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# Characterizations for Certain Subclasses of Starlike and Convex Functions Associated with Lommel, Struve and Bessel Functions

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**Abstract:** *In this paper we have determined some necessary and sufficient conditions of normalized Lommel function* $s_{\mu,\nu}$ *,Struve function* $h_\nu$ *and Bessel function* $j_\nu$ *of the first kind to be in the subclasses* $S(\alpha, \beta, \gamma)$ *and* $K(\alpha, \beta, \gamma)$ *of starlike and convex functions of order* $\alpha$ *and type* $\beta, \gamma$, *in the unit disk U.*

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## 1 Introduction

Let $\mathcal{S}$ be the class of univalent functions $f$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{T}$ be subclass of $\mathcal{S}$ consisting of functions of the form;

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{1.2}$$

**Definition 1.1.** [7] A function $f$ of the form (1.1) is said to be in the class $S(\alpha, \beta, \gamma)$ if it satisfies following condition:

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(2\gamma - 1)\frac{zf'(z)}{f(z)} + (1 - 2\gamma\alpha)} \right| < \beta; \qquad z \in U, \tag{1.3}$$

where $0 \leq \alpha < 1, 0 < \beta \leq 1$ and $1/2 < \gamma \leq 1$.

**Definition 1.2.** [6] A function $f$ of the form (1.1) is said to be in the class $K(\alpha, \beta, \gamma)$ if it satisfies following condition:

$$\left| \frac{\frac{zf''z)}{f'(z)}}{(2\gamma - 1)\frac{zf''(z)}{f'(z)} + 2\gamma(1-\alpha)} \right| < \beta; \qquad z \in U, \tag{1.4}$$

where $0 \le \alpha < 1, 0 < \beta \le 1$ and $1/2 < \gamma \le 1$.

The subclasses $S(\alpha, \beta, \gamma)$ and $K(\alpha, \beta, \gamma)$ are the well-known subclasses of starlike and convex functions of order $\alpha$ and type $\beta, \gamma$, respectively introduced by Kulkarni [7] and Joshi et. al. [6]. Let $T^*(\alpha, \beta, \gamma)$ and $C(\alpha, \beta, \gamma)$ be subclasses of $\mathcal{T}$ defined by

$$T^*(\alpha, \beta, \gamma) = S(\alpha, \beta, \gamma) \cap \mathcal{T} \textit{ and } C(\alpha, \beta, \gamma) = K(\alpha, \beta, \gamma) \cap \mathcal{T}.$$

Also

$$f(z) \in C(\alpha, \beta, \gamma) \Leftrightarrow zf'(z) \in T^*(\alpha, \beta, \gamma).$$

We note that $S(\alpha, \beta, 1) = S(\alpha, \beta)$ and $K(\alpha, \beta, 1) = K(\alpha, \beta)$ are well-known subclasses of starlike and convex functions of order $\alpha$ and type $\beta$, respectively introduced by Gupta and Jain [8]. Also $S(\alpha, 1, 1) = S(\alpha)$ and $K(\alpha, 1, 1) = K(\alpha)$ are well-known subclasses of starlike and convex functions of order $\alpha$, respectively introduced by Robertson [13], MacGregor [9] and Schild [14]. Further, $T^*(\alpha, 1, 1) = T^*(\alpha)$ and $C(\alpha, 1, 1) = C(\alpha)$ are the subclasses of starlike and convex functions of order $\alpha$ with negative coefficients introduced by Silverman [15].

In this paper, we consider three special functions, the Struve function of the first kind $H_\nu$, the Lommel function of the first kind $S_{\mu,\nu}$ and the Bessel function of the first kind $J_\nu$. We know that the Bessel, Struve and Lommel functions can be expressed as the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}; \qquad \nu \notin \mathbb{Z}^-, \tag{1.5}$$

$$H_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\nu+1}; \qquad \nu + \frac{1}{2} \notin \mathbb{Z}^- \tag{1.6}$$

and

(1.7)
$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{\mu-\nu+1}{2})\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(n+\frac{\mu-\nu+3}{2})\Gamma(n+\frac{\mu+\nu+3}{2})} \left(\frac{z}{2}\right)^{2n}; \qquad \frac{\mu \pm \nu + 1}{2} \notin \mathbb{Z}^-,$$

for $\mu, \nu \in \mathbb{C}$.

