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Approximation Solutions for Solving Some Special Functions of Fractional Calculus via Caputo - Fabrizio Sense

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Abstract: Fractional calculus is a novel idea that has found wide use in modern science across many fields. The numerical and analytical solutions to the arbitrary orders of fractional derivatives and integrations have not been studied. The purpose of this work is to compare the approximate solutions of the non-fractional equation when $\alpha = 1$ and other approximation solutions of the fractional equation when $\alpha =$ 0 to 1 and usually some systemic graphs in 2 or 3-dimensional space can be found. The fractional differentiation equation(FDE) of some special functions that are identity, sine, and cosine functions in the sense of Caputo Fabrizio are investigated. By employing the Grunwald-Letinikov (G-L) numerical solution, non-zero fractional derivatives of a constant function are avoided in this work. This work makes a significant contribution in that when the orders of the fractional derivatives and integrations approach 1, the approximation result of the derivatives and integrations of some special functions of fractional calculus via the Caputo - Fabrizio Sense is comparable to conventional calculus and the operator is linear; otherwise, the result differs from the conventional derivative, and the applied operator is non-linear. A comparison of the smoothness of the curves in the $2D$ and $3D$ situations, as well as various values of α at different step sizes, are examined using a number of the instances mentioned above as examples to illustrate the ideas.

Keywords: Caputo fractional derivatives, Grunwald-Letinikov, Numerical solution, Conventional Calculus.

1 Introduction

A system with differential order is characterized by a set of fractional differential equations, fractional integral equations, or both moreover an equation with fractional derivatives is known as a fractional differential equation, and an equation with fractional integrals is a fractional integral equation [23]. Fractional Calculus (F-C) has gained a significant amount of study interest recently due to applying to more and more science and technology disciplines to which scientists used to simulate a variety of physical, biological, and chemical processes [8, 23]. To explain physical phenomena and complicated dynamic systems, an increasing number of fractional-order differential equation-based models were developed [4]. With the tremendous work of researchers, the theory of F-C and its applications have advanced quickly. The rule of F-C is not a universal one, even though several mathematicians have contributed various formulations to it.

The Riemann-Liouvellie (R-L) fractional calculus is one of these definitions that are most frequently used [9]. However, the R-L result indicates that the derivatives of a constant term are not zero, making it challenging to utilize classical calculus to analyze F-C. Jumarie [23] updated the concept of the F-C of the R-L type to address this issue, with this newer version. The most helpful definitions for avoiding the non-zero derivatives of a constant function are Caputo, Grunwald-Letinikov (G-L) [8] and Jumarie's who modified R-L fractional derivatives [2]. While Grunwald-Letinikov's (G-L) definition is helpful for numerical purposes, R-L and Caputo's definitions are particularly helpful for analytical approaches.

The linear fractional differential equations cannot be solved using a single method whereas using the Jumarie [13] modified definition of fractional derivative, we can derive the derivative of the Mittag-Leffler function. Consequently, there are no standard procedures for solving fractional differential equations (FDE). The fractional derivatives of specific fractional functions, such as polynomial, exponential, sine, and cosine functions, are obtained in this work using the fractional differential approach. Furthermore, and in comparison to Podlubny [23] these results can be used to determine the specific solution of a non-homogeneous linear FDE with constant coefficients in a comprehensible way. This is a continuation of the approach suggested by Aleroev et al. [2].

L' Hospital [12] wrote to Leibniz to inquire about the n^{th} derivative of the linear function $f(t)= t$ and what would occur when $n= \frac{1}{2}$. This question led to the creation of fractional calculus. In response, Leibniz [12] replied, " This is an apparent paradox; one day it brings useful implications to all of mankind".The conversation between the two mathematicians showed that on September 30, 1695, a new branch of mathematics was established [18]. After 34 years, Eular [24] introduced the beta and gamma functions to fractional calculus. Some of the important definitions of F-C from several renowned mathematicians are enumerated here, in fact, in 1772 Lagrange [23] formulated and published the exponent principle for differential operators of integer order. Laplace [24] in 1812, Lacroix using Legendre's symbol Γ for the generalized factorial, built the n^{th} order derivative easily. Numerous national and international mathematicians contributed to recent advancements in F-C and many of them have created iterative and finite difference methods to solve linear and non-linear F-C [6].

