



The Parsimonious Gompertzian Mortality Parameters: Evidence From Advanced Actuarial Numerical Estimation Technique

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Abstract: Efforts in obtaining distribution function which reasonably models life table functions have always constituted a major challenge in mortality construction and attempts with many estimating functions have not solved the problem. Numerous distribution functions have been experimented for this purpose but do not seem to be of interest because they are not parsimonious in actuarial representation and consequently could not adequately describe mortality table data. In order to overcome this problem, it is possible to estimate parameters by adopting less sophisticated mortality distribution functions. In traditional mortality models like Gompertz function, mortality rate increases exponentially as ages advance chronologically. The parametric estimation is meant to produce an estimated value of μ_x .

This work aims to estimate the Gompertzian mortality model parameters from a new numerical perspective. We present a parametrization that focuses on actuarial information based on Gompertzian mortality from indirect approach. The paper intends to develop an acceptable level of approximating the Gompertz model parameters for mortality rates. The objectives are to estimate (i) the level of mortality at initial age (ii) the rate of mortality increase across ages (iii) construct the modal age at death and (iv) the ageing rate. In order to compute the parameters, a numerical method that is more advanced than the traditional methods is adopted to demonstrate that the Gompertz model describes the behaviour of mortality trajectory with all its actuarial parameters. Based on W function under logarithmic link conjecture, our investigation shows that the expected value of the random life time X is given by

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \exp\left[-\frac{1}{x}\right] (\log_e x) dx$$

Keywords: Gompertz Mortality, Gumbel function, Parameters, W -Function, Modal-age

1. Introduction

The need for a mathematical model of age in mortality law constitutes the most proven area in actuarial literature. The quest to appraise the experienced mortality risk by life offices has grown over the years, although pricing, the regulatory environment and solvency requirements are the core concerns for constructing mortality tables which mirrors the experience of the policy holders so as to permit life offices quantify the underlying mortality risk for premium computation purposes. Apparently, Life insurance premium computations depend on life table. The mortality tables are usually constructed on the empirical observations in big human population structure which seems to have reliable mortality data. The problem of which instantaneous mortality law to consider for pricing and risk analysis is of paramount importance

in mortality literature. In order to deal with this issue, we need to address parsimony considered here using Gompertz for ease of computations and then use mortality data to build life table functions. A number of parametric techniques have been used to construct mortality table in actuarial architecture and which models are important to life offices especially where there is mortality data. These techniques would then fit the mortality data to the underlying parametric structure of interest and consequently the graduated mortality can be applied to estimate potential liabilities associated with the population of the assured.

In Pham[23], it was reported that because human lifespan is increasing, actuarial scientists have been keen to develop actuarial models for analyzing mortality rates. Cheung, Robine, Tu, & Caselli [3], Kannisto [12], Kannisto[13] and Robine [24] observe that researches in modal age at death give an opportunity to have a diverse overview of the changes in the distribution of deaths so as to describe the changes in mortality at old ages . In Cheung, Robine, Tu, & Caselli [4], Canudas-Romo [6], Canudas-Romo & Schoen [7], Canudas-Romo & Wilmoth [8] ,and Cheung & Robine [9] mortality models express the age schedule of mortality in a given year but vary in the number of parameters used and in the age band for which they model mortality. The more parameters they use, the more flexible they can fit mortality at different ages but the more difficult they are to analyze mathematically (Cohen, [5]).

Life table represents a brief summary tool for appraising and comparing prevailing mortality conditions in a group. Its construction requires reliable data on a population's mortality rates, by age and sex. The vital registration system where all deaths and births are registered remains the most dependable source of such data collection. The resulting age-sex-specific death rates are then used to compute a life table. The variations in the two types of life table; Cohort and Periodic life tables are based on the nature of data used for construction Van der Mulen, [26]. The radix of the hypothetical cohort, l_0 is arbitrary in deterministic life tables and is usually designated, 1000 or 100,000 as reported in (Li & Tuljapurkar [7] and Neil, [18]).

Following Stoeldraijer, Duin, Wissen, & Janssen [25], mortality forecasts are based on these tables, aside from the life underwriting information and the existing analytic mortality models. Bowers, Newton Jr, Gerba, Hickman, Jones & Nesbitt [2] proposed analytical framework for examining determinants of mortality for the following reasons: philosophical, practical and ease of estimation. The philosophical justification assumes that, since many physical phenomena are described sufficiently by simple function. Consequently, using biological arguments, human survival can be governed by an equally simple law, one that is convenient to estimate and interpret for practical purposes. For practical reasons, using the probability tool of model parsimony makes it relatively easier to communicate a function with few parameters than to communicate a life table with too many parameters or mortality probabilities. Lastly, a simple analytic survival function will have fewer parameters to be estimated from the mortality data.

Actuaries provide actuarial technique for mortality analysis which can be used in valuation, compute premium and construct life table so as to give a better overview of the underlying phenomena governing the mortality pattern. In Booth and Tickle [1], mortality modeling is classified into three core domains, which are expectation, explanation and extrapolative approaches. The expectation approach is subjective and conservative. Explanation approach is restricted to certain causes of death with known determinants and the extrapolative approach use probability method of estimation which focus on age patterns and trends over time. In most cases of mortality forecasting, some aspects of each approach is applied, hence exact distinction between the three approaches is not clearly spelt out.

In Keilman [15], it was reported that actuarial models are typically aggregate models which describe how the mortality of groups of individuals evolves over time. It particularly uses past data to estimate changes over time usually so as to break down mortality risks by demographic variables such as age and sex. Non-demographic or causal models emphasize causal factors in mortality such as health, socio-economic status, environmental conditions and access to health care, (Hudson, [11]).