We know that the Bessel function $J_\nu$ is a solution of homogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0.$$

Also Struve function $H_\nu(z)$ and Lommel function $S_{\mu,\nu}(z)$ are a particular solutions of the following non-homogeneous Bessel differential equations (see [12]).

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = \frac{(z/2)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}$$

and

$$z^2 w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1}.$$

Also note that the functions $J_\nu(z), H_\nu(z)$ and $S_{\mu,\nu}(z)$ are expressed in terms of hypergeometric functions ${}_1F_2$ as follows.

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_1F_2\left(1; 1, \nu+1; -\frac{z^2}{4}\right); \nu \notin \mathbb{Z}^-,$$

$$H_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\frac{\pi}{4}}\Gamma\left(\nu+\frac{3}{2}\right)} {}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right); \nu + \frac{1}{2} \notin \mathbb{Z}^-,$$

and

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right); \frac{\mu \pm \nu + 1}{2} \notin \mathbb{Z}^-.$$

For more information about these functions please see [16].

In this paper, we are interested in the normalized Bessel function of the first kind $j_\nu : U \to \mathbb{C}$, the normalized Struve function of the first kind $h_\nu : U \to \mathbb{C}$, and normalized Lommel function of the first kind $s_{\mu,\nu} : U \to \mathbb{C}$, which are defined as follows

(1.8) $$j_\nu(z) := \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1)_n(\nu+1)_n} z^{n+1}; \nu \notin \mathbb{Z}^-,$$

(1.9)
$$h_\nu(z) := \Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right) z^{1-\frac{\nu}{2}} H_\nu(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{3}{2}\right)_n \left(\nu+\frac{3}{2}\right)_n} z^{n+1}; \nu + \frac{1}{2} \notin \mathbb{Z}^-,$$

and

(1.10) $$s_{\mu,\nu}(z) := (\mu-\nu+1)(\mu+\nu+1) z^{-\mu} S_{\mu,\nu}(2\sqrt{z})$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{\mu-\nu+3}{2}\right)_n \left(\frac{\mu+\nu+3}{2}\right)_n} z^{n+1}; \ \frac{\mu \pm \nu + 1}{2} \notin \mathbb{Z}^-.$$

Observe that

$$s_{\nu,\nu}(z) = h_\nu(z) \text{ and } s_{\nu-1,\nu}(z) = j_\nu(z). \tag{1.11}$$

Recently, El-Ashwah et al. [3], Cho et al. [1] and Murugusundaramoorthy and Janani [10] introduced some characterization of generalized Bessel functions of first kind to be in certain subclasses of uniformly starlike and uniformly convex functions. Motivated by various authors ([1],[2], [3], [4], [5], [10], [11]), in the present paper, we have determined necessary and sufficient conditions for the normalized Bessel function of the first kind, the normalized Struve function of the first kind and normalized Lommel function of the first kind to be in Subclasses of analytic functions $S(\alpha, \beta, \gamma)$ and $K(\alpha, \beta, \gamma)$ of starlike and convex functions of order $\alpha$ and type $\beta, \gamma$.

## 2 Characterizations on Lommel Functions

Following lemmas are useful to establish our main results.

**Lemma 2.1.** *[7] (i) A sufficient condition for a function $f$ of the form (1.1) to be in the class $S(\alpha, \beta, \gamma)$ is that*

$$\sum_{n=2}^{\infty} [n - 1 + \beta(1 - n + 2\gamma n - 2\gamma\alpha)] \, |a_n| \leq 2\beta\gamma(1 - \alpha). \tag{2.1}$$

*(ii) A necessary and sufficient condition for a function $f$ of the form (1.2) to be in the $T^*(\alpha, \beta, \gamma)$ is that*

$$\sum_{n=2}^{\infty} [n - 1 + \beta(1 - n + 2\gamma n - 2\gamma\alpha)] a_n \leq 2\beta\gamma(1 - \alpha). \tag{2.2}$$

**Lemma 2.2.** *[6] (i) A sufficient condition for a function $f$ of the form (1.1) to be in the class $K(\alpha, \beta, \gamma)$ is that*

$$\sum_{n=2}^{\infty} n[n - 1 + \beta(1 - n + 2\gamma n - 2\gamma\alpha)] |a_n| \leq 2\beta\gamma(1 - \alpha). \tag{2.3}$$

