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On Generalized Form of Difference Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function

Gyan Prasad Paudel[1], Suresh Kumar Sahani[2] and Narayan Prasad Pahari[3]

[1]*Central Campus of Science and Technology, Mid-Western University, Surkhet, Nepal*
[2]*Department of Mathematics, MIT Campus, Janakpurdham, Nepal*
[3]*Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.*

Email: [1]*gyan.math725114@gmail.com*, [2]*sureshkumarsahani35@gmail.com* & [3]*nppahari @gmail.com*

Abstract: *Lotfi A. Zadeh developed the concept of fuzzy logic in 1965, and it has since been applied to the study of various class spaces. This research aims to investigate the linearity and some topological properties of the generalized form of difference sequence spaces* $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$, $F(\bar{\rho}, \mathcal{M}, p, A)$ *and* $F_0(\bar{\rho}, \mathcal{M}, p, A)$ *of fuzzy real numbers defined by the Orlicz function in its generalized form by defining new metrics.*

Keywords: *Fuzzy real numbers, Orlicz function, Difference sequence space.*

AMS Subject Classification 03B52, 46A45.

1. Introduction

A large number of research projects have been carried out in mathematical structures built with real or complex numbers so far. In recent years, many researchers have investigated many results of replacing these mathematical structures with fuzzy numbers and interval numbers. Prior to the introduction of fuzzy logic and fuzzy sets, mathematics could only reach two conclusions: true or false (denoted by 1 and 0). In 1965, Zadeh [26] introduced the concept of fuzzy logic and fuzzy sets based on the notion of the relative degree of membership, which is inspired by the process of human perception and knowledge. Then, slowly and gradually, the use of fuzzy logic and fuzzy sets is increased. Many researchers are motivated towards further investigation and application of it. The fuzzy set and fuzzy real numbers have been studied by a wide number of academics in many classes of sequence space with their different properties. Matloka [13] analyzed bounded and convergent sequences of fuzzy numbers in 1986 and proved that every convergent sequence of fuzzy numbers is bounded. In 2004, Savas and Savas [21] proposed a new idea of λ-strong convergence with regard to an Orlicz function and investigated some of its features. Rifat, and colleagues [18], in 2009 proposed the difference operator Δ^m and an Orlicz function to analyze several sequence spaces of fuzzy numbers and studied some of their properties such as completeness, solidity, symmetry, and so on and also provided some relationships connected to these spaces. In 2010, Faried and Barkey [7] proposed the Orlicz-Cesaro difference sequence space with distinct paranormed sequences. Also, in 2010, Faried and Barkey [8] presented the Musielak-Orlicz difference sequence space, paranormed the difference, and examined several inclusion relations. Sarma [19] in 2012 proposed some I-convergent sequence spaces of fuzzy real numbers defined by the Orlicz function to investigate some of their features. Pahari [14] developed a new class of sequences with values in 2- Banach space as an extension of the conventional spaces of summable sequences in 2014 and investigated some topological structures. Also in 2014, Tripathy and Borgohain [23] studied sequence space of fuzzy real numbers using metric space. Sarma [20] developed certain fuzzy sequence spaces formed by the Orlicz function in 2015, studied their various features, and established some inclusion properties among them. In 2016, Dutta and et al.[3] examined classes of real numbers sequence over 2-normed spaces defined by means of Orlicz function, bounded sequence of strict positive real numbers. In 2018, Ebadullah [4] created a class of sequence spaces defined by a sequence of modulii and examined the topology that evolved in those spaces. Basu [2] formed a new class of fuzzy number sequences in 2018 applying the Orlicz function to produce numerous useful classes with rich structural features, and topological behavior was examined for these new classes. Recently, in 2021, Kim and Lee [11] studied the sequence spaces of fuzzy normed space with new concept.

Wladyslaw Orlicz, firstly introduced Orlicz spaces in 1932. Later on, Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty \; for\ some\ \rho > 0 \right\}$$

The space l_M with the norm $\|x\|$ defined by $\|x\| = \inf \left\{ \rho : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) \leq 1 \right\}$ becomes a Banach space and it is called Orlicz Sequence Space.