2. Preliminaries Measures of Mortality

In Ogungbenle & Adeyele [20], Ogungbenle & Ogungbenle [21] mortality is measured to draw inferences about the probability of death occurring within a defined group over definite period of time. Thus, for the measure to be defined proportionally as a rate of mortality, the number of deaths occurring per unit population over a defined time interval must be specified. The unit of population is usually specified in the order 100,000; 100,000 or 1,000,000 for computational convenience. Furthermore, it is also necessary to specify the kind of deaths and kind of population such as the general population, population of a particular sex or age since the risk of death varies with gender, age and other determinants which affect the physical environment such as place of birth, occupation, place of residence, marital status of the insured. In computing the rate of mortality, we need to distinguish the effect of these determinants as well as to differentiate the contribution of diverse decrement causes of death either through disease or injury. Mortality rate could be categorized as general or specific, the first relating to the cause of death and to the general population, while the latter relates to special causes of death or to deaths in a specific part of the population. Where general or specific death rate is being considered, we need to ensure that the population considered in the computation is exactly the one which produces the death data used in the computation and conversely the deaths should consist of those occurring in the population. The denominator of the rate of which the numerator is the relevant number of deaths is the population at risk. It is apparent that the rate of mortality is a measure of population expressing the relative frequency of an occurrence of an event such as death or of characteristics. In order to obtain an accurate measure, it is necessary to define what type of population and which period of time the rate is applicable. In other to examine a defined rate which avoids lack of specifics about sex and age and about the population from which the deaths are obtained, it is important to check the mortality table function q_x which define the proportion of life exactly aged x who die before attaining age $x + 1$.

In Ogungbenle [19], mortality table is computed differently for both male and female because of the differences in mortality of both male and female.

$\mu(x)$: In actuarial mathematics, the force of mortality describes the instantaneous rate of mortality at a certain age measured on an annual basis.

l_0 : Entry population in a study known as radix.

l_x = Expected number of survivors attaining age x ;

l_{x+1} = Expected number of survivors attaining age $x + 1$

$d_x = l_x - l_{x+1}$ = The number of lives dying in the interval between age x and $x + 1$

$T_x = \int_x^{\infty} l_{x+s} ds$ = The total number of years survived beyond age x by the survivorship group

$$L_x = \int_0^1 l_{x+s} ds, L_x = \int_x^{\infty} l_y dy - \int_{x+1}^{\infty} l_y dy \Rightarrow L_x = T_x - T_{x+1}$$

$e_x = E(T(x))$ is the expected future life time E measuring expected time remaining until death.

$$e_x = \int_0^{\infty} s {}_s p_x \mu_{x+s} dx \Rightarrow e_x = \int_0^{\infty} s \left(-\frac{d}{dx} {}_s p_x \right) ds \tag{1}$$

$$e_x = -[s \times {}_s p_x]_0^{\omega-x} \times \int_0^{\infty} {}_s p_x ds \tag{2}$$

$$e_x = -[s \times {}_s p_x]_0^{\omega-x} \times \int_0^{\infty} e^{-\int_0^s \mu_{x+u} du} ds = \sum_{k=1}^{\infty} k p_x + \frac{1}{2} \tag{3}$$

The curtate future lifetime $K(x) = [T(x)]$ of a life aged exactly x is the whole number of years lived after age x . The curtate future life time of a life age x representing integral part of $T(x)$ is

$$K(x) = [T(x)] = \sum_{k=1}^{\infty} k p_x \tag{4}$$

$$\mu(x) = \lim_{\delta x \rightarrow 0} \left(\frac{\Pr\left(x \leq X < x + \frac{\delta x}{X} \geq x\right)}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left(\frac{\Pr(x \leq X < x + \delta x)}{\delta x \Pr(X \geq x)} \right) \tag{4a}$$

$$\mu(x) = \lim_{\delta x \rightarrow 0} \left(\frac{F_X(x + \delta x) - F(x)}{\delta x \Pr(X \geq x)} \right) = \lim_{\delta x \rightarrow 0} \left(\frac{F_X(x + \delta x) - F_X(x)}{\delta x S_X(x)} \right) = \frac{d}{dx} \frac{F_X(x)}{S_X(x)} = \frac{f_X(x)}{S_X(x)} \tag{4b}$$

3. Material and Methodology

The exponential functional relationship between mortality rate and age is usually set as the standard in actuarial literature. Consequently, delineating the causes of death within the age ranges which both leads to mortality rate and in particular complies with Gompertz structure could lend support to the method of numerical technique adopted here. Any functional departure from this established functional relationship shows that in a sequence of ages, the population of the assured may experience an unexpected mortality risk and hence it is necessary to use numerical technique in order to estimate the model parameters. There are numerous methods which estimates life table functions but they are too complex to apply in estimating the parameters which are not parsimonious.

A preferred approach is needed if we use this technique for certain parsimonious mortality functions like Gompertz’s law which makes it possible to compute the parameters numerically. The reason why Gompertz’s law is justified in mortality is that it is parsimonious because within a known group of mortality laws, mortality models in relation to parametric parsimony are most preferred to the sophisticated models hence parsimony refers to the cardinality of efficient model parameters. The Gompertz’s law offers a good trade-off between a simple analytic procedures and efficient numerical algorithms. Gompertz’s is intuitive, parsimonious and consequently removes the effect of the risk of cumbersome numerical computations. A prime observation in respect of the behavior of the force of mortality is to assume positive rates that increase with age from age 0 to omega age for all future time such that the survival probabilities ${}_s p_x$ are well defined for all ages, however, the stochastic short rate trajectories seem to evince negative rates with positive death probabilities.

Mortality functions which have been built recently possess sophisticated actuarial techniques with many parameters and consequently they are very complex to estimate numerically making it difficult to fit to mortality data. In order to overcome this problem, we need to adopt novel numerical algebraic technique to estimate the appropriate values of model parameters which could permit us fit the underlying model to mortality data. There is no continuous registration system where we can source data locally, consequently, the mortality data used was sourced from the mortality of the population of England and wales during the

years 1990, 1991 and 1992 because the data is believed to have been validated and hence would be more reliable. The data obtained from English life table 15 male and female is tested to confirm the interval of validity where the estimates would fall.