Agrawal [1] and Erdely (10) first developed the Mittage - Leffler function in the F-C with parameters and also Humbert and Agarwal [10] developed numerous linkages for this investigation by applying the Laplace transform approach. In the discussion of F-C, Gerenflo and Mainardi [11] investigated the simple fractional relaxation and vibration equations. It is well known that the first and second orders of linear differential equations, respectively, regulate the traditional method of solving the oscillation and relaxation equations [4, 20]. Leibniz was the first to take steps in this approach after L' Hospital's questioning on the order of differentiation. In his letters to Wallis and Bernoulli from 1695 to 1697, Leibniz discussed a potential strategy for the differentiation of fractional orders in the sense that the definition may be as follows for non-integers [13, 23]. Alexopoulos and Weinberg's [14] tried to explain the relevance of fractional order derivatives using the power rule and tried to establish their geometric interpretation in the distance, velocity, and acceleration because there isn't a clear justification for fractional calculus. However, the area of the triangle decreases if the value of the fractional order derivatives (FOD) increases and the area of the triangle increases if the value of FOD decreases. Therefore, the values of fractional order derivatives and the area of triangles are inversely proportional [3, 14, 21]. From the perspective of Caputo Fabrizio fractional derivative and integration, the objective of this work is to investigate the approximate solutions of the identity, sine, and cosine functions.

2 Mathematical Model

One of the more logical extensions of the definition of the ordinary derivative, the power rule, is one of the three basic definitions of fractional derivatives with mathematical expressions that are presented.

2.1 Grunwald-Letinikove (G-L)

G-L fractional derivative is one of the powerful definitions of the ordinary derivative and it is the extension of classical calculus. In the following, we give an analytical solution for the polynomial function.

$$\begin{aligned} \text{Let } y &= x^n \\ \frac{d^k y}{dx^k} &= \frac{n!}{(n-k)!} \cdot x^{n-k} \quad \text{As } k \in \mathbf{N} \to \alpha \in \mathbf{R} \\ \frac{d^\alpha y}{dx^\alpha} = D^\alpha(y) &= \frac{n!}{(n-\alpha)!} \cdot x^{n-\alpha} = \frac{n!}{\Gamma(n-\alpha+1)} \cdot x^{n-\alpha} \end{aligned} \tag{1}$$

where gamma function is

$$\Gamma(n) = \int_0^\infty e^t \cdot (1-t)^{n-1} \ \ (n > 0).$$

2.1.1 Half order fractional derivatives of a constant function

When $y = C$

$$D^{\frac{1}{2}}(C) = \frac{0!}{\Gamma(0-\frac{1}{2}+1)} \cdot x^{0-\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}$$

The result does not agree with classical calculus, and, indeed, it is not continuous at x = 0. As far as we are concerned, half-order derivatives of a polynomial are never continuous. The relation described below is established to further the aforementioned contradictions [15].

$$g'(x) = \lim_{h\to 0}\left(\frac{g(x)-g(x-h)}{h}\right)$$

The α order differentiation is

$$g^{\alpha}(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^{\alpha}\sum_{i=0}^{n}(-1)^i\begin{bmatrix}\alpha\\ i\end{bmatrix}g(x-ih)$$

$$g^{\alpha}(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^{\alpha}\sum_{i=0}^{n}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}.g(x-ih)$$

$$g^{\alpha}(x) = \lim_{n\to\infty}\left(\frac{n}{(x-a)}\right)^{\alpha}\sum_{i=0}^{n}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}.g(x-\frac{i}{n}(x-a)) \quad [8]\ (2)$$

which is known as G-L fractional derivatives. We must determine and evaluate this limit for every function, which is a difficult task, therefore we must make some additional modifications to this formulation as we can see that it is rather complex and that it only makes sense if the limit exists.
As $\alpha \to$ - α, then equation 2 reduces to

$$g^{-\alpha}(x) = \lim_{n\to\infty}\left(\frac{1}{h}\right)^{-\alpha}\sum_{i=0}^{n}(-1)^i\begin{bmatrix}-\alpha\\ i\end{bmatrix}g(x-ih)$$

$$I^{\alpha}g(x) = h^{\alpha}\lim_{n\to\infty}\left(\frac{1}{h}\right)^{-\alpha}\sum_{i=0}^{n}\frac{\Gamma(\alpha+i)}{i!\Gamma(\alpha)}.g(x-i.h) \quad (3)$$

This expression gives the fractional integration. The fractional derivative or fractional integration can be well approximated if h is small enough. Furthermore, it can be established that this solution's accuracy is o(h). In general, the Grunwald-Letnikov fractional derivative is not practical for a non-integer term. The Riemann-Liouville definition is in this situation the most well-known definition [19].

