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k-Product Cordial Labeling of Napier Bridge Graphs

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Abstract: *Let f be a map from $V(G)$ to $\{0, 1, ..., k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x = 0, 1, ..., k-1)$. It is yet another study on k-product cordial labeling. In this paper, we define a new graph $P_n(t)$ namely Napier bridge graph and find some results on 3-product cordial and 4-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.*

Keywords: *Cordial labeling, Product cordial labeling, k-Product cordial labeling, 3-Product cordial labeling, 4-Product cordial labeling, Napier bridge graph.*
AMS Subject Classification (2010): 05C78.

1 Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3]. The concepts of labeling of graph has gained a lot of popularity in the field of graph theory during the last 60 years due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [15] published a pioneering paper on graph labeling problems. Thereafter, many types of graph labeling techniques have been studied by several authors. All these labelings are beautifully classified by Gallian [2] in his survey. Cordial labeling is a weaker version of graceful and harmonious labeling was defined by Cahit [1]: Let f be a function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Motivated by the concept of cordial labeling, Sundaram et al. [16] introduced the concept of product cordial labeling: Let f be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0)-v_f(1)|\leq 1$ and $|e_f(0)-e_f(1)|\leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i=0,1)$. Several results have been published on this topic (see [2]).

In 2012, Ponraj et al. [14] extended the concept of product cordial labeling and introduced k-product cordial labeling: Let f be a map from $V(G)$ to $\{0,1,...,k-1\}$ where k is an integer, $1\leq k\leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i)-v_f(j)|\leq 1$, and $|e_f(i)-e_f(j)|\leq 1$, $i,j\in\{0,1,...,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x $(x=0,1,...,k-1)$. They proved that k-product cordial labeling of stars, bistars and also 4-product cordial labeling behavior of paths, cycles, complete graphs and combs. Javed and Jamil [4] proved that rhombic grid graphs are 3-total edge product cordial graphs. Jeyanthi and Maheswari [12] gave the maximum number of edges in a 3-product cordial graph of order p is $\frac{p^2-3p+6}{3}$ if $p\equiv 0(mod\ 3)$, $\frac{p^2-2p+7}{3}$ if $p\equiv 1(mod\ 3)$ and $\frac{p^2-p+4}{3}$ if $p\equiv 2(mod\ 3)$ and also they showed that paths and cycles are 3-product cordial graphs. The same authors [13] proved that the graph P_n^2 is 3-product cordial. Inspired by the concept of k-product cordial labeling and also the results in [12, 13, 14], we further studied on k-product cordial labeling and established that the following graphs admit/ do not admit k-product cordial labeling: union of graphs [5]; cone and double cone graphs [6]; fan and double fan graphs [7]; powers of paths [8]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil\frac{p-1}{4}\rceil\lfloor\frac{p-1}{4}\rfloor+3$ [9]; product of graphs [10] and paths [11]. In this paper, we define a new graph $P_n(t)$ namely Napier bridge graph, since the image of the graph looks like the Napier bridge in Chennai city, India. This graph is obtained from the path P_n by joining all the pairs of vertices u, v of P_n with $d(u,v)=t$. Clearly, $P_n(t)\cong P_n$ if $n\leq t$ and $P_n(t)\cong C_n$ if $n=t+1$. In addition, we study the k-product cordial behavior of Napier bridge graph $P_n(t)$.

2 3-product cordial labeling of Napier bridge graphs

In this section, we study the 3-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.

Theorem 2.1. *For $n \geq 5$, the graph $P_n(3)$ is 3-product cordial if and only if $n \equiv 1$ or 2 $(mod\ 3)$.*

Proof. Let the vertex and edge set of $P_n(3)$ be $V(P_n(3)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(3)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\}$ respectively.
We have the following three cases.
Define $f : V(P_n(3)) \to \{0, 1, 2\}$ as follows:
Case (i): If $n \equiv 1(mod\ 3)$, then
$f(v_i) = 0\ ;\ 1 \leq i \leq \lfloor \frac{n}{3} \rfloor$.
For $i = \lfloor \frac{n}{3} \rfloor + j\ ;\ 1 \leq j \leq n - \lfloor \frac{n}{3} \rfloor$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 2(mod\ 4) \\ 2 & \text{if } j \equiv 3, 0(mod\ 4). \end{cases}$$