*(ii) A necessary and sufficient condition for a function $f$ of the form (1.2) to be in the $C(\alpha, \beta, \gamma)$ is that*

$$\sum_{n=2}^{\infty} n[n - 1 + \beta(1 - n + 2\gamma n - 2\gamma\alpha)] a_n \leq 2\beta\gamma(1 - \alpha). \tag{2.4}$$

If

$$f_0(z) = \frac{z}{1+z} = z + \sum_{n=1}^{\infty}(-1)^n z^{n+1} \quad (z \in U), \tag{2.5}$$

then using convolution principle, we define

$$\mathfrak{s}_{\mu,\nu}(z) = s_{\mu,\nu}(z) * f_0(z).$$

Now we prove our main results.

**Theorem 2.3.** *If $\mu > \nu - 3$, then the condition*

$$(1-\beta+2\beta\gamma)\mathfrak{s}'_{\mu+2,\nu}(1) + 2\beta\gamma(1-\alpha)\mathfrak{s}_{\mu+2,\nu}(1) \le \frac{8\beta\gamma(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)} \tag{2.6}$$

*suffices that $\mathfrak{s}_{\mu,\nu}(z) \in S(\alpha,\beta,\gamma)$.*

*Proof.* Since

$$\mathfrak{s}_{\mu,\nu}(z) = z + \sum_{n=2}^{\infty} \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}} z^n; \quad \frac{\mu \pm \nu + 1}{2} \notin \mathbb{Z}^-.$$

By virtue of (i) in Lemma 2.1 , it is suffices to show that

$$\sum_{n=2}^{\infty}[n-1+\beta(1-n+2\gamma n-2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}} \le 2\beta\gamma(1-\alpha).$$

By simple computation, we have

$$\begin{aligned}
M(\alpha,\beta,\gamma;\mu,\nu) &= \sum_{n=2}^{\infty}[n-1+\beta(1-n+2\gamma n-2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}} \\
&= \sum_{n=0}^{\infty}[n+1+\beta(-1-n+2\gamma(n+2)-2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+1}} \\
&= \sum_{k=0}^{\infty}[(n+1)(1-\beta+2\beta\gamma)+2\beta\gamma(1-\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+1}} \\
&= \frac{(1-\beta+2\beta\gamma)}{\left(\frac{\mu-\nu+3}{2}\right)\left(\frac{\mu+\nu+3}{2}\right)}\sum_{n=0}^{\infty}(n+1)\frac{1}{\left(\frac{\mu+2-\nu+3}{2}\right)_n\left(\frac{\mu+2+\nu+3}{2}\right)_n} \\
&\quad + \frac{2\beta\gamma(1-\alpha)}{\left(\frac{\mu-\nu+3}{2}\right)\left(\frac{\mu+\nu+3}{2}\right)}\sum_{n=0}^{\infty}\frac{1}{\left(\frac{\mu+2-\nu+3}{2}\right)_n\left(\frac{\mu+2+\nu+3}{2}\right)_n} \\
&= \left(\frac{\mu-\nu+3}{2}\right)\left(\frac{\mu+\nu+3}{2}\right)\left[(1-\beta+2\beta\gamma)\mathfrak{s}'_{\mu+2,\nu}(1)\right. \\
&\quad \left.+2\beta\gamma(1-\alpha)\mathfrak{s}_{\mu+2,\nu}(1)\right].
\end{aligned}$$

Thus, we see that the last expression is bounded above by $2\beta\gamma(1-\alpha)$ if condition (2.6) is satisfied. This completes the proof of Theorem 2.3 . ☐

If

$$f_1(z) = z\left(2 - \frac{1}{1+z}\right) = z + \sum_{n=1}^{\infty}(-1)^{n+1}z^{n+1} \quad (z \in U) \tag{2.7}$$

and the function

$$\mathfrak{t}_{\mu,\nu}(z) := s_{\mu,\nu}(z) * f_1(z),$$

then we have the following result.