2.2 Riemann-Liouville definition

A fractional derivative is typically defined by the Riemann-Liouville formula [3].

$${}_a\mathbf{D}_x^{\alpha}(v(x)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\cdot\left(\frac{d}{dx}\right)^n\int_a^x(x-t)^{n-\alpha-1}v(t).dt; & (n-1)<\alpha<n \\ \frac{d^n}{dx^n}v(x) & \alpha = n. \end{cases} \quad (4)$$

It is established that the G-L fractional derivative and the Caputo F-C are equivalent for analytical purposes. In the R-L, a constant function's F-C is zero, but not in the Caputo sense. The Caputo fractional derivative is frequently utilized in F-C [3].

2.3 Caputo Fractional derivative definition

The Caputo Fractional derivative is commonly used [22]

$${}_a\mathbf{D}_x^{\alpha}(u(x)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\cdot\left(\frac{d}{dx}\right)^n u(t).dt; & (n-1)<\alpha<n \\ \frac{d^n}{dx^n}u(x) \quad \alpha = n \end{cases}$$

2.4 Caputo and Fabrizio Fractional derivative definition

Caputo and Fabrizio [5] have introduced a unique fractional derivative without any singularities in its kernel. The kernel of the new fractional derivative resembles an exponential function.
Let $\alpha \in$ [0,1], f(x) $\in H^1(a,b)$ for a $<$ b. Then the Caputo-Fabrizio fractional derivative is defined as

$$ {}_{a}^{CF}\mathbf{D}_x^{\alpha}(f(x)) = \frac{M(\alpha)}{(1-\alpha)} \int_a^x (e)^{\frac{-\alpha(x-t)}{(1-\alpha)}} \cdot f'(t) \cdot dt; \quad M(0) = M(1) = 1. $$

2.5 Linearity

Let the function g(x) and h(x) be such that a linear operator such that [20]

$$ {}_{a}^{c}D_x^{\alpha} g(x) \quad \text{and} \quad {}_{a}^{c}D_x^{\alpha} h(x) $$

exits. Let $(m-1) < m$, $m \in \mathbf{R}$, α, $\mu \in \mathbf{C}$. The linear operator of the Caputo functional derivative is

$$ {}_{a}^{c}D_x^{\alpha}(\mu g(x) + \gamma h(x)) = \mu \cdot_{a}^{c} D_x^{\alpha} g(x) + \gamma \cdot_{a}^{c} D_x^{\alpha} h(x) $$

Like wise, the R-L operator holds the following condition,

$$ {}_{a}^{RL}D_x^{\alpha}(\mu g(x) + \gamma h(x)) = \mu \cdot_{a}^{RL} D_x^{\alpha} g(x) + \gamma \cdot_{a}^{RL} D_x^{\alpha} h(x) $$

3 Result and Discussion

The fractional derivatives and integrations of linear functions are discussed below along with their analytical and approximation solutions.

3.1 Solution of Fractional Derivative

The analytical and approximation solutions of the half-order derivatives of x are described below

3.1.1 Analytical Solution of the half order derivative of x

The integer order derivatives in fractional calculus are used by various approaches in equation 1. In particularly, choose $n = 1$ and $\alpha = \frac{1}{2}$:

$$ D^{\frac{1}{2}}(y) \quad = \quad \frac{1!}{\Gamma(1-\frac{1}{2}+1)!} \cdot x^{1-\frac{1}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. $$

It applies to all polynomial functions, and it may be computed to any order derivatives. The simplest analytical solutions for the differentiation and integration of an identity function with the form $x(t) = t$ are described below. We also note many stages of transition from $x(t)= t$ to $x'(t) = 1$ and from $x(t) = t$ to $\int x(t)dt = \frac{t^2}{2}$ for the various values of α between $0 < \alpha < 1$. Calculating the n^{th} derivative of a function means taking the derivatives at the function n times, which makes sense given that the first derivative displays the slope of a graph, the second derivative indicates concavity, and so on. However, what does it mean to take fractional derivatives for different values of α between 0 to 1?