The Gompertz force of mortality

$$\mu(x) = ae^{bx}, l_x = \left(\frac{a}{b} - \frac{a}{b} e^{bx} \right) \quad (5)$$

$$\mu(x) = ah^x, h = e^b \quad (6)$$

where a is the level of mortality at age $x = 0$, while b describes the rate of mortality increase across x and $e = 2.71$. If x is the current age of (x) and α is the starting age of mortality analysis, then

$$x \rightarrow x - \alpha, \mu(x) = ae^{b(x-\alpha)} \quad (7)$$

Since, $l_x = \exp(-\int \mu(x)dx)$

$$\text{Therefore } l_x = \kappa g^{\zeta^x} \quad (8)$$

The exponential nature of Gompertz mortality law fully describes a log-link in general linear modelling

Where κ, g and ζ are parameters; $\kappa = e^{c_1}$, c_1 is an integrating constant, $g = e^{-\frac{a}{b}}$ and $\zeta = \exp b$

Given that $l_{20} = 98496, l_{30} = 97645, l_{40} = 96500$ from English life table 15 male, the male mortality data yielded the following parameter estimates in our model as follows.

$$\text{Let } y = \log_e l_x \quad (9)$$

$$\text{Therefore } y = \log_e \kappa + \zeta^x \log_e g \quad (10)$$

$$y_1 = A + B\zeta^{x_1}, y_2 = A + B\zeta^{x_2} \text{ and } y_3 = A + B\zeta^{x_3}$$

$$y_2 - y_1 = A + B\zeta^{x_2} - A - B\zeta^{x_1} \quad (11)$$

$$y_2 - y_1 = B(\zeta^{x_2} - \zeta^{x_1}) \quad (12)$$

$$y_3 - y_2 = B(\zeta^{x_3} - \zeta^{x_2}) \quad (13)$$

$$\frac{y_3 - y_2}{y_2 - y_1} = \frac{B(\zeta^{x_3} - \zeta^{x_2})}{B(\zeta^{x_2} - \zeta^{x_1})} = \zeta^m \quad (14)$$

$$\zeta = \left(\frac{y_3 - y_2}{y_2 - y_1} \right)^{\frac{1}{m}}, \text{ and } g = \exp \left[\frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1} \left(\frac{y_2 - y_1}{y_3 - y_2} \right)^{\frac{x_1}{m}} \right] \quad (15)$$

$$\kappa = \exp \left[y_1 - \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1} \right] \quad (16)$$

$$x_1 = 20, x_2 = 30, x_3 = 40 \quad (17)$$

$$y_1 = \log_e l_{x_1} = \log_e 98496 \quad (18)$$

$$y_2 = \log_e l_{x_2} = \log_e 97645 \quad (19)$$

$$y_3 = \log_e l_{x_3} = \log_e 96500 \quad (20)$$

$$m = x_2 - x_1 = x_3 - x_2 = 30 - 20 = 40 - 30 = 10 \quad (21)$$

$$\zeta = \left(\frac{\log 96500 - \log 97645}{\log 97645 - \log 98496} \right)^{\frac{1}{10}} \quad (22)$$

$$\zeta = \left(\frac{11.47729829 - 11.48909373}{11.48909373 - 11.49777122} \right)^{\frac{1}{10}} \quad (23)$$

$$\zeta = \left(\frac{-0.01179544}{-0.00867749} \right)^{0.1} = (1.359314733)^{0.1} \quad (24)$$

$$\zeta = 1.031174114 \quad (25)$$

$$g = \exp \left[\frac{(11.48909373 - 11.49777122)^2}{11.47729829 - 2(11.48909373) + 11.49777122} \left(\frac{11.48909373 - 11.49777122}{11.47729829 - 11.48909373} \right)^{\frac{20}{10}} \right] \quad (26)$$

$$g = \exp \left[\frac{(-0.00867749)^2}{11.47729829 - 22.97818746 + 11.49777122} \left(\frac{11.48909373 - 11.49777122}{11.47729829 - 11.48909373} \right)^2 \right] \quad (27)$$

$$g = \exp \left[\frac{0.0000752988327}{-0.00311795} (0.735664799)^2 \right] \quad (28)$$

$$g = \exp(-0.024150109 \times 0.541202696) \quad (29)$$

$$g = \exp(-0.013070104) = 0.987014938 \quad (30)$$

$$\kappa = \exp \left[11.49777122 - \frac{(11.48909373 - 11.49777122)^2}{11.47729829 - 2(11.48909373) + 11.49777122} \right] \quad (31)$$

$$\kappa = \exp \left[11.49777122 - \frac{0.0000752988327}{-0.00311795} \right] \quad (32)$$

$$\kappa = \exp(11.49777122 + 0.024150109) \quad (33)$$

$$\kappa = \exp(11.52192133) = 100903.6449 \quad (34)$$

$$\mu(x) = ae^{bx} \quad (35)$$

$$\zeta = e^b, \text{ therefore } \log_e \zeta = b \quad (36)$$

$$\log_e \zeta = 0.030698069, \text{ hence, } b = 0.030698069 \quad (37)$$

$$g = e^{-\frac{a}{b}}, \log_e g = -\frac{a}{b} \Rightarrow a = -b \log_e g \quad (38)$$

$$\log_e g = -0.013070105 \quad (39)$$

$$a = -0.030698069 \times -0.013070105, \text{ and, } a = 0.000401227 \quad (40)$$

$$\text{Therefore } \mu_x = 0.000401227 e^{0.030698069x} \quad (41)$$

$$l_x = 100903.6449 \times 0.987014938^{1.031174114x} \quad (42)$$

It was observed also that other functions such as l_x of the life table decreases with age except for the force of mortality $\mu(x)$ which increases with age.

From English life table 15 Female

$$l_{20} = 98957, l_{30} = 98617, l_{40} = 97952 \tag{43}$$

$$\zeta = \left(\frac{11.49223284 - 11.49899894}{11.49899894 - 11.50244069} \right)^{0.1} \tag{44}$$

$$\zeta = \left(\frac{-0.0067661}{-0.00344175} \right)^{0.1} = (1.965889446)^{0.1} = 1.069931341 \tag{45}$$

$$g = \exp \left[\frac{(11.49899894 - 11.50244069)^2}{11.49223284 - 2(11.49899894) + 11.50244069} \left(\frac{11.49899894 - 11.50244069}{11.49223284 - 11.49899894} \right)^{\frac{20}{10}} \right] \tag{46}$$

$$g = \exp \left[\frac{(-0.00344175)^2}{11.49223284 - 22.99799788 + 11.50244069} \left(\frac{-0.00344175}{-0.0067661} \right)^2 \right] \tag{47}$$

$$g = \exp \left[\frac{0.00001184564306}{-0.00332435} (0.508675603)^2 \right] \tag{48}$$

$$g = \exp[-0.003563296001 \times 0.258750869] \tag{49}$$

$$g = \exp(-0.0009220059368) = 0.999078419 \tag{50}$$

$$\kappa = \exp \left[11.50244069 - \frac{(11.49899894 - 11.50244069)^2}{11.49223284 - 2(11.49899894) + 11.50244069} \right] \tag{51}$$

$$\kappa = \exp \left[11.50244069 - \frac{(-0.00344175)^2}{-0.00332435} \right] \tag{52}$$