**Theorem 2.4.** *For* $\mu > \nu - 3$,

$$(1 - \beta + 2\beta\gamma)\mathfrak{t}'_{\mu+2,\nu}(1) + 2\beta\gamma(1-\alpha)\mathfrak{t}_{\mu+2,\nu}(1) \le \frac{8\beta\gamma(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)}, \tag{2.8}$$

*is the necessary and sufficient condition for* $\mathfrak{t}_{\mu,\nu}(z)$ *to be in the class* $T^*(\alpha,\beta,\gamma)$.

*Proof.* Since

$$\mathfrak{t}_{\mu,\nu}(z) = z - \sum_{n=2}^{\infty}\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}}z^n;\ \frac{\mu \pm \nu + 1}{2} \notin \mathbb{Z}^-, \tag{2.9}$$

then by using Lemma 2.1, togehter with the same techniques of Theorem 2.3, we complete the proof.

□

**Theorem 2.5.** *If* $\mu > \nu - 3$, *then the condition*

$$(1 - \beta + 2\beta\gamma)\mathfrak{s}''_{\mu+2,\nu}(1) + 2(3\beta\gamma - \alpha\beta\gamma - \beta + 1)\mathfrak{s}'_{\mu+2,\nu}(1) \tag{2.10}$$

$$+2\beta\gamma(1-\alpha)\mathfrak{s}_{\mu+2,\nu}(1) \le \frac{8\beta\gamma(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)}$$

*suffices that* $\mathfrak{s}_{\mu,\nu}(z) \in K(\alpha,\beta,\gamma)$.

*Proof.* By virtue of (i) in Lemma 2.2, it is suffices to show that

$$\sum_{n=2}^{\infty} n[n - 1 + \beta(1 - n + 2\gamma n - 2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}} \le 2\beta\gamma(1-\alpha).$$

By simple computation, we have

$$
\begin{aligned}
&G(\alpha,\beta,\gamma;\mu,\nu)\\
&=\sum_{n=2}^{\infty} n[n-1+\beta(1-n+2\gamma n-2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n-1}\left(\frac{\mu+\nu+3}{2}\right)_{n-1}}\\
&=\sum_{n=0}^{\infty}(n+2)[n+1+\beta(-1-n+2\gamma n+4\gamma-2\gamma\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+1}}\\
&=\sum_{n=0}^{\infty}[(1-\beta+2\beta\gamma)(n+1)(n)+2(1-\beta+3\beta\gamma-\alpha\beta\gamma)(n+1)\\
&\qquad+2\beta\gamma(1-\alpha)]\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{n+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+1}}\\
&=\frac{1}{\left(\frac{\mu-\nu+3}{2}\right)\left(\frac{\mu+\nu+3}{2}\right)}\left[(1-\beta+2\beta\gamma)\mathfrak{s}''_{\mu+2,\nu}(1)\right.\\
&\qquad\left.+2(1-\beta+3\beta\gamma-\alpha\beta\gamma)\mathfrak{s}'_{\mu+2,\nu}(1)+2\beta\gamma(1-\alpha)\mathfrak{s}_{\mu+2,\nu}(1)\right].
\end{aligned}
$$

Thus, we see that the last expression is bounded above by $2\beta\gamma(1-\alpha)$ if (2.10) is satisfied. Hence, the proof. □

**Theorem 2.6.** *For $\mu>\nu-3$,*

$$
(2.11)\qquad (1-\beta+2\beta\gamma)\mathfrak{t}''_{\mu+2,\nu}(1)+2(3\beta\gamma-\alpha\beta\gamma-\beta+1)\mathfrak{t}'_{\mu+2,\nu}(1)
$$
$$
+2\beta\gamma(1-\alpha)\mathfrak{t}_{\mu+2,\nu}(1)\le\frac{8\beta\gamma(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)},
$$

*is the necessary and sufficient condition for $\mathfrak{t}_{\mu,\nu}(z)$ to be in the class $C(\alpha,\beta,\gamma)$.*

Putting $\gamma=1$ in Theorems 2.4 and 2.6, we obtain the following corollaries.