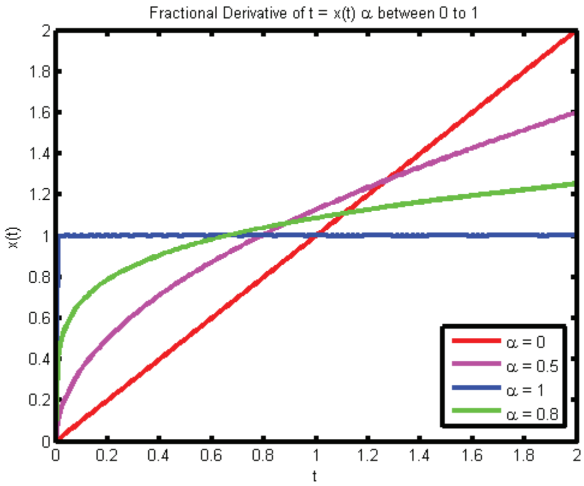

Figure 1: Frictional derivative of linear fuction $x(t)= t$ of order $(\alpha)= 0$ 0.5, 0.8, 1.

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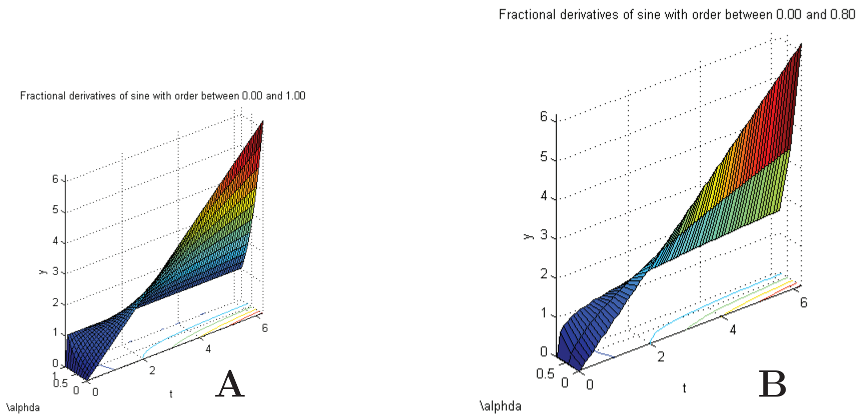

Figure 2: $3D$ graphic of fractional derivatives of linear function: $x(t) = t$ when **A:** steps size $(h) = 0.01$, order $(\alpha) = 0 : 0.1 : 1$ and **B:** steps size(h)= 0.01, order$(\alpha) = 0 : 0.5 : 1$

Several unique and non-uniform behaviors of an identity function $x(t) = t$ as well as their fractional derivatives in two dimensional for various values of α are examined. Here, in Fig. 1, the simple geometric formulation of the fractional order of an identity function, $x(t) = t$ provides and discusses their behavior patterns in the two-dimensional case for the various orders, namely 0, 0.5, 0.8, 1 and is plotted them for different values of x. Additionally, we can see that according to a well-known fact in classical calculus, the derivatives of $x(t) = t$ at 0, 0.5, 0.8, 1 are approaches to $x'(t) = 1$. The identity function $x(t) = t$ in calculus satisfies the property $x'(t) = 1$.

In Fig. 2 A, the family of fractional derivatives of the given function $x(t) = t$ with steps size $h = 0.01$ and order $\alpha = 0.1$ in $0 < \alpha < 1$, all curves tend to approach the line x'(t) = 1. The identity function x(t) = t in classical calculus satisfies the property x'(t) = 1. The family of fractional derivatives of the given function, x(t) = t, with steps size h = 0.01 and order $\alpha = 0.5$ in $0 < \alpha < 1$ (in Fig. 2 B, all curves on the right side of the 3D graphic, tend to approach the line x'(t) = 1. The property x'(t) = 1 is satisfied by the identity function in classical calculus, $x(t) = t$.

3.1.2 Approximation Solution for Fractional Derivatives

Table 1 shows the solutions for the different values of α. As we can see in the adjacent table, whenever the value of α is close to 1, the approximation values are fairly close to the analytical solutions. The solution provided by the Grunwald-Letinikov method is reliable and effective for resolving differential equations of fractional order, and it exhibits reduced error as α approaches 1. The analytical answer for the arbitrary nearest value of α is hence often the approximate solution.

Table 1: Fractional derivatives for different orders of α.