$$\kappa = \exp \left(11.50244069 - \frac{0.00001184564306}{-0.00332435} \right) \tag{53}$$

$$\kappa = \exp(11.50244069 + 0.003563296001) \tag{54}$$

$$\kappa = 99310.24193 \tag{55}$$

Therefore

$$\mu(x) = ae^{bx}, \zeta = e^b \Rightarrow \log_e \zeta = b \tag{56}$$

$$b = \log_e 1.069931341 \Rightarrow b = 0.067594479 \tag{57}$$

$$g = e^{-\frac{a}{b}}, \log_e g = -\frac{a}{b} \text{ and } -b \log_e g = a \tag{58}$$

$$a = -0.067594479 \log 0.999078419 = -0.067594479 \times -0.0009220059368 \tag{59}$$

$$a = 0.00006232251093 \tag{60}$$

$$\mu(x) = (0.00006232251093) e^{0.067594479x} \tag{61}$$

$$l_x = 99310.24193 (0.999078419)^{1.069931341x} \tag{62}$$

Koppelaar [16] defined the Gompertz distribution in terms of Lambert W function as

$$G(x; \sigma, \mu) = W \left(\frac{x - a}{b} \right)$$

with mean μ and standard deviation σ where

$$W\left(\frac{x-a}{b}\right) = 1 - \exp\left(e^{-\left(\frac{x-a}{b}\right)}\right) \tag{63}$$

with probability density function

$$w(x) = \exp\left[\left(\frac{x-a}{b}\right) - e^{-\left(\frac{x-a}{b}\right)}\right] \tag{64}$$

$$\Rightarrow \left[\left(\frac{x-a}{b}\right) - e^{-\left(\frac{x-a}{b}\right)}\right] = \log_e(w(x))$$

$$\text{When } a = 0, b = 0, \text{ then } w(x) = \exp[x - e^{-x}] \tag{65}$$

$$\text{If } \gamma = \left[\Gamma(1) + \gamma\Gamma(1) + \Gamma(1) \sum_{n \geq 1} \left(\frac{1}{n+1} - \frac{1}{n}\right)\right] = \lim_{n \rightarrow \infty} \left(\sum_{1 \leq k \leq n} \frac{1}{k} - \log_e n\right) \cong 0.57722 \tag{66}$$

is the Euler-Mascheront constant, and if X is the random life time of (x) , then using the first two cumulants κ_1, κ_2 , we have

$$b = \frac{\sqrt{6\kappa_2}}{\pi}, a = \kappa_1 + 0.57722b, \kappa_1 = \mu; \kappa_2 = \sigma^2; \pi = \frac{355}{113} \tag{67}$$

From above definitions,

$$W\left(\frac{x-a}{b}\right) = 1 - g^{C^x}, \text{ when } g = \exp\left(-e^{\frac{a}{b}}\right) \text{ and } C = \exp\left(\frac{1}{b}\right) \tag{68}$$

$$l_x = l_0 g^{C^x}, -\infty < x < \infty; 0 < g < 1; 1 < c$$

$$\log_e \frac{1}{g} = \left[e^{\frac{a}{b}}\right]; \log_e c = \frac{1}{b} \tag{69}$$

Consequences of Koppelaar [16] Constructions on Random Life Time

3.1 Theorem 1

Let X be the random life time with pdf $w(y)$. If we define

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{ty} w(y) dy \tag{70}$$

then

$$(i) E(1) = \int_{-\infty}^{\infty} w(y) dy = 1 \tag{71}$$

$$(ii) \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} (\log_e x) \exp\left[-\frac{1}{x}\right] dx \tag{72}$$

$$(iii) \int_0^{\infty} x^2 f(x) dx = \int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy \tag{73}$$

Proof

$$E(e^{0xt}) = \int_{-\infty}^{\infty} e^{0xt} w(y) dy = \int_{-\infty}^{\infty} w(y) dy = 1$$

since $w(y)$ is the probability density function

$$\frac{d}{dt} E(e^{tX}) = E\left(\frac{d}{dt} e^{tX}\right) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{ty} w(y) dy = \int_{-\infty}^{\infty} \frac{d}{dt} e^{ty} w(y) dy = \int_{-\infty}^{\infty} y e^{ty} w(y) dy \quad (74)$$

$$\frac{d}{dt} E(e^{tX}) = E(xe^{tX}) = \int_{-\infty}^{\infty} y e^{ty} w(y) dy \quad (75)$$

$$\frac{d}{dt} E(e^{tX})|_{t=0} = E(x) = \int_{-\infty}^{\infty} y w(y) dy \quad (76)$$

$$\int_0^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} y w(y) dy \quad (77)$$

$$\frac{d^2}{dt^2} E(e^{tX}) = E\left(\frac{d^2}{dt^2} e^{tX}\right) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{ty} w(y) dy = \int_{-\infty}^{\infty} \frac{d^2}{dt^2} e^{ty} w(y) dy = \int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy \quad (78)$$

$$E(x^2 e^{tX}) = \int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy \Rightarrow E(x^2)|_{t=0} = \int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy \quad (79)$$

$$\int_0^{\infty} x^2 f(x) dx = \int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy \quad (80)$$

$$\text{Standard deviation } \sigma_x = \left[\int_{-\infty}^{\infty} y^2 e^{ty} w(y) dy - \left(\int_{-\infty}^{\infty} y w(y) dy \right)^2 \right]^{\frac{1}{2}} \quad (81)$$

The Euler-Mascheront Constant

$$\Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u} du \quad (81a)$$

$$\frac{d}{dz} \Gamma(z) = \int_0^{\infty} u^{z-1} e^{-u} \log_e u du \quad (81b)$$

$$\frac{\Gamma'(z)}{\Gamma(z)} = \psi(z) \quad (81c)$$

$$\frac{d^{m+1} \log_e \Gamma(z)}{dz^{m+1}} = \psi^{(m)}(z) \quad (81d)$$

$$\frac{d^1 \log_e \Gamma(z)}{dz^1} = \psi^{(0)}(z) \quad (81e)$$

When $m = 0; z = 1$

$$\gamma = -\psi(1) = \Gamma'(1) = -\int_0^{\infty} e^{-u} \log_e u du \quad (81f)$$

$$\int_0^1 \left(\frac{1 - e^{-t} - e^{-\frac{1}{t}}}{t} \right) dt = \gamma = -\psi(1) \quad (81g)$$