The following is how the work is organized: in Section 2, we review some background material on fuzzy set theory, the concept of Orlicz function, set of fuzzy real numbers, sequence of fuzzy numbers, Cauchy sequence, convergent sequence, and bounded fuzzy sequence. In Section 3, we will look at different difference sequence spaces of fuzzy numbers that have been examined by researchers. In Section 4, we will look at the paper's primary findings. Section 5 contains a discussion and conclusion.

2. Preliminaries and Definitions

Definition 2.1 [15]: A function $\mathcal{M} : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing, and convex with the properties that $\mathcal{M}(0) = 0$, $M(t) > 0$ for $t > 0$ and $\mathcal{M}(t) \to \infty$ as $t \to \infty$, is called an Orlicz function.

Example 1: The function defined by $\mathcal{M}(t) = |t|^{\eta}$ for all $t \in \mathbb{R}$ and $\eta \geq 0$ is an Orlicz function.

Example 2: The function $\mathcal{M}: \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\mathcal{M}(t) = e^{|t|} - t - 1, \forall\, t \in \mathbb{R}^+$ is an Orlicz function. The Orlicz function $\mathcal{M}$ is said to be *convex* if

$$\mathcal{M}(\alpha\, t_1 + (1-\alpha)t_2 \leq \alpha\mathcal{M}(t_1) + (1-\alpha)\mathcal{M}(t_2)$$

If we replace the Orlicz function $\mathcal{M}$ by $\mathcal{M}(t+s) \leq \mathcal{M}(t) + \mathcal{M}(s)$, then the function $\mathcal{M}$ is a **modulus** function.

Definition 2.2 An Orlicz function $\mathcal{M}$ is said to satisfy Δ_2 −condition for all values of t if there exists a constant $\lambda > 0$ such that $\mathcal{M}(2t) \leq \lambda\mathcal{M}(t), \forall\, t \geq 0$.

We note that an Orlicz function $\mathcal{M}$ satisfies the relation $\mathcal{M}(kt) \leq k\mathcal{M}(t), \forall t$ with $0 < k < 1$.

Let, $\mathfrak{T}$ be the set of all bounded interval $\mathcal{B}(b_1, b_2)$ on the real line $\mathbb{R}$. For any $\mathcal{B}$, $\mathcal{C}$ with $\mathcal{B} = [b_1, b_2)$ and $\mathcal{C} = [c_1, c_2]$, we have $\mathcal{B} \leq \mathcal{C}$ if $c_1 \leq b_1$ and $b_1 \leq c_2$. Define a relation ρ on $\mathfrak{T}$ by $\mathfrak{T}(\mathcal{B}, \mathcal{C}) = \max\{|b_1 - c_1|, |b_2 - c_2|\}$. Then clearly, ρ defines a metric on $\mathfrak{T}$ and $(\mathfrak{T}, \rho)$ a complete metric space.

Definition 2.3: A fuzzy real number is a fuzzy set, that is, a mapping $\mathcal{F}: \mathbb{R} \to I = [0,1]$ associating each real number $t \in \mathbb{R}$ with the membership function $\mathcal{F}(t)$ such that
the fuzzy number $\mathcal{F}$ is said to be

i. Normal if there exists $t \in \mathbb{R}$ such that $\mathcal{F}$ (t) = 1

ii. Convex if for $t, s \in \mathbb{R}$ and $0 \leq \theta \leq 1$, $\mathcal{F}(\theta t + (1-\theta) s) \geq \min\{\mathcal{F}(t), \mathcal{F}(s)\}$

iii. A fuzzy real number $\mathcal{F}$ is said to be *upper semi-continuous* if for $\varepsilon > 0, \mathcal{F}^{-1}([0, a+\varepsilon)), \forall a \in I$ is open in the usual topology of $\mathbb{R}$.

The α- *level* set on fuzzy set $\mathcal{F}$ is denoted by $\mathcal{F}^{\alpha}$ and defined by $\mathcal{F}^{\alpha} = \{ t \in \mathbb{R} : \mathcal{F}(t) \geq \alpha \}$.

Support of a fuzzy number is the set of all those fuzzy numbers having membership values greater than zero.