Iterations	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1.0$
0.0	0.0000	0.0000	0.0000	1.0000
0.2	0.0100	0.1000	0.3981	1.0000
0.4	0.0200	0.1500	0.4777	1.0000
0.6	0.0300	0.1875	0.5255	1.0000
0.8	0.0400	0.2187	0.5605	1.0000
1.0	0.0500	0.2461	0.5886	1.0000
1.2	0.0600	0.2707	0.6121	1.0000
1.4	0.0700	0.2933	0.6325	1.0000
1.6	0.0800	0.3142	0.6506	1.0000
1.8	0.0900	0.3338	0.6668	1.0000
2.0	0.1000	0.3524	0.6817	1.0000

3.2 Solution for Fractional Integration

3.2.1 Analytical solution for fractional integration

The fractional integration's properties are given below [20]

$$\begin{aligned} {}_aI_x^\alpha(f) &= \frac{1}{\Gamma(\alpha)}.\int_a^x (x-t)^{\alpha-1}f(t).dt \\ {}_aI_x^\alpha(Cf) &= C_aI_x^\alpha(f) \\ {}_aI_x^\alpha(f\pm g) &= {}_aI_x^\alpha(f)\pm_a I_x^\alpha(g) \\ {}_aI_x^\alpha({}_aI_x^\beta)) &= {}_aI_x^{\alpha+\beta}(f) ={}_a I_x^\beta({}_aI_x^\alpha(f)) \end{aligned} \tag{5}$$

A. Integration of a constant function at any fractional orders: From the existing relation,

$$\begin{aligned} {}_aI_x^\alpha(Cf) &= C\cdot_a I_x^\alpha(f) \\ {}_aI_x^\alpha(C) &= \frac{1}{\Gamma(\alpha)}.\int_a^x (x-t)^{\alpha-1}C.dt \\ {}_aI_x^\alpha(C) &= \frac{C}{\Gamma(\alpha)}.\int_a^x (x-t)^{\alpha-1}.dt \\ {}_aI_x^\alpha(C) &= \frac{(x-a)^\alpha}{\alpha\Gamma(\alpha)} \\ \text{For } a=0;\ {}_0I_x^1(C) &= C\cdot\frac{x^\alpha}{\alpha\Gamma(\alpha)} \\ \text{For } \alpha=1;\ {}_0I_x^1(C) &= C.x \\ \text{For } \alpha=2;\ {}_0I_x^2(C) &= \frac{Cx^2}{2}. \end{aligned}$$

The result ${}_aI_x^\alpha$ (C) $= \frac{(x-a)^\alpha}{\alpha\Gamma(\alpha)}$ is used to determine the constant term integral because the prior two results were compatible with classical calculus. A constant function's antiderivatives of any integer order can be found using this formula (5).

B. Analytical Solution of the half order Integration of x: The following is a definition of the fractional integration of any order:

$$
\begin{aligned}
{}_aI_x^{\alpha} &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{(\alpha-1)}\cdot t^n dt \\
{}_aI_x^{\alpha} &= \frac{1}{\Gamma(\alpha)}\int_a^x x^{\alpha-1}(1-\frac{t}{x})^{(\alpha-1)}\cdot t^n dt \\
{}_aI_x^{\alpha} &= \frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_a^x (1-\frac{t}{x})^{\alpha-1} t^n dt \\
{}_aI_x^{\alpha} &= \frac{x^{\alpha+n}}{\Gamma(\alpha)}\int_{\frac{a}{x}}^1 (1-u)^{\alpha-1} u^n du \\
{}_aI_x^{\alpha} &= \frac{x^{\alpha+n}}{\Gamma(\alpha)}\beta(n+1,\alpha) \\
{}_0I_x^{\alpha} &= \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\cdot x^{(\alpha+n)}
\end{aligned}
$$

The result from the condition satisfied the classical calculus when the values of $n = 1$ and $n = 2$. So, the arbitrary order integration can be found using a formula :

$$
{}_0I_x^{\frac{1}{2}} = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{a}{2}+\frac{1}{2}+1)}\cdot x^{(\frac{a}{2}+\frac{1}{2})} = \frac{\sqrt{\pi}x}{2}
$$

where $\alpha = n = \frac{1}{2}$. Any order integration can be computed, and it applies to any polynomial equations.

3.2.2 Approximation Solution for fractional integration

Figure 3 shows how the integration orders at 0, 0.5, 0.7, and 1 are approaches to the integration of $\int \mathrm{x(t)} = \frac{t^2}{2}$. As a result, the approximate answer is very closer to the analytical solution. From the preceding result, it is clear that analyzing a function's fractional integration will provide us with the fractional derivative of that function.