$\gamma \cong 0.5772156649015328606065$ is the Euler-Mascheront constant

$$\psi(z) = \frac{d \log_e \Gamma(z)}{dz} \quad (81h)$$

is the digamma function.

$$Ei(x) = \int_{-\infty}^x \frac{e^u}{u} du; x \in R^+ \quad (81i)$$

$$Ei(x) = \gamma + \log_e x + \sum_{j=1}^{\infty} \frac{x^j}{j \times j!} \quad (81j)$$

3.2 Theorem 2

Suppose the link function is $y = \log_e x$ in Theorem 1, then

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} (\log_e x) \exp\left[-\frac{1}{x}\right] dx \quad (82)$$

Observe that $dy = \frac{dx}{x}$ (82a)

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{1}{x} (\log_e x) w(\log_e x) dx \quad (83)$$

$$w(\log_e x) = \exp\left[\log_e x - e^{-\log_e x}\right] \quad (84)$$

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{1}{x} (\log_e x) \exp\left[\log_e x - e^{-\log_e x}\right] dx \quad (85)$$

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{1}{x} (\log_e x) \exp\left[\log_e x - \frac{1}{x}\right] dx \quad (86)$$

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{1}{x} (\log_e x) \exp[\log_e x] \exp\left[-\frac{1}{x}\right] dx \quad (87)$$

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \frac{1}{x} (\log_e x) x \exp\left[-\frac{1}{x}\right] dx \quad (88)$$

$$\int_0^{\infty} x f_X(x) dx = \int_0^{\infty} (\log_e x) \exp\left[-\frac{1}{x}\right] dx \quad (89)$$

Suppose in RHS of equation (89), $x \rightarrow \frac{1}{x}$, in the integrand, then

$$\int_0^{\infty} \left(\log_e \frac{1}{x} \right) \exp[-x] dx = \int_0^{\infty} (\log_e x^{-1}) \exp[-x] dx = \int_0^{\infty} -(\log_e x) \exp[-x] dx = -\int_0^1 \log_e |\log_e x| dx = \gamma \quad (90)$$

3.3 The Superimposition Principle For The Gompertz Force of Mortality

3.4 Theorem 3

The principle asserts that whenever m different mortality cohorts of same age group are under the intensity of Gompertzian mortality, then the resulting aggregate probability of survival is independent of the initial mortality. That is

$$\mu(x) = BC^x + AD^x + GE^x \text{ then } {}_t p_x = h_1^{(C^{x+t}-C^x)} h_2^{(D^{x+t}-D^x)} h_3^{(E^{x+t}-E^x)} \quad (91)$$

Proof

We take three distinct forces of mortalities

$$\mu(x) = BC^x + AD^x + GE^x \quad (92)$$

$$\Rightarrow \int_0^x \mu_t dt = \int_0^x BC^t + AD^t + GE^t dt = \int_0^x BC^t dt + \int_0^x AD^t dt + \int_0^x GE^t dt$$

$$\int_0^x \mu_t dt = \left[\frac{BC^t}{\log_e C} \right]_0^x + \left[\frac{AD^t}{\log_e D} \right]_0^x + \left[\frac{GE^t}{\log_e E} \right]_0^x \quad (93)$$

$$\int_0^x \mu_t dt = \frac{BC^x}{\log_e C} - \frac{B}{\log_e C} + \frac{AD^x}{\log_e D} - \frac{A}{\log_e D} + \frac{GE^x}{\log_e E} - \frac{G}{\log_e E} \quad (94)$$

$$\begin{aligned} \int_0^x \mu_t dt &= \left(\frac{BC^x}{\log_e C} - \frac{B}{\log_e C} \right) + \left(\frac{AD^x}{\log_e D} - \frac{A}{\log_e D} \right) + \left(\frac{GE^x}{\log_e E} - \frac{G}{\log_e E} \right) \\ &= \frac{B}{\log_e C} (C^x - 1) + \frac{A}{\log_e D} (D^x - 1) + \frac{G}{\log_e E} (E^x - 1) \end{aligned} \quad (95)$$

We define

$$\log_e h_1 = \frac{-B}{\log_e C}, \log_e h_2 = \frac{-A}{\log_e D} \text{ and } \log_e h_3 = \frac{-G}{\log_e E} \quad (96)$$

$$\begin{aligned} \int_0^x \mu_t dt &= -\log_e h_1 (C^x - 1) - \log_e h_2 (D^x - 1) - \log_e h_3 (E^x - 1) \\ &= -\left[(C^x - 1) \log_e h_1 + (D^x - 1) \log_e h_2 + (E^x - 1) \log_e h_3 \right] \end{aligned} \quad (97)$$

$$\int_0^x \mu_t dt = -\left[\log_e h_1^{(C^x-1)} + \log_e h_2^{(D^x-1)} + \log_e h_3^{(E^x-1)} \right] \quad (98)$$

$$\int_0^x \mu_t dt = -\left[\log_e h_1^{(C^x-1)} h_2^{(D^x-1)} h_3^{(E^x-1)} \right] \quad (99)$$

$$l_x = l_0 e^{-\int_0^x \mu_t dt} = l_0 e^{\log_e h_1^{(C^x-1)} h_2^{(D^x-1)} h_3^{(E^x-1)}} = l_0 h_1^{(C^x-1)} h_2^{(D^x-1)} h_3^{(E^x-1)} \quad (100)$$

$$l_x = \frac{l_0}{g} g^{c^x} = k g^{c^x}, k = \frac{l_0}{g} \quad (101)$$

$$e^{\int_0^t \mu_{x+s} ds} = \frac{l_0 h_1^{(C^{x+t}-1)} h_2^{(D^{x+t}-1)} h_3^{(E^{x+t}-1)}}{l_0 h_1^{(C^x-1)} h_2^{(D^x-1)} h_3^{(E^x-1)}} = \frac{l_0 h_1^{(C^{-1})} h_1^{(C^{x+t})} h_2^{(D^{-1})} h_2^{(D^{x+t})} h_3^{(E^{-1})} h_3^{(E^{x+t})}}{h_1^{(C^{-1})} l_0 h_1^{(C^x)} h_2^{(D^{-1})} h_2^{(D^x)} h_3^{(E^{-1})} h_3^{(E^x)}} = {}_t P_x$$

$$\frac{h_1^{(C^{x+t})} h_2^{(D^{x+t})} h_3^{(E^{x+t})}}{h_1^{(C^x)} h_2^{(D^x)} h_3^{(E^x)}} = h_1^{(C^{x+t}-C^x)} h_2^{(D^{x+t}-D^x)} h_3^{(E^{x+t}-E^x)}$$

Hence, we conclude that this will be true for m cohort groups.