From the above labeling we get,
$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor$.
Hence, $P_n(3)$ is a 3-product cordial graph if $n \equiv 1(mod\ 3)$.
Case (ii): If $n \equiv 2(mod\ 3)$.
For $n = 5$,

$$f(v_i) = \begin{cases} 0 & \text{if } i = \lfloor \frac{n}{3} \rfloor \\ 1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 2 \\ 2 & \text{if } \lfloor \frac{n}{3} \rfloor + 3 \leq i \leq \lfloor \frac{n}{3} \rfloor + 4. \end{cases}$$

For $n \geq 8$,

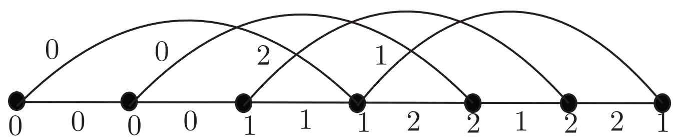
$$f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ 1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 3 \\ 2 & \text{if } \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 6. \end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + j\ ;\ 1 \leq j \leq 2\left(\lfloor \frac{n}{3} \rfloor - 2\right)$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 2, 4, 5, 7(mod\ 8) \\ 2 & \text{if } j \equiv 1, 3, 6, 0(mod\ 8). \end{cases}$$

Thus we get,
$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) = 2\lfloor \frac{n}{3} \rfloor$.
Hence, $P_n(3)$ is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 5$.
Case (iii): If $n \equiv 0(mod\ 3)$ for $n \geq 6$, then $|V(P_n(3))| = 3t$ and $|E(P_n(3))| = 6t-4$. Thus, $v_f(i) = t$ $(i = 0, 1, 2)$ and $e_f(i) = 2t - 1$ *or* $2t - 2$ $(i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, $P_n(3)$ is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n > 3$. □

An example of 3-product cordial labeling of $P_7(3)$ is shown in Figure 1.



$Figure\ 1:\ 3-product\ cordial\ labeling\ of\ P_7(3).$

Theorem 2.2. *For $n \geq 6$, the graph $P_n(4)$ is 3-product cordial if and only if $n \equiv 2\ (mod\ 3)$ or $n = 6$.*

Proof. Let the vertex and edge set of $P_n(4)$ be $V(P_n(4)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(4)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+4})\ ;\ 1 \leq i \leq n-4\}$ respectively. We have the following four cases.
Define $f : V(P_n(4)) \to \{0, 1, 2\}$ as follows:

Case (i): If $n \equiv 2(mod\ 3)$, then
$f(v_i) = 0\ ;\ 1 \leq i \leq \lfloor \frac{n}{3} \rfloor$.
Subcase (i): If $n = 8$.
For $i = \lfloor \frac{n}{3} \rfloor + j\ ;\ 1 \leq j \leq n - \lfloor \frac{n}{3} \rfloor$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 0(mod\ 4) \\ 2 & \text{if } j \equiv 2, 3(mod\ 4). \end{cases}$$

From the above labeling we get,
$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor$.
Subcase (ii): If $n \geq 11$.
For $1 \leq i \leq 8$,

$$f(v_{\lfloor \frac{n}{3} \rfloor + i}) = \begin{cases} 1 & \text{if } i = 1, 2, 4, 5 \\ 2 & \text{if } i = 3, 6, 7, 8. \end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + 8 + j\ ;\ 1 \leq j \leq n - 8 - \lfloor \frac{n}{3} \rfloor$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 0(mod\ 4) \\ 2 & \text{if } j \equiv 2, 3(mod\ 4). \end{cases}$$

From the above labeling we get,
$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor$.
Hence, $P_n(4)$ is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 8$.

Case (ii): If $n = 6$, then the 3-product cordial labeling of $P_n(4)$ is shown in Table 1.

Table 1: 3-product cordial labeling of $P_n(4)$ for $n = 6$.

n	v_1	v_2	v_3	v_4	v_5	v_6
6	1	1	0	0	2	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, $P_n(4)$ is a 3-product cordial graph if $n = 6$.

Case (iii): If $n \equiv 1(mod\ 3)$ for $n \geq 7$, then $|V(P_n(4))| = 3t + 1$ and $|E(P_n(4))| = 6t - 3$. Thus, $v_f(i) = t\ or\ t + 1\ (i = 0, 1, 2)$ and $e_f(i) = 2t - 1\ (i = 0, 1, 2)$. If

$v_f(0) = t\ or\ t + 1$, then $e_f(0) > 2t - 1$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, $P_n(4)$ is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$ for $n \geq 7$.