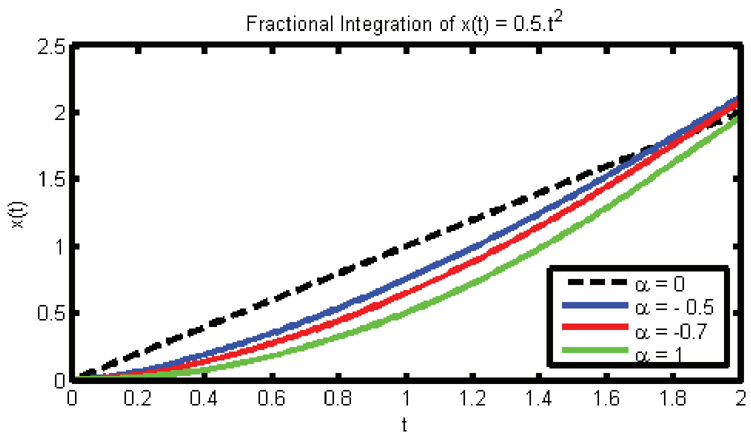

Figure 3: Fractional integration of the identity function.

3.3 Approximation Solution for fractional integration:

Table 2 below shows the solutions for the different values of α. As we can see in the table 3.3, whenever the value of α is close to 1, the approximation values are fairly close to the analytical solutions, The solution

Table 2: Frictional Integration for various values of α

Iterations	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1.0$
0.0	0.0000	0.0000	0.0000	1.0000
0.2	0.0100	0.0010	0.0004	1.0000
0.4	0.0200	0.0025	0.0011	1.0000
0.6	0.0300	0.0044	0.0020	1.0000
0.8	0.0400	0.0066	0.0031	1.0000
1.0	0.0500	0.0090	0.0044	1.0000
1.2	0.0600	0.0117	0.0059	1.0000
1.4	0.0700	0.0147	0.0076	1.0000
1.6	0.0800	0.0178	0.0095	1.0000
1.8	0.0900	0.0211	0.0115	1.0000
2.0	0.1000	0.0247	0.0137	1.0000

provided by the Grunwald-Letinikove method is reliable and effective for resolving integration equations of integrataion order, and it exhibits reduced error as α approaches 1 when the steps are increasing. The analytical answer for the arbitrary nearest value of α is hence often the approximate solution.

3.3.1 Analytical Solution of the half order antiderivative of sin(x):

In Caputo Fabrizio sense [9], the half-order fractional derivative of sin(x) can be calculated analytically as

$$\begin{aligned}
{}_a^cD_x^\alpha g(x) &= \frac{N(\alpha)}{(1-\alpha)}\int_a^x e^{\frac{-\alpha(x-\alpha)}{1-\alpha}}.f'(t).dt \\
\text{When } a &= 0, N(\alpha)=1,\ \ f(x)=sin(x),\ \ f'(t)=cos(t). \\
{}_0^cD_x^\alpha sin(x) &= \frac{1}{(1-\alpha)}\int_0^x e^{\frac{-\alpha(x-\alpha)}{1-\alpha}}.cos(t).dt \\
{}_0^cD_x^\alpha sin(x) &= \frac{1}{(1-\alpha)}.e^{\frac{-\alpha}{(1-\alpha)x}}\int_0^x .e^{\frac{\alpha t}{1-\alpha}}cost.dt. \\
{}_0^cD_x^\alpha sin(x) &= \frac{1-\alpha}{(\alpha)^2+(1-\alpha)^2} \\
{}_0^cD_x^\alpha sin(x) &= \frac{\alpha}{1-\alpha}.cosx+sinx-\frac{\alpha}{1-\alpha}e^{\frac{-\alpha}{1-\alpha}x}
\end{aligned}$$

whenever $\alpha = \frac{1}{2}$, then ${}_0^cD_x^\alpha sin(x)$ = cos(x) + sin(x) - e^{-x}, which is the half-order fractional derivatives of the sine function in the Caputo Fabrizio sense [10]. Furthermore, using the Riemann and Liouville [6] formula or the Caputo Fabrizio sense [12] for different values for α, one may determine the fractional derivatives and fractional integrations of the functions cosines, e^x, cosh x and sinh x.