Suppose $\mathbb{R}(I)$ denotes the set of all fuzzy numbers which are upper semi-continuous and have compact support. In other words, $\mathcal{F} \in \mathbb{R}(I)$ then for any $\alpha \epsilon [0,1]$,

$$\mathcal{F}^{\alpha} = \begin{cases} t : \mathcal{F}(t) \geq \alpha \; for \; \alpha \epsilon (0,1] \\ t : \mathcal{F}(t) > \alpha \; for \; \alpha = 0 \end{cases}$$

Let us consider a mapping $\bar{\rho}(\mathcal{F}, \mathcal{G}) = \sup \rho(\mathcal{F}^{\alpha}, \mathcal{G}^{\alpha}) \; for \; 0 \leq \alpha \leq 1$. Then clearly, $\bar{\rho}$ defines a metric on $\mathbb{R}(I)$ and $(\mathbb{R}(I), \bar{\rho})$ forms a complete metric space.

A sequence of fuzzy real numbers $\mathcal{F} = (\mathcal{F}_k)$ is said to be *convergent* to a fuzzy real number $\mathcal{F}_o \in \mathbb{R}$ if $\forall \varepsilon > 0, \exists n_o \in \mathbb{N}$ such that $\bar{\rho}(\mathcal{F}_k, \mathcal{F}_o) < \varepsilon, \forall k \geq n_o$ and we write $\lim_{k\to\infty} \mathcal{F}_k = \mathcal{F}_o$.

It is said to be *Cauchy* sequence if $\forall \varepsilon > 0, \exists n_o \in \mathbb{N} : \forall m, n \geq n_o \implies \bar{\rho}(\mathcal{F}_m, \mathcal{F}_n) < \varepsilon$.

3. Difference Sequence Spaces

The idea of difference sequence space was firstly used by Kizmaz [10] in 1981 $l_{\infty}(\Delta)$, $C(\Delta), C_o(\Delta)$ as follows

$$l_{\infty}(\Delta) = \{ X = X_k : \Delta X \in l_{\infty}\}$$
$$C(\Delta) = \{ X = X_k : \Delta X \in C\}$$
$$C_o(\Delta) = \{ X = X_k : \Delta X \in C_o\}, \text{ where } \Delta X = X_k - X_{k+1}$$

and showed that these from s Banach Space with respect to the norm $\|X\|_{\Delta} = |X_1| + \|\Delta X\|_{\infty}$.
In 1995, Colka and Et.[6] generalized the difference sequence spaces of real and complex numbers to study different properties of the sequence spaces $l_{\infty}(\Delta^m)$, $C(\Delta^m), C_o(\Delta^m)$ defined as follow

$$l_{\infty}(\Delta^m) = \{ X = X_k \in \omega : (\Delta^m X) \in l_{\infty}\}$$
$$C(\Delta) = \{ X = X_k \in \omega : (\Delta^m X) \in C\}$$
$$C_o(\Delta) = \{ X = X_k \in \omega : (\Delta^m X) \in C_o\}.$$

To examine the diverse properties of the difference sequence spaces $l_{\infty}^F(\Delta_m)$, $C^F(\Delta_m)$ and $C_o^F(\Delta_m)$ of fuzzy numbers, Baruah and Tripathy [1] established the notation of difference operator Δ_m in 2009. Tripathy and Sharma [25] studied different properties of the convergent, null, and bounded sequence of fuzzy real numbers defined by the Orlicz function as follows:

$$(l_{\infty})_F(\mathcal{M}) = \left\{ \mathcal{F} = (\mathcal{F}_k) : \sup_k \mathcal{M}\left(\frac{\bar{\rho}(\mathcal{F}_k, \bar{0})}{\xi}\right) < \infty \; for \; some \; \xi > 0 \right\}$$
$$C_F(\mathcal{M}) = \left\{ \mathcal{F} = (\mathcal{F}_k) : \lim_k \mathcal{M}\left(\frac{\bar{\rho}(\mathcal{F}_k, L)}{\xi}\right) = 0 \text{ for some } \xi > 0 \; and \; L \in \mathbb{R} \right\}$$
$$(C_o)_F(\mathcal{M}) = \left\{ \mathcal{F} = (\mathcal{F}_k) : \lim_k \mathcal{M}\left(\frac{\bar{\rho}(\mathcal{F}_k, 0)}{\xi}\right) = 0 \text{ for some } \xi > 0 \right\}.$$

and studied different topological properties of the spaces.