Case (iv): If $n \equiv 0(mod\ 3)$ for $n \geq 9$, then $|V(P_n(4))| = 3t$ and $|E(P_n(4))| = 6t - 5$. Thus, $v_f(i) = t\ (i = 0, 1, 2)$ and $e_f(i) = 2t - 1\ or\ 2t - 2\ (i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t > 2$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, $P_n(4)$ is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n \geq 9$. □

An example of 3-product cordial labeling of $P_8(4)$ is shown in Figure 2.

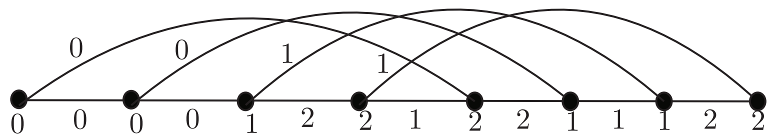


Figure 2 : 3 − product cordial labeling of $P_8(4)$.

Theorem 2.3. *For $n \geq 7$, the graph $P_n(5)$ is 3-product cordial if and only if $n \equiv 2\ (mod\ 3)$ or $n = 7$.*

Proof. Let the vertex and edge set of $P_n(5)$ be $V(P_n(5)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(5)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n - 1\} \cup \{(v_i, v_{i+5})\ ;\ 1 \leq i \leq n - 5\}$ respectively. We have the following four cases.

Define $f : V(P_n(5)) \to \{0, 1, 2\}$ as follows:

Case (i): If $n = 7, 8\ or\ 11$, then the 3-product cordial labelings of $P_n(5)$ are shown in Table 2.

Table 2: 3-product cordial labelings of $P_n(5)$ for $n = 7, 8\ and\ 11$.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
7	1	1	0	0	2	2	1				
8	1	1	1	0	0	2	2	2			
11	0	0	0	1	1	1	2	2	2	1	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, $P_n(5)$ is a 3-product cordial graph if $n = 7, 8\ or\ 11$.

Case (ii): If $n \equiv 2(mod\ 3)$ for $n \geq 14$, then

$$f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\ 1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 3 \\ 2 & \text{if } \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 6. \end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + 6 + j$; $1 \leq j \leq n - 6 - \lfloor \frac{n}{3} \rfloor$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 2, 4, 5, 7(mod\ 8) \\ 2 & \text{if } j \equiv 1, 3, 6, 0(mod\ 8). \end{cases}$$

From the above labeling we get,
$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1$,
$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor$.
Hence, $P_n(5)$ is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 14$.

Case (iii): If $n \equiv 1(mod\ 3)$ for $n \geq 10$, then $|V(P_n(5))| = 3t + 1$ and $|E(P_n(5))| = 6t - 4$. Thus, $v_f(i) = t\ or\ t + 1$ $(i = 0, 1, 2)$ and $e_f(i) = 2t - 1\ or\ 2t - 2$ $(i = 0, 1, 2)$. If $v_f(0) = t\ or\ t + 1$, then $e_f(0) > 2t - 1$ for $t \geq 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, $P_n(5)$ is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$ for $n \geq 10$.

Case (iv): If $n \equiv 0(mod\ 3)$ for $n \geq 9$, then $|V(P_n(5))| = 3t$ and $|E(P_n(5))| = 6t - 6$. Thus, $v_f(i) = t$ $(i = 0, 1, 2)$ and $e_f(i) = 2t - 2$ $(i = 0, 1, 2)$. If $v_f(0) = t$, then $e_f(0) > 2t - 2$ for $t \geq 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for j=1,2. Hence, $P_n(5)$ is not a 3-product cordial graph if $n \equiv 0(mod\ 3)$ for $n \geq 9$. □

An example of 3-product cordial labeling of $P_8(5)$ is shown in Figure 3.

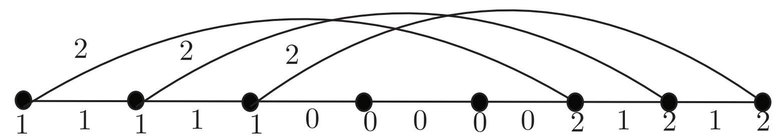


Figure 3 : 3 − product cordial labeling of $P_8(5)$.

3 4-product cordial labeling of Napier bridge graphs

In this section, we study the 4-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.

Theorem 3.1. *For* $n \geq 5$*, the graph* $P_n(3)$ *is 4-product cordial if and only if* $5 \leq n \leq 11$ *except* $n = 8$.

Proof. Let the vertex and edge set of $P_n(3)$ be $V(P_n(3)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(3)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+3})\ ;\ 1 \leq i \leq n-3\}$ respectively. We have the following five cases.
Define $f : V(P_n(3)) \to \{0, 1, 2, 3\}$ as follows:
Case (i): If $5 \leq n \leq 11$ except $n = 8$, then the 4-product cordial labelings of $P_n(3)$ are shown in Table 3.

Table 3: 4-product cordial labelings of $P_n(3)$ for $5 \leq n \leq 11$ except $n = 8$.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
5	0	2	1	3	3						
6	0	2	1	1	3	3					
7	0	2	2	1	1	3	3				
9	0	0	2	1	1	3	3	3	2		
10	0	0	2	1	1	1	3	3	3	2	
11	0	0	2	2	1	1	1	3	3	3	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(3)$ is a 4-product cordial graph if $5 \leq n \leq 11$ except $n = 8$.

Case (ii): If $n \equiv 0(mod\ 4)$ for $n \geq 8$, then $|V(P_n(3))| = 4t$ and $|E(P_n(3))| = 8t - 4$. Thus, $v_f(i) = t\ (i = 0, 1, 2, 3)$ and $e_f(i) = 2t - 1\ (i = 0, 1, 2, 3)$. If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(3)$ is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$.

Case (iii): If $n \equiv 1(mod\ 4)$ for $n \geq 13$, then $|V(P_n(3))| = 4t + 1$ and $|E(P_n(3))| = 8t - 2$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0, 1, 2, 3)$ and $e_f(i) = 2t\ or\ 2t - 1\ (i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t\ or\ t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Hence, $P_n(3)$ is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 13$.

Case (iv): If $n \equiv 2(mod\ 4)$ for $n \geq 14$, then $|V(P_n(3))| = 4t+2$ and $|E(P_n(3))| = 8t$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0, 1, 2, 3)$ and $e_f(i) = 2t\ (i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t\ or\ t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Thus, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Hence, $P_n(3)$ is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 14$.

Case (v): If $n \equiv 3(mod\ 4)$ for $n \geq 15$, then $|V(P_n(3))| = 4t+3$ and $|E(P_n(3))| = 8t+2$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0,1,2,3)$ and $e_f(i) = 2t\ or\ 2t+1\ (i = 0,1,2,3)$. Obviously, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t+1$. Thus, $e_f(0) = 2t$. Clearly, $v_f(2) = t+1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t+1$. Then, $e_f(2) > 2t+1$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i,j = 0,1,2,3$. Hence, $P_n(3)$ is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 15$. □

An example of 4-product cordial labeling of $P_5(3)$ is shown in Figure 4.

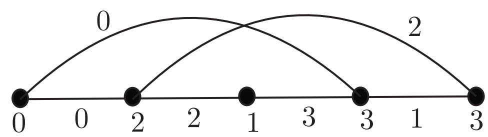


Figure 4 : $4 - product\ cordial\ labeling\ of\ P_5(3)$.

Theorem 3.2. *For $n \geq 6$, the graph $P_n(4)$ is 4-product cordial if and only if $n = 6\ or\ 10$.*

Proof. Let the vertex and edge set of $P_n(4)$ be $V(P_n(4)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(4)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+4})\ ;\ 1 \leq i \leq n-4\}$ respectively. We have the following five cases.
Define $f : V(P_n(4)) \rightarrow \{0,1,2,3\}$ as follows:

Case (i): If $n = 6\ or\ 10$, then the 4-product cordial labelings of $P_n(4)$ are shown in Table 4.

Table 4: 4-product cordial labelings of $P_n(4)$ for $n = 6\ and\ 10$.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
6	0	2	1	1	3	3				
10	0	0	2	1	1	1	3	3	3	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i,j = 0,1,2,3$. Hence, $P_n(4)$ is a 4-product cordial graph if $n = 6\ or\ 10$.

Case (ii): If $n \equiv 0(mod\ 4)$ for $n \geq 8$, then $|V(P_n(4))| = 4t$ and $|E(P_n(4))| = 8t-5$. Thus, $v_f(i) = t\ (i = 0,1,2,3)$ and $e_f(i) = 2t-1\ or\ 2t-2\ (i = 0,1,2,3)$. If $v_f(0) = t$, then $e_f(0) > 2t-1$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i,j = 0,1,2,3$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 8$.

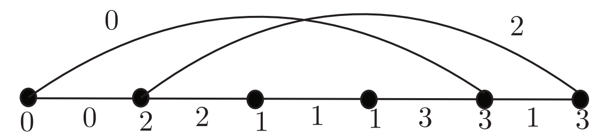
Case (iii): If $n \equiv 1(mod\ 4)$ for $n \geq 9$, then $|V(P_n(4))| = 4t+1$ and $|E(P_n(4))| = 8t-3$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0,1,2,3)$ and $e_f(i) = 2t\ or\ 2t-1\ (i = 0,1,2,3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t\ or\ t+1$. If

$v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t - 1$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 9$.

Case (iv): If $n \equiv 2(mod\ 4)$ for $n \geq 14$, then $|V(P_n(4))| = 4t + 2$ and $|E(P_n(4))| = 8t - 1$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0, 1, 2, 3)$ and $e_f(i) = 2t\ or\ 2t - 1\ (i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t\ or\ t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 14$.

Case (v): If $n \equiv 3(mod\ 4)$ for $n \geq 7$. then $|V(P_n(4))| = 4t + 3$ and $|E(P_n(4))| = 8t+1$. Thus, $v_f(i) = t\ or\ t+1\ (i = 0, 1, 2, 3)$ and $e_f(i) = 2t\ or\ 2t+1\ (i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Clearly, $v_f(2) = t + 1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t + 1$. Then, $e_f(2) \geq 2t + 1$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 7$. □

An example of 4-product cordial labeling of $P_6(4)$ is shown in Figure 5.



$Figure\ 5:\ 4 - product\ cordial\ labeling\ of\ P_6(4).$

Theorem 3.3. *For $n \geq 7$, the graph $P_n(5)$ is 4-product cordial if and only if $n = 7\ or\ 10$.*

Proof. Let the vertex and edge set of $P_n(5)$ be $V(P_n(5)) = \{v_i\ ;\ 1 \leq i \leq n\}$ and $E(P_n(5)) = \{(v_i, v_{i+1})\ ;\ 1 \leq i \leq n - 1\} \cup \{(v_i, v_{i+5})\ ;\ 1 \leq i \leq n - 5\}$ respectively. We have the following five cases.

Define $f : V(P_n(5)) \to \{0, 1, 2, 3\}$ as follows:

Case (i): If $n = 7\ or\ 10$, then the 4-product cordial labelings of $P_n(5)$ are shown in Table 5.

Table 5: 4-product cordial labelings of $P_n(5)$ for $n = 7$ *and* 10.

n	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
7	1	1	2	0	2	3	3			
10	0	2	1	1	3	0	3	3	1	2

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is a 4-product cordial graph if $n = 7$ *or* 10.

Case (ii): If $n \equiv 0(mod\ 4)$ for $n \geq 8$, then $|V(P_n(5))| = 4t$ and $|E(P_n(5))| = 8t-6$. Thus, $v_f(i) = t$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t-1$ *or* $2t-2$ $(i = 0, 1, 2, 3)$. If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$ for $n \geq 8$.

Case (iii): If $n \equiv 1(mod\ 4)$ for $n \geq 9$, then $|V(P_n(5))| = 4t + 1$ and $|E(P_n(5))| = 8t - 4$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t - 1$ $(i = 0, 1, 2, 3)$. If

$v_f(0) = t$ *or* $t + 1$, then $e_f(0) > 2t - 1$ for $t \geq 2$ Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n \geq 9$.

Case (iv): If $n \equiv 2(mod\ 4)$ for $n \geq 14$, then $|V(P_n(5))| = 4t + 2$ and $|E(P_n(5))| = 8t - 2$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ *or* $2t - 1$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t$ *or* $t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t+1$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n \geq 14$.

Case (v): If $n \equiv 3(mod\ 4)$ for $n \geq 11$. then $|V(P_n(5))| = 4t+3$ and $|E(P_n(5))| = 8t$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 2t$ $(i = 0, 1, 2, 3)$. Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Clearly, $v_f(2) = t + 1$ and 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 11$. □

An example of 4-product cordial labeling of $P_7(5)$ is shown in Figure 6.

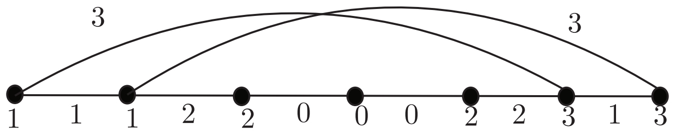


Figure 6 : $4-$ *product cordial labeling of* $P_7(5)$.

4 Conclusion

In this paper, we find the 3-product and 4-product cordial labeling of Napier bridge graphs $P_n(3),\ P_n(4)$ and $P_n(5)$. In future, we propose to find the k-product cordial labeling of $P_n(m)$ for $k \geq 5$ and $m \geq 2$.

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