and θ is

$$f_{GGZ}(x)=\theta\lambda e^{\alpha x}e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\left[1-\exp\left(-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)\right)\right]^{\theta-1}\;;\;\;\lambda,\theta>0,\alpha\geq 0,x\geq 0$$

(iii). **Weibull Extension (WE) distribution:**

The probability density function of Weibull extension (Tang et al., 2003) with three parameters (α, β, λ) is given by

$$f_{WE}(x) = \lambda\beta\left(\frac{x}{\beta}\right)^{\beta-1} \exp\left(\frac{x}{\beta}\right)^{\beta} \exp\left\{-\lambda\alpha\left(\exp\left(\frac{x}{\beta}\right)^{\beta}-1\right)\right\}; x > 0$$

(iv). **Flexible Weibull (FW) distribution:**

The density function of Flexible Weibull (FW) extension distribution [Bebbington, 2007] with α and β are given as

$$f_{FW}(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left(\alpha x - \frac{\beta}{x}\right) \exp\left\{-\exp\left(\alpha x - \frac{\beta}{x}\right)\right\} \quad ; x \geq 0, \quad \alpha, \beta \geq 0$$

(v). **Generalized Exponential (GE) distribution:**

The PDF of GE model [Gupta & Kundu, 1999] is given by

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}\left\{1 - e^{-\lambda x}\right\}^{\alpha-1}; \quad \alpha > 0, \lambda > 0, x > 0$$

(vi). **Gompertz distribution (GZ) distribution:**

PDF of Gompertz distribution [Murthy et al., 2003] is

$$f_{GZ}(x) = \theta\ e^{\alpha x}\ exp\left\{\frac{\theta}{\alpha}\left(1 - e^{\alpha x}\right)\right\} \quad ; x \geq 0, \theta > 0, -\infty < \alpha < \infty.$$

Here, we have also shown the estimated fitted cdf curve of the MGE model with the empirical cdf curve in figure 6.

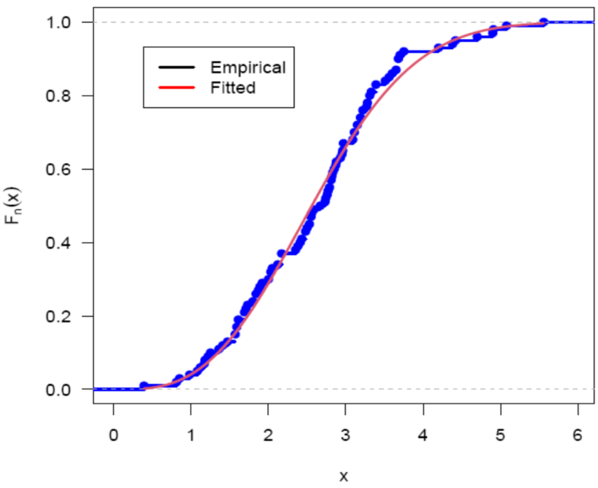

***Figure 6**: Empirical distribution plot and fitted distribution curve of MGE*

6. Conclusion

Here, we have formulated a new model called *Modified Generalized Exponential distribution* containing three parameters. Here we have presented and plotted some important statistical as well as mathematical properties of the model. We also derived the expression for hazard function, reliability function, skewness and kurtosis etc. Three important method of estimation are used for the estimation of the parameters of the model. The probability density curve of MGE have shown that its shape is increasing-decreasing having right skewed. The curve is found to be more flexible for modeling for a real-life data also. Hazard function shows that inverted bathtub or reverse j-shaped depending on the values of parameters of the model. A real set data and different model validation criteria show that the proposed model fits data better than the other models taken in consideration.

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