Tripathy and Borgohain [24] introduced the classes of generalized difference bounded, convergent, and null sequences of fuzzy real numbers defined by an Orlicz function and some properties of these sequence spaces like solidness, symmetricity and convergence are studied, and obtain some inclusion relations involving these sequence spaces.

In 2012, to study the various propertied of difference sequence spaces using the difference operator Δ^m, where m is fixed positive integer , and Orlicz function N. Subramaninan et al. [22] introduced following sequence spaces

$$\Lambda_{\mathcal{M}}^{F}(\Delta^m) = \left\{ (X_k) \in \omega(F): \sup_k \mathcal{M}\left(\frac{\rho\left(|\Delta^m X_k|^{\frac{1}{k}}, \bar{0} \right)}{\xi} \right) < \infty \text{ for some } \xi > 0 \right\}$$

$$X_{\mathcal{M}}^{F}(\Delta^m) = \left\{ (X_k) \in \omega(F): \mathcal{M}\left(\frac{\rho\left((k!|\Delta^m X_k|)^{\frac{1}{k}}, \bar{0} \right)}{\xi} \right) \to 0 \text{ as } k \to \infty, \text{ for some } \xi > 0 \right\}$$

$$\Gamma_{\mathcal{M}}^{F}(\Delta^m) = \left\{ (X_k) \in \omega(F): \mathcal{M}\left(\frac{\rho\left(|\Delta^m X_k|^{\frac{1}{k}}, \bar{0} \right)}{\xi} \right) \to 0 \text{ as } k \to \infty \text{ for some } \xi > 0 \right\}$$

$$\chi_{\mathcal{M}}^{F}(\Delta^m, \emptyset) = \left\{ (X_k) \in \omega(F): \frac{1}{\emptyset_s}\mathcal{M}\left(\frac{\rho\left((k!|\Delta^m X_k|)^{\frac{1}{k}}, \bar{0} \right)}{\xi} \right) \to 0 \text{ as } k \to \infty, s \to 0 \text{ for } k \in \sigma \in P_s \right\}$$

$$\Gamma_{\mathcal{M}}^{F}(\Delta^m, \emptyset) = \left\{ (X_k) \in \omega(F): \frac{1}{\emptyset_s}\mathcal{M}\left(\frac{\rho\left(|\Delta^m X_k|^{\frac{1}{k}}, \bar{0} \right)}{\xi} \right) \to 0 \text{ as } k \to \infty, s \to 0 \text{ for } k \in \sigma \in P_s \right\}$$

In 2012, Sarma [20] introduced I – convergent sequence of fuzzy real numbers defined by Orlicz function as follow:

$$(C^I)^F(\mathcal{M}) = \left\{ \mathcal{F} = (\mathcal{F}_k) : \left\{ k : \mathcal{M}\left(\frac{\bar{\rho}(\mathcal{F}_k, L)}{r} \right) \geq \varepsilon \text{ for some } r > 0 \text{ and } L \in \mathbb{R}(I) \right\} \in I \right\}$$

$$(l_\infty^I)^F(\mathcal{M}) = \left\{ \mathcal{F} = (\mathcal{F}_k) : \left\{ k : \mathcal{M}\left(\frac{\bar{\rho}(\mathcal{F}_k,)}{r} \right) \geq \varepsilon \text{ for some } r > 0 \right\} \in I \right\}$$

and studied different properties like solidity and symmetricity.

Savas and Altinok [5] introduced some classes of fuzzy number sequences, looked at them, and studied some of their characteristics including completeness, solidity, symmetry and convergence free. They also looked at some inclusion relations that were relevant to these classes. Further, in 2017, Ebadullah [4] introduced certain classes of sequence spaces of fuzzy numbers defined by sequence ofmodulii and studied their topology. Then, Sarma [21] introduced the notations $W^F(\mathcal{M}, P)$, $W_o^F(\mathcal{M}, P)$ and $W_\infty^F(\mathcal{M}, P)$ of fuzzy sequence spaces defined by Orlicz function and studied various properties such as completeness, symmetricity and established some inclusion relation. By using the sequence for modulo function, Ganie and Gupkari [9] introduced and investigated various new ways of difference sequences of fuzzyreal numbers in 2021.