**Corollary 2.7.** *The function $\mathfrak{t}_{\mu,\nu}(z)$ is a starlike function of order $\alpha(0\le\alpha<1)$ and type $\beta(0<\beta\le1)$, if and only if*

$$
(1+\beta)\mathfrak{t}'_{\mu+2,\nu}(1)+2\beta(1-\alpha)\mathfrak{t}_{\mu+2,\nu}(1)\le\frac{8\beta(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)}.
$$

**Corollary 2.8.** *The function $\mathfrak{t}_{\mu,\nu}(z)$ is a convex function of order $\alpha(0\le\alpha<1)$ and type $\beta(0<\beta\le1)$, if and only if*

$$
(1+\beta)\mathfrak{t}''_{\mu+2,\nu}(1)+2(2\beta-\alpha\beta+1)\mathfrak{t}'_{\mu+2,\nu}(1)
$$
$$
+2\beta(1-\alpha)\mathfrak{t}_{\mu+2,\nu}(1)\le\frac{8\beta(1-\alpha)}{(\mu-\nu+3)(\mu+\nu+3)}.
$$

## 3 Characterizations on Struve Functions

Taking $\mu = \nu$ in Theorems 2.3-2.6, we obtain the corresponding results of Struve function $h_\nu$.

**Theorem 3.1.** *If $\nu > -\frac{1}{2}$, then the condition*

$$(1-\beta+2\beta\gamma)\mathfrak{h}'_{\nu+2}(1)+2\beta\gamma(1-\alpha)\mathfrak{h}_{\nu+2}(1) \le \frac{8\beta\gamma(1-\alpha)}{3(2\nu+3)} \tag{3.1}$$

*suffices that $\mathfrak{h}_\nu(z) \in S(\alpha,\beta,\gamma)$, where*

$$\mathfrak{h}_\nu(z) := h_\nu(z) * f_0(z) = z + \sum_{n=2}^{\infty} \frac{1}{\left(\frac{3}{2}\right)_{n-1}\left(\nu+\frac{3}{2}\right)_{n-1}} z^{n+1} \left(\nu > -\frac{1}{2}\right). \tag{3.2}$$

**Theorem 3.2.** *For $\nu > -\frac{3}{2}$,*

$$(1-\beta+2\beta\gamma)\hbar'_{\nu+2}(1)+2\beta\gamma(1-\alpha)\hbar_{\nu+2}(1) \le \frac{8\beta\gamma(1-\alpha)}{3(2\nu+3)}, \tag{3.3}$$

*is the necessary and sufficient condition for $\hbar_\nu(z)$ to be in the class $T^*(\alpha,\beta,\gamma)$, where*

$$\hbar_\nu(z) := (h_\nu(z) * f_1(z)) = z - \sum_{n=2}^{\infty} \frac{1}{\left(\frac{3}{2}\right)_{n-1}\left(\nu+\frac{3}{2}\right)_{n-1}} z^{n+1} \left(\nu > -\frac{1}{2}\right). \tag{3.4}$$

**Theorem 3.3.** *If $\nu > -\frac{3}{2}$, then the condition*

$$\begin{aligned}(1-\beta+2\beta\gamma)\mathfrak{h}''_{\nu+2}(1)+2(3\beta\gamma-\alpha\beta\gamma-\beta+1)\mathfrak{h}'_{\nu+2}(1)\\ +2\beta\gamma(1-\alpha)\mathfrak{h}_{\nu+2}(1) \le \frac{8\beta\gamma(1-\alpha)}{3(2\nu+3)}\end{aligned} \tag{3.5}$$

*suffices that $\mathfrak{h}_\nu(z) \in K(\alpha,\beta,\gamma)$.*

**Theorem 3.4.** *For $\nu > -\frac{3}{2}$,*

$$\begin{aligned}(1-\beta+2\beta\gamma)\hbar''_{\nu+2}(1)+2(3\beta\gamma-\alpha\beta\gamma-\beta+1)\hbar'_{\nu+2}(1)\\ +2\beta\gamma(1-\alpha)\hbar_{\nu+2}(1) \le \frac{8\beta\gamma(1-\alpha)}{3(2\nu+3)}\end{aligned} \tag{3.6}$$

*is the necessary and sufficient condition for $\hbar_\nu(z)$ to be in the class $C(\alpha,\beta,\gamma)$.*

Putting $\gamma = 1$ in Theorem 3.2 and 3.4, we have the following corollaries.