Let us consider the fractional-order derivative of the function y=sin(t) at the range [0, 1]. Then, the fractional derivative in the range t $\in [0, 2\pi]$ can be computed as as shown in Fig. 4.**A**, Fig. 4.**B** and Fig. 4.**C**. Here, in Fig. 4. **A**, the simple geometric formulation of the fractional order of an identity function, $x(t) = t$ provides and discusses their behavior patterns in the two-dimensional case for the various orders, namely 0^{th}, 0.5^{th}, 1^{st} and is plotted them for different values of x. Additionally, we can see that according to a well-known fact in conventional calculus, the derivatives of $x(t) = t$ at 0, 0.5, 1 are the coincidence to FOD, $x'(t) = 1$. The identity function $x(t) = t$ in calculus satisfies the property $x'(t) = 1$. For the monotonically increasing function, whenever the order of derivatives is greater than 1, the result is greater than first-order derivatives. The green curve on the Fig. 4.**A** illustrates the fractional derivatives in $2D$ at $\alpha = 0.5$, which is halfway between 0 and 1. The G-L formulation [1] is used to provide the fractional derivatives of the $3D$ figure of the sine function in Fig. 4. **B** with the order of $\alpha = 0 : 0.1 : 1$ at the step size $h = 0.01$ on a regular grid. Fig 4 **C** is a three-dimensional of fractional derivatives with steps of size $h = 0.08$ and an order of alpha $= 0 : 0.5 : 1$, using G-L formulation. We find that all sets of curves converge to $x'(t) = 1$.

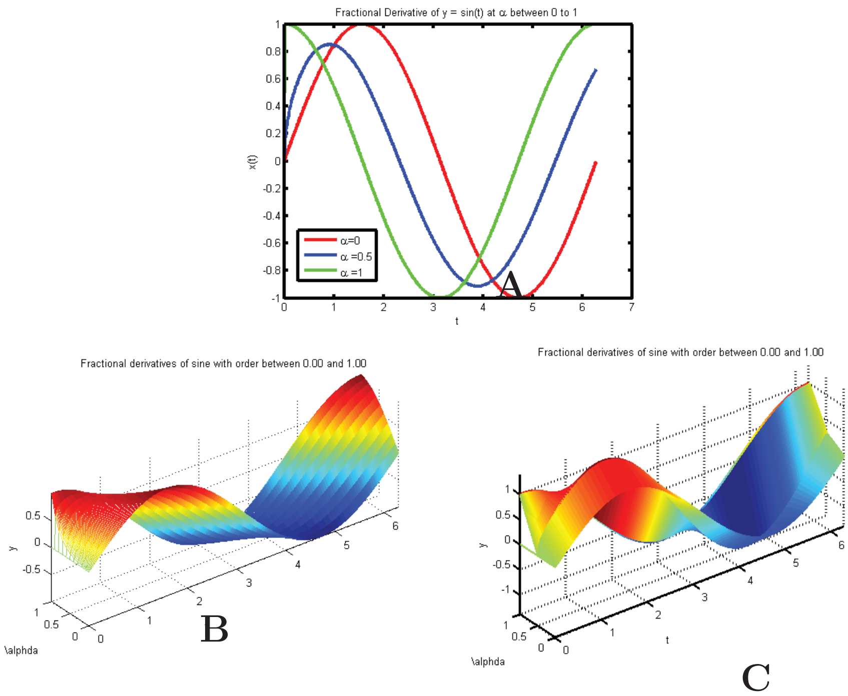

Figure 4: Fractional derivatives of sine function in **A:** 2D case, **B:** 3D case when $\alpha = 0 : 0.2 : 1$ and **C:** 3D case when $\alpha = 0 : 0.0.01 : 1$.

The geometrical interpretation and the results diverge from velocity are not seen clearly for other values of fractional order derivatives, when the order of derivative diverges from one.

4 Conclusion

Many academic fields are using fractional calculus more frequently. On the other hand, it is challenging to compute fractional integrals and fractional derivatives numerically due to a lack of numerical approaches. The fractional calculus can be efficiently computed, though, by using the well-known Caputo-Fabrizio and Riemann Liouville procedures using computer software. The G-L formulation is used to develop the fractional order derivatives and integration of the identity function and sine function in $2D$ and $3D$, Simple analytical and approximation solutions are produced. Numerical simulations used to analyze geometrical techniques demonstrate that over a range of all values close to the first-order derivatives and integrations of a function, there are minimal inaccuracies. When the first-order fractional order derivatives converge to 1, the given result agrees with the first-order derivatives. In addition, the derivative operator is linear for order 1 but non-linear for higher orders. In order to show the validity of our findings, various examples are provided in this work that deal with the fractional calculus of specific functions. FC is also a generalization of traditional calculus.