Paudel and Pahari [16] studied some topological structures of fuzzy metric space in 2021 using the concept of fuzzy. Additionally, in 2022, Paudel and et al.[17] introduced the generalized form of the p-bounded variation of fuzzy real numbers, which is defined as follows:

$$bV_p{}^F(\Delta_m\mathcal{F}) = \{ \mathcal{F} = (\mathcal{F}_k) \in \omega(F): \textstyle\sum_{k=1}^{\infty} \{\bar{d}(\Delta_m\mathcal{F}_k, \bar{0})\}^p < \infty \}$$

for $1 \leq p < \infty$, where $\Delta_m\mathcal{F}_k = \mathcal{F}_k - \mathcal{F}_{k+m}$

and looked at a few findings that describe the topological structure of the novel class of P-bounded variants of the generalized difference sequence space of fuzzy real numbers.

4. Main Results

In this paper, we introduce generalized difference sequence classes of fuzzy real numbers defined by Orlicz function as follow:

$$F_\infty(\bar{\rho}, \mathcal{M}, p, A) = \left\{\mathcal{F} = (\mathcal{F}_k) \in \omega^F : \sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k), \bar{0})}{\xi}\right)\right]^{p_k} < \infty\right\}$$

$$F(\bar{\rho}, \mathcal{M}, p, A) = \left\{\mathcal{F} = (\mathcal{F}_k) \in \omega^F : \lim_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k), \bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k} = 0\right\}$$

$$F_o(\bar{\rho}, \mathcal{M}, p, A) = \left\{\mathcal{F} = (\mathcal{F}_k) \in \omega^F : \lim_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k), \bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k} = 0 \ for\ some\ \rho > 0\right\}$$

where $p = (p_k)$, a sequence of strictly positive real number $A = (a_k)$, a sequence of real numbers and $\Delta_m \mathcal{F}_k = \mathcal{F}_k - \mathcal{F}_{k+m}$

Theorem 4.1:

The classes $F_\infty(\bar{\rho}, \mathcal{M}, p, A), F(\bar{\rho}, \mathcal{M}, p, A)$ and $F_o(\bar{\rho}, \mathcal{M}, p, A)$ are linear spaces.

Proof:

To prove the linearity of the classes , we firstly show that $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$ is linear and then other proof is similar.

Let $\mathcal{F} = (\mathcal{F}_k)$ and $\mathcal{H} = (\mathcal{H}_k)$ are from $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$. Then there exists $\xi_1 > 0$ $\xi_2 > 0$ such that $\sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}), \bar{0})}{\xi_1}\right)\right]^{p_k} < \infty$ and $\sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{H}), \bar{0})}{\xi_2}\right)\right]^{p_k} < \infty$.

Then for any α , β and $\xi = \max\{2\alpha\xi_1, 2\beta\xi_2\}$, we have

$$\begin{aligned}
\sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \alpha \mathcal{F}_k) + a_k(\Delta_m \beta \mathcal{H}_k), \bar{0})}{\xi}\right)\right]^{p_k} &\leq \sup_k \left[\mathcal{M}\left(\alpha \frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k)\,\bar{0}))}{\xi} + \beta \frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{H}_k)\,\bar{0}))}{\xi}\right)\right]^{p_k} \\
&\leq \sup_k \left[\mathcal{M}\left(\frac{1}{2}\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k)\,\bar{0}))}{\xi_1} + \frac{1}{2}\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{H}_k)\,\bar{0}))}{\xi_2}\right)\right]^{p_k} \\
&\leq G \sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k)\,\bar{0}))}{\xi_1}\right)\right]^{p_k} + G \sup_k \left[\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{H}_k)\,\bar{0}))}{\xi_2}\right]^{p_k} \\
&< \infty
\end{aligned}$$

where, $G = \max\{1, 2^{\mu-1}\}$, $\mu = \sup_k p_k$.

So that $\sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \alpha \mathcal{F}_k) + a_k(\Delta_m \beta \mathcal{H}_k), \bar{0})}{\xi}\right)\right]^{p_k} < \infty$ and so $\alpha\mathcal{F} + \beta\mathcal{H} \in F_\infty(\bar{\rho}, \mathcal{M}, p, A)$ and hence the sequence of class $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$ is linear.