**Corollary 3.5.** *The function $\hbar_\nu(z)$ is a starlike function of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 < \beta \le 1)$ if and only if*

$$(1+\beta)\hbar'_{\nu+2}(1)+2\beta(1-\alpha)\hbar_{\nu+2}(1) \le \frac{8\beta(1-\alpha)}{3(2\nu+3)}.$$

**Corollary 3.6.** *The function $\hbar_\nu(z)$ is a convex function of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 < \beta \le 1)$ if and only if*

$$(1+\beta)\hbar''_{\nu+2}(1)+2(2\beta-\alpha\beta+1)\hbar'_{\nu+2}(1)+2\beta(1-\alpha)\hbar_{\nu+2}(1) \le \frac{8\beta(1-\alpha)}{3(2\nu+3)}.$$

# 4 Characterizations on Bessel Functions

Taking $\mu = \nu - 1$ in Theorems 2.3-2.6, then we obtain the corresponding results of Bessel function $j_\nu$ as following:

**Theorem 4.1.** *If $\nu > -1$, then the condition*

$$(4.1) \qquad (1-\beta+2\beta\gamma)\mathrm{j}'_{\nu+1}(1) + 2\beta\gamma(1-\alpha)\mathrm{j}_{\nu+1}(1) \leq \frac{2\beta\gamma(1-\alpha)}{(\nu+1)}$$

*suffices that $\mathrm{j}_\nu(z) \in S(\alpha, \beta, \gamma)$, where*

$$(4.2) \qquad \mathrm{j}_\nu(z) := j_\nu(z) * f_0(z) = z + \sum_{n=2}^{\infty} \frac{1}{(1)_{n-1}(\nu+1)_{n-1}} z^n (\nu > -1).$$

**Theorem 4.2.** *For $\nu > -1$,*

$$(4.3) \qquad (1-\beta+2\beta\gamma)\jmath'_{\nu+1}(1) + 2\beta\gamma(1-\alpha)\jmath_{\nu+1}(1) \leq \frac{2\beta\gamma(1-\alpha)}{(\nu+1)},$$

*is the necessary and sufficient condition for $\jmath_\nu(z)$ to be in the class $T^*(\alpha, \beta, \gamma)$, where*

$$(4.4) \qquad \jmath_\nu(z) := j_\nu(z) * f_1(z) = z - \sum_{n=2}^{\infty} \frac{1}{(1)_{n-1}(\nu+1)_{n-1}} z^n (\nu > -1).$$

**Theorem 4.3.** *If $\nu > -1$, then the condition*

$$(4.5) \qquad (1-\beta+2\beta\gamma)\mathrm{j}''_{\nu+1}(1) + 2(3\beta\gamma - \alpha\beta\gamma - \beta + 1)\mathrm{j}'_{\nu+1}(1) + 2\beta\gamma(1-\alpha)\mathrm{j}_{\nu+1}(1) \leq \frac{2\beta\gamma(1-\alpha)}{(\nu+1)}$$

*suffices that $\mathrm{j}_\nu(z) \in K(\alpha, \beta, \gamma)$.*

**Theorem 4.4.** *For $\nu > -1$,*

$$(4.6) \qquad (1-\beta+2\beta\gamma)\jmath''_{\nu+1}(1) + 2(3\beta\gamma - \alpha\beta\gamma - \beta + 1)\jmath'_{\nu+1}(1) + 2\beta\gamma(1-\alpha)\jmath_{\nu+1}(1) \leq \frac{2\beta\gamma(1-\alpha)}{(\nu+1)}$$

*is the necessary and sufficient condition for $\jmath_\nu(z)$ to be in the class $C(\alpha, \beta, \gamma)$.*

Putting $\gamma = 1$ in Theorems 4.2 and 4.4, we obtain the following corollaries.

**Corollary 4.5.** *The function $\jmath_\nu(z)$ is a starlike function of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$, if and only if*

$$(1+\beta)\jmath'_{\nu+1}(1) + 2\beta(1-\alpha)\jmath_{\nu+1}(1) \leq \frac{2\beta(1-\alpha)}{(\nu+1)}.$$

**Corollary 4.6.** *The function $\jmath_\nu(z)$ is a convex function of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 < \beta \le 1)$, if and only if*

$$(1+\beta)\jmath''_{\nu+1}(1) + 2(2\beta - \alpha\beta + 1)\jmath'_{\nu+1}(1) + 2\beta(1-\alpha)\jmath_{\nu+1}(1) \le \frac{2\beta(1-\alpha)}{(\nu+1)}.$$

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