If equation (56) holds, then we have

$$\log_e \left(\log_e \frac{1}{p_x} \right) = \log_e c^x - \log_e \left(\frac{\log_e c}{b(c-1)} \right) = \log_e \left(\frac{b(c-1)c^x}{\log_e c} \right)$$

3.5 Theorem 4

Suppose that there is a uniform distribution of death among the insured in the interval of age $x \in [a, b]$

such that $s = x - a$ and $\omega_T = \inf \{t \geq 0 : S_{T(x)}(t) = 0\}$ is the omega age. Assuming the expected number of the assured surviving to age x is a third degree approximating polynomial given by

$$\int_0^{\omega-x} l_{x+s} \mu_{x+s} ds = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

then,

$$(i) 3\beta_4 x = 3\beta_4 a + \left[-(3a\beta_4 + \beta_3) \pm \sqrt{\beta_3^2 + 3\beta_4\beta_3 b + 3\beta_4^2 b^2 + 3\beta_4^2 ba + 3\beta_4^2 a^2} \right]$$

is the maximum error in $\int_0^{\omega} l_{x+s} \mu_{x+s} ds$ where μ_{x+s} is the force of mortality

(ii) If $S(s)$, the sum assured which the life office pays on death is an increasing function $S'(s) > 0$ and that we discount by the continuous interest rate function $e^{-\delta s}$. Given the boundary conditions $S(0) = 0$ and $S(1) = S$. Then the death benefits is

$$DB(s) = \frac{1}{\delta} \sum_{R=0}^{\omega-x-1} v^{R+1} {}_{R/1} q_x \left(\int_0^1 e^{\delta-\delta t} S'(t) dt - S \right)$$

Proof of (i)

We recognize that $\int_0^{\omega-x} l_{x+s} \mu_{x+s} ds = l_x$ consequently l_x is an approximating polynomial and β_i

$i = 1, 2, 3, 4$ are real constants and $\mu(x)$ is the Gompertz's force of mortality.

$$l_a = \beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3$$

$$l_b = \beta_1 + \beta_2 b + \beta_3 b^2 + \beta_4 b^3$$

$$l_0 = \beta_1$$

The assumption of the first condition of uniform distribution of death implies that

$$\frac{\bar{l}_{a+s} - l_a}{l_b - l_a} \cong \frac{x - a}{b - a}$$

$$\Rightarrow \bar{l}_{a+s} - l_a = \frac{s(l_b - l_a)}{b - a} = \frac{sl_0 \left(\frac{{}_b P_0 - {}_a P_0}{l_0} \right)}{b - a} \tag{109}$$

$$\bar{l}_{a+s} - (\beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3) = \frac{s(\beta_1 + \beta_2 b + \beta_3 b^2 + \beta_4 b^3 - (\beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3))}{b - a} \tag{110}$$

$$\bar{l}_{a+s} - (\beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3) = \frac{s(\beta_1 + \beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_1 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)}{b - a} \tag{111}$$

$$\bar{l}_{a+s} - (\beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3) = \frac{s(\beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)}{b - a} \tag{112}$$

$$\bar{l}_{a+s} = \beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3 + \frac{s(\beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)}{b - a} \tag{113}$$

$$\begin{aligned} l_{a+s} &= \beta_1 + \beta_2(a+s) + \beta_3(a+s)^2 + \beta_4(a+s)^3 \\ &= \beta_1 + a\beta_2 + \beta_2 s + a^2\beta_3 + s^2\beta_3 + 2a\beta_3 s + \beta_4 a^3 + \beta_4 s^3 + 3a^2\beta_4 s + 3a\beta_4 s^2 \end{aligned} \tag{114}$$

$$\begin{aligned} l_{a+s} - \bar{l}_{a+s} &= (\beta_1 + a\beta_2 + \beta_2 s + a^2\beta_3 + s^2\beta_3 + 2a\beta_3 s + \beta_4 a^3 + \beta_4 s^3 + 3a^2\beta_4 s + 3a\beta_4 s^2) \\ &\quad - \left(\beta_1 + \beta_2 a + \beta_3 a^2 + \beta_4 a^3 + \frac{s(\beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)}{b - a} \right) \end{aligned} \tag{115}$$

$$\begin{aligned} (l_{a+s} - \bar{l}_{a+s}) &= (\beta_2 s + a^2\beta_3 + s^2\beta_3 + 2a\beta_3 s + \beta_4 a^3 + \beta_4 s^3 + 3a^2\beta_4 s + 3a\beta_4 s^2) \\ &\quad - \beta_3 a^2 - \beta_4 a^3 - \left[\frac{(\beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)s}{b - a} \right] \end{aligned} \tag{116}$$

Differentiating both sides with respect to s , we obtain the maximum error in l_x

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = \beta_2 + 2s\beta_3 + 2a\beta_3 + 3\beta_4 s^2 + 3a^2\beta_4 + 6a\beta_4 s - \left[\frac{(\beta_2 b + \beta_3 b^2 + \beta_4 b^3 - \beta_2 a - \beta_3 a^2 - \beta_4 a^3)}{b - a} \right] \tag{117}$$

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = \beta_2 + 2\beta_3(s+a) + 3\beta_4 s^2 + 3a^2\beta_4 + 6a\beta_4 s - \left[\frac{\beta_2(b-a) + \beta_3(b^2 - a^2) + \beta_4(b^3 - a^3)}{b - a} \right] \tag{118}$$

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = \beta_2 + 2\beta_3(s+a) + 3\beta_4 s^2 + 3a^2\beta_4 + 6a\beta_4 s \left[\frac{\beta_2(b-a) + \beta_3(b-a)(b+a) + \beta_4(b-a)(b^2 + ba + a^2)}{b - a} \right] \tag{119}$$

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = \beta_2 + 2\beta_3(s+a) + 3\beta_4 s^2 + 3a^2\beta_4 + 6a\beta_4 s - [\beta_2 + \beta_3(b+a) + \beta_4(b^2 + ba + a^2)] \tag{120}$$