Theorem 4.2 :

Suppose $p = (p_k)$ and $q = (q_k)$ be two positive strictly increasing sequence such that $0 < p_k \leq q_k < \infty$ for all values of k, then the following relation holds:

(i) $F(\bar{\rho}, \mathcal{M}, p, A) \subseteq F(\bar{\rho}, \mathcal{M}, q, A)$ (ii) $F_o(\bar{\rho}, \mathcal{M}, p, A) \subseteq F_o(\bar{\rho}, \mathcal{M}, q, A)$

Proof :

Suppose $p = (p_k)$ and $q = (q_k)$ be two positive strictly increasing sequence such that $0 < p_k \leq q_k < \infty$ for all values of k . Let $\mathcal{F} = (\mathcal{F}_k) \in F(\bar{\rho}, \mathcal{M}, p, A)$. Then there exists $\xi > 0$ such that

$$\lim_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k), \bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k} = 0.$$

This relation is true only for $\left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}), \bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k} \leq 1$ for sufficiently large value of <u>k</u>. Since, $p_k \leq q_k$ for all values of k, we have

$\left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}),\bar{\mathcal{F}}_o)}{\xi}\right)\right]^{q_k} \leq \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}),\bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k}.$

Taking limit $k \to \infty$,

$\lim_{k \to \infty}\left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k),\bar{\mathcal{F}}_o)}{\xi}\right)\right]^{q_k} \leq \lim_{k \to \infty}\left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k),\bar{\mathcal{F}}_o)}{\xi}\right)\right]^{p_k} = 0$

$\Rightarrow \lim_{k \to \infty}\left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k),\bar{\mathcal{F}}_o)}{\xi}\right)\right]^{q_k} = 0$

$\Rightarrow \mathcal{F} = (\mathcal{F}_k) \in F(\bar{\rho}, \mathcal{M}, q, A)$

$\Rightarrow F(\bar{\rho}, \mathcal{M}, p, A) \subseteq F(\bar{\rho}, \mathcal{M}, q, A)$

Similarly, we can show that $F_o(\bar{\rho}, \mathcal{M}, p, A) \subseteq F_o(\bar{\rho}, \mathcal{M}, q, A)$. This completes the proof.

Theorem 4.3:

The space $F_\infty(\bar{\rho}, \mathcal{M}, p, A) = \left\{\mathcal{F} = (\mathcal{F}_k) \in \omega^F : \sup_k \left[\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k),\bar{0})}{\xi}\right)\right]^{p_k} < \infty\right\}$ is a complete metric space with the metric defined as

$$\bar{d}_G(\mathcal{F}, \mathcal{H}) = \inf\left\{\xi^{\frac{p_k}{T}} : \sup_k \left[\left\{\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k),\; a_k(\Delta_m \mathcal{H}_k))}{\xi}\right)\right\}^{p_k}\right]^{\frac{1}{T}} \leq 1\right\}$$

where, $p = p_k$, be a sequence of strictly positive real numbers , $A = (a_k)$, a sequence of real numbers.

Proof:

Let $\{\mathcal{F}^i\}$ be a Cauchy sequence in $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$. Then for $\varepsilon > 0$, let us choose $u > 0$, and $\eta > 0$ such that

$$\mathcal{M}\left(\frac{u\eta}{2}\right) \geq 1 \ldots\ldots\ldots.(*)$$

Then for $\varepsilon > 0, \exists n_o \in \mathbb{N} : \forall\, i, j \geq n_o$, we have

$$\bar{d}_G(\mathcal{F}^i, \mathcal{F}^j) < \frac{\varepsilon}{u\eta}$$

By the definition of $\bar{d}_G$ we have,

$$\bar{d}_G(\mathcal{F}^i, \mathcal{F}^j) = \inf\left\{\xi^{\frac{p_k}{T}} : \sup_k \left[\left\{\mathcal{M}\left(\frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi}\right)\right\}^{p_k}\right]^{\frac{1}{T}}\right\} \leq 1$$

$$\Rightarrow \left\{\mathcal{M}\left(\frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi}\right)\right\}^{p_k} \leq 1$$

$$\Rightarrow \left\{\mathcal{M}\left(\frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi}\right)\right\}^{p_k} \leq 1 \leq \mathcal{M}\left(\frac{u\eta}{2}\right)$$

Since, $\mathcal{M}$ is a non-decreasing function,

$$\frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi} \leq \frac{u\eta}{2}$$

Since, $\xi > 0$ let us choose $= \frac{\varepsilon}{u\eta}$, so that

$$\frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi} \leq \frac{u\eta}{2}.\frac{\varepsilon}{u\eta} < \varepsilon$$

$\therefore \frac{\bar{\rho}\left(a_k(\Delta_m \mathcal{F}_k^i),\; a_k(\Delta_m \mathcal{F}_k^j)\right)}{\xi} < \varepsilon \;\forall\, i, j \geq n_o$

This shows that $\{a_k(\Delta_m \mathcal{F}_k^i)\}$ is a Cauchy sequence in $\mathbb{R}(\mathrm{I})$ for all $k \in \mathbb{N}$. Since $\mathbb{R}(\mathrm{I})$ is complete, the sequence $\{a_k(\Delta_m \mathcal{F}_k^i)\}$ converges in $\mathbb{R}(\mathrm{I})$ and say

$$\lim_{i \to \infty} a_k \Delta_m \mathcal{F}_k^i = a_k \Delta_m \mathcal{F}_k$$

$$\Rightarrow \lim_{i \to \infty} \left(\mathcal{F}_k^i - \mathcal{F}_{k+m}^i\right) = \mathcal{F}_k - \mathcal{F}_{k+m}$$

$$\Rightarrow \lim_{i \to \infty} \mathcal{F}_k^i = \mathcal{F}_k$$

$$\therefore \lim_{i \to \infty} \mathcal{F}^i = \mathcal{F} \text{ , where } \mathcal{F} = \mathcal{F}_k$$

To complete proof , we show that $\mathcal{F} \in F_\infty(\bar{\rho}, \mathcal{M}, p, A)$. For , we have,

$$\bar{d}_G(\mathcal{F}, \mathcal{H}) = \inf\left\{\xi^{\frac{p_k}{T}} : \sup_k \left[\left\{\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k)\,,\ a_k(\Delta_m \mathcal{H}_k))}{\xi}\right)\right\}^{p_k}\right]^{\frac{1}{T}} \le 1\right\}$$

Now, $\inf\left\{\xi^{\frac{p_k}{T}} : \sup_k \left[\left\{\mathcal{M}\left(\frac{\bar{\rho}\,(a_k(\Delta_m \mathcal{F}_k)\,,0)}{\xi}\right)\right\}^{p_k}\right]^{\frac{1}{T}}\right\} = \bar{d}_G(\mathcal{F}, \bar{0})$

$$\le \bar{d}_G(\mathcal{F}, \mathcal{H}) + \bar{d}_G(\mathcal{F}^i, \bar{0})$$

$$< \varepsilon' + \bar{d}_G(\mathcal{F}^i, \bar{0}) < \infty$$

$\Rightarrow \mathcal{F} \in F_\infty(\bar{\rho}, \mathcal{M}, p, A)$. Hence $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$ is complete.

Similarly, we can show that space $F(\bar{\rho}, \mathcal{M}, p, A)$ and $\mathcal{F} \in F_o(\bar{\rho}, \mathcal{M}, p, A)$ are complete.

Conclusion

In this paper, we have used the fuzzy real numbers and Orlicz function to study the generalized form of difference sequence spaces of fuzzy real numbers. We have shown that the classes $(\bar{\rho}, \mathcal{M}, p, A)$, $F_o(\bar{\rho}, \mathcal{M}, p, A)$ and $F_\infty(\bar{\rho}, \mathcal{M}, p, A)$ are linear and complete metric spaces with the metric $\bar{d}_G$. More over the inclusion relation $F(\bar{\rho}, \mathcal{M}, p, A) \subseteq F(\bar{\rho}, \mathcal{M}, q, A)$ and $F_o(\bar{\rho}, \mathcal{M}, p, A) \subseteq F_o(\bar{\rho}, \mathcal{M}, q, A)$ are shown.

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