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References

[1] Agarwal, R.P., A propos d'une note de M.Pierre Humbert, C.R., **(1953)**. *Séances Acad. Sci.*, 236(21): 2031-2032.

[2] Aleroev, T. V., Eronkhin, S. V. **(2019)**. Parameteric Identification of the Fractional-Derivative Order in the Bagley-Torvik Model.*Math. Models Comp. Simul.*, 2: 219 - 225.

[3] Bulut, H., Baskonus, H. M. and Belgacem, F. B. M. **(2013)**. The analytical solution of some fractional ordinary differential equations by the Sumudu transform method, *Abstr. Appl. Anal.*, 1–6.

[4] Caputo, M. **(2001)**. Distributed order differential equations modelling, dielectric induction and diffusion. *Frac. Calc. Appl. Anal.*, 4: 421-442.

[5] Caputo, M., Fabrizio, M. **(2015)**. *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl. 1(2), 1–13.

[6] Carpinteri, A. **(1997)**. Fractals and Fractional Calculus in Continuum Mechanics CISM International Centre for Mechanical Sciences. Springer, Wien.

[7] Das, S. **(2011)**. Functional fractional calculus, *Springer-Verlag*, Second Edition.

[8] Davis, H. T. **(2008)**. The Theory of Linear Operator, *Myers Press*, Bloomington, USA.

[9] Duff, G.F.D and Naylor, D. **(1966)**. Differential Equations of Applied Mathematics. John Wiley and Sons, Hoboken, 42. http://dx.doi.org/10.1119/1.1972713.

[10] Erdelyi, A. **(1954)**. *Tables of Integral Transforms*, McGraw-Hill, York,1.

[11] Gorenflo, R. and Mainardi, F. **(1997)**. *Fractional Calculus Integral and Differential Equations of Fractional Order*, Springer Verlag Wien and New York,:223-276.

[12] Hilfer, R. **(2021)**. Application of Fractional calculus in physics, *world Scientic publishing*, Second Edition.

[13] Jumarie, G. **(2005)**. On the solution of the stochastic differential equation of exponential growth driven by fractional Brownian motion, *Applied Mathematics Letters*, 18(7), 817–826.

[14] Karci, A., **(2015)**. The Physical and Geometrical Interpretation of Fractional Oder Derivativates *Universal Journal of Engineering Science*, Turkey: 53-63.

[15] Lacrois, S.F. **(1819)**. Traite du Calculus Differential et du Calcul Integral, *Courcier*, 3: 409-410.

[16] Laplace, P.S. **(1820)**. *Theorie Analytique des Probabilities*, 3 (85): Paris (Courcier).

[17] Lavoi, J.L., Osler, T.J. and Tremblay, R. **(1976)**, Fractional Derivatives and Special Functions, *SIAM Review*, 18(2): 240-268.

[18] Luchko, Y. **(2021)**. *General Fractional Integrals and Derivatives of Arbitrary Order*, Symmetry, 13(5): 755. https://doi.org/10.3390/sym13050755.

[19] Mattai, A. M., Saxena, R.K., Hans, J. H. **(2009)**. *theory and application,* Springer.

[20] Olagunju, A., Joseph, L. **(2021)**. Construction of Functions for Fractional Derivatives using Matlab. *Journal of Advances in Mathematics and Computer Science*, 36(6):1-10, ISSN: 2456-9968.

[21] Oldham, K.B. and Spanier, J. **(1974)**. The Fractional Calculus, Academic Press, New York, London.

[22] Petráš, I. (**2009**). Fractional-Order Feedback Control of a DC Motor, *Journal of Electrical Engineering*, 60(3), 117–128.

[23] Podlubny, I. (**1999**). Fractional Differential Equation;*Academic Press*, New York, USA. http://www.sciepub.com/reference/3051.

[24] Romero, L.G., Medina, G.D., Ojeda, N.R., Pereira, J.H. **(2016)**. A new alfa-Integral Laplace Transform, Asian Journal of Current Engineering andMaths., 59-62.