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = \beta_2 + 2\beta_3(s+a) + 3\beta_4s^2 + 3a^2\beta_4 + 6a\beta_4s - \beta_2 - \beta_3(b+a) - \beta_4(b^2 + ba + a^2) \quad (121)$$

$$\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = 2\beta_3s + 2\beta_3a + 3\beta_4s^2 + 3a^2\beta_4 + 6a\beta_4s - \beta_3b - \beta_3a - \beta_4b^2 - \beta_4ba - \beta_4a^2 \quad (122)$$

At turning point, $\frac{d(l_{a+s} - \bar{l}_{a+s})}{ds} = 0$, consequently,

$$2\beta_3s + 2\beta_3a + 3\beta_4s^2 + 3a^2\beta_4 + 6a\beta_4s - \beta_3b - \beta_3a - \beta_4b^2 - \beta_4ba - \beta_4a^2 = 0 \quad (123)$$

$$3\beta_4s^2 + (6a\beta_4 + 2\beta_3)s + (2\beta_3a + 3a^2\beta_4 - \beta_3b - \beta_3a - \beta_4b^2 - \beta_4ba - \beta_4a^2) = 0 \quad (124)$$

$$s = \frac{-(6a\beta_4 + 2\beta_3) \pm \sqrt{(6a\beta_4 + 2\beta_3)^2 - 12\beta_4 \left(2\beta_3a + 3a^2\beta_4 - \beta_3b - \beta_3a - \beta_4b^2 - \beta_4ba - \beta_4a^2 \right)}}{6\beta_4} \quad (125)$$

$$s = \frac{-(6a\beta_4 + 2\beta_3) \pm \sqrt{4\beta_3^2 + 12a\beta_3\beta_4 - 12\beta_4\beta_3a + 12\beta_4\beta_3b + 12\beta_4^2b^2 + 12\beta_4^2ba + 12\beta_4^2a^2}}{6\beta_4} \quad (126)$$

$$s = \frac{-(3a\beta_4 + \beta_3) \pm \sqrt{\beta_3^2 + 3a\beta_3\beta_4 - 3\beta_4\beta_3a + 3\beta_4\beta_3b + 3\beta_4^2b^2 + 3\beta_4^2ba + 3\beta_4^2a^2}}{3\beta_4} \quad (127)$$

$$s = \frac{-(3a\beta_4 + \beta_3) \pm \sqrt{\beta_3^2 + 3\beta_4\beta_3b + 3\beta_4^2b^2 + 3\beta_4^2ba + 3\beta_4^2a^2}}{3\beta_4} \quad (128)$$

provided $\beta_3^2 + 3\beta_4\beta_3b + 3\beta_4^2b^2 + 3\beta_4^2ba + 3\beta_4^2a^2 > 0$ (129)

$$x - a = \frac{-(3a\beta_4 + \beta_3) \pm \sqrt{\beta_3^2 + 3\beta_4\beta_3b + 3\beta_4^2b^2 + 3\beta_4^2ba + 3\beta_4^2a^2}}{3\beta_4} \quad (130)$$

$$x = a + \frac{-(3a\beta_4 + \beta_3) \pm \sqrt{\beta_3^2 + 3\beta_4\beta_3b + 3\beta_4^2b^2 + 3\beta_4^2ba + 3\beta_4^2a^2}}{3\beta_4} \quad (131)$$

By the assumptions of the theorem, this is the maximum error in $\int_0^{\omega-x} l_{x+s} \mu(x+s) ds$

Proof of (ii)

The probability that the life aged x dies in the small time interval $(x+s; x+s+\delta s)$ is approximately given by ${}_sP_x \mu(x+s) \delta s$ as $\delta s \rightarrow 0$. So if $S(s)$, the sum assured which the life office pays on death is

an increasing function $S'(s) > 0$ and discounting by the continuous interest function $e^{-\delta s}$, then the present value of death benefits as a function of s is

$$DB(s) = \int_0^{\omega-x} e^{-\delta s} S(s) {}_sP_x \mu(x+s) ds \tag{132}$$

By reason of equation (153) below, $E(I_\beta) = {}_{R/1}q_x$ (132a)

$$DB(s) = \sum_{R=0}^{\omega-x-1} \left(\int_R^{R+1} e^{-\delta s} S(s) ({}_sP_x \mu(x+s)) ds \right) \tag{132b}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} {}_R P_x v^{R+1} \left(\int_0^1 e^{\delta} e^{-\delta t} S(t) ({}_t P_{x+R} \mu(x+t+R)) dt \right) \tag{133}$$

The assumption of the second condition of uniform distribution of death implies that

$${}_t P_{x+R} \mu(x+t+R) = q_{x+R} \tag{133a}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} v^{R+1} {}_R P_x q_{x+R} \left(\int_0^1 e^{\delta} e^{-\delta t} S(t) dt \right) \tag{134}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} v^{R+1} \Pr(K_x = R) \left(\int_0^1 e^{\delta} e^{-\delta t} S(t) dt \right) \tag{135}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} v^{R+1} {}_{R/1}q_x \left(\int_0^1 e^{\delta} e^{-\delta t} S(t) dt \right) \tag{136}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} v^{R+1} E(I_\beta) \left(\left[\frac{e^{\delta-\delta t}}{-\delta} S(t) \right]_0^1 + \frac{1}{\delta} \int_0^1 e^{\delta-\delta t} S'(t) dt \right) \tag{137}$$

$$DB(s) = \sum_{R=0}^{\omega-x-1} v^{R+1} E(I_\beta) \left(\frac{e^\delta}{\delta} S(0) - \frac{1}{\delta} S(1) + \frac{1}{\delta} \int_0^1 e^{\delta-\delta t} S'(t) dt \right) \tag{138}$$

By the initial conditions,

$$DB(s) = \left(\sum_{R=0}^{\omega-x-1} v^{R+1} \right) \times E(I_\beta) \left(\frac{1}{\delta} \int_0^1 e^{\delta-\delta t} S'(t) dt - \frac{1}{\delta} S \right) \tag{139}$$

4. Discussion of Results

Adopting the parametrization technique, the model can be changed to a more informative mathematical form as

$$\mu(x) = \frac{1}{\sigma} \exp \left\{ \frac{x-m}{\sigma} \right\} \tag{139a}$$

where $b = \frac{1}{\sigma} \exp \left(-\frac{m}{\sigma} \right)$, $c = \exp \left\{ \frac{1}{\sigma} \right\}$ with modal age $m > \sigma$.

However, the more general modal age at death

$$m_{x+t} = x + \frac{d_{x+t} - d_{x-1+t}}{2d_{x+t} - d_{x-1+t} - d_{x+1+t}} \tag{139b}$$

where t is the time, x is the age with the highest number of death in the life table.

From the Gompertz model, the theoretical modal age at which most deaths occur can be analytically obtained as follows

$$m = \alpha = \frac{1}{b} \log_e \frac{b}{a} = \log_e \left[\frac{b}{a} \right]^{\frac{1}{b}}$$

by differentiating the curve of death

If $\mu(x) = ae^{bx}$, then at modal age at death α is given as

$$\frac{d\mu(x)l_x}{dx} = 0 \Rightarrow \frac{l_x d\mu(x)}{dx} + \mu(x) \frac{dl_x}{dx} = 0 \quad (140)$$

$$\frac{l_x d\mu(x)}{dx} + \mu(x) \frac{dl_x}{dx} = 0 \Rightarrow \frac{1}{\mu(x)} \frac{d\mu(x)}{dx} + \frac{1}{l_x} \frac{dl_x}{dx} = 0 \quad (141)$$

$$\frac{1}{\mu(x)} \frac{d\mu(x)}{dx} = -\frac{1}{l_x} \frac{dl_x}{dx} = \mu(x) \quad (142)$$

$$\frac{1}{ae^{bx}} \frac{dae^{bx}}{dx} = ae^{bx} \quad (143)$$

$$\frac{1}{ae^{bx}} abe^{bx} = ae^{bx} \Rightarrow b = ae^{bx} \quad (144)$$

$$x = \frac{1}{b} \log_e \frac{b}{a} = \log_e \left[\frac{b}{a} \right]^{\frac{1}{b}} \quad (145)$$

$$\text{Again, } e^{bx} = \frac{b}{a} \Rightarrow \frac{1}{e^{bx}} = \frac{a}{b} \Rightarrow a = \frac{b}{e^{bx}} \quad (146)$$

$$\text{when } x = \alpha \text{ modal age } a = \frac{b}{e^{b\alpha}} \quad (147)$$

$$\text{The modal age at death therefore is given by } \alpha = \frac{\log_e b - \log_e a}{b} \quad (148)$$

$$\mu(x) = \frac{b}{e^{b\alpha}} e^{bx} \quad (149)$$

$$\alpha_{Male} = 141.29 \text{ and } \alpha_{Female} = 103.40 \quad (150)$$

The mode of the density function is expected to be vanishingly zero as $m \leq 0$ but the mode will be m as $m > 0$. Although, the male modal age at death is comparatively bigger than the female modal age at death while their respective forces of mortality are completely defined, it is therefore imperative for us to exercise caution here since in practice, it may not signify that male live longer than females. The mean life expectancy of females is usually larger than males definitely and this seems to mean that females age less than the males. Nonetheless, the variation in the level of ageing among males and females sounds actuarially myopic. The mathematical reasoning to demonstrate that female usually age less than male accounts for the fact that male undergoes higher mortality rates at every integral age and hence experience more hazards and this is responsible for their comparatively smaller life expectancy at birth because male seem to be less gender robust and hence experience high back-ground mortality $\mu_b(x)$.

The above argument is part of the reasons why life table is constructed separately for both men and women to distinguish the substantial degree of variation in their level of mortality rates. The results in equation

(150) however, opposes the conjecture of the hypothesis that female age less and by the results in (150), the hypothesis might not be *entirely overwhelming* because men undergo excruciating hardship from multiple decrements at every integral age.

Whenever the insured aged x suffers a decrement such as death, it is assumed that such death must have happened within the anniversary interval period. Since we are dealing with mortality of the assured aged x , we can let α be the event that the live (x) dies in an interval $(R, R + 1)$. Suppose that I_α is an indicator function on event α with appropriate probabilities, then we define

$$I_\alpha = \begin{cases} 1 & \text{with } Pr(\text{assured dies}) \\ 0 & \text{with } Pr(\text{assured does not die}) \end{cases} \quad (151)$$

$$I_\alpha = \begin{cases} 1 & \text{with } {}_{R/1}q_x \\ 0 & (1 - {}_{R/1}q_x) \end{cases} \quad (152)$$

Therefore, our arguments in equations (137)-equations (139) can hence be justified in terms of indicator function. Consequently, the mathematical expectation of the indicator function in equation (152) will result in the death probability as

$$E(I_\alpha) = 1 \times ({}_{R/1}q_x) + 0 \times (1 - {}_{R/1}q_x) = {}_{R/1}q_x = \frac{l_{x+k} - l_{x+k+1}}{l_x} \quad (153)$$

Therefore, using the male values we have the following male probability of death

$$\begin{aligned} E(I_\alpha) &= 1 \times ({}_{R/1}q_x) + 0 \times (1 - {}_{R/1}q_x) = {}_{R/1}q_x \\ &= \frac{99310.24193(0.999078419)^{1.069931341^{x+k}} - 99310.24193(0.999078419)^{1.069931341^{x+k+1}}}{99310.24193(0.999078419)^{1.069931341^x}} \end{aligned} \quad (154)$$

Suppose the α is the event $s < T(x) \leq s + \delta s$ where $T(x)$ is the complete future life time

$$\text{Then, } E(I(s < T(x) \leq s + \delta s)) = Pr(s < T(x) \leq s + \delta s) \quad (155)$$

$$E(s < T(x) \leq s + \delta s) = Pr(T(x) > s) \times Pr(T(x) < s + \delta s | T(x) > s) \quad (156)$$

$$E(s < T(x) \leq s + \delta s) = {}_sP_x \mu(x+s) \delta s = f_{T(x)}(s) \quad (157)$$

$$E(s < T(x) \leq s + \delta s) = \left(\frac{99310.24193(0.999078419)^{1.069931341^{x+s}}}{99310.24193(0.999078419)^{1.069931341^x}} (0.00006232251093) e^{0.067594479(x+s)} \right) \delta s \quad (158)$$

where δs is an infinitesimal time. Therefore suppose a life office pays a sum assured S_1 after m years if (x) dies during that time period and the same life underwriter pays sum assured S_2 after $2m$ years if (x) dies during the second m year period and that no benefit is paid if otherwise, then the present value random variable is given by

$$PV = S_1 I(T(x) < m) e^{-10\delta} + S_2 I(m < T(x) \leq 2m) e^{-20\delta} \quad (159)$$

Consequently, the actuarial present value becomes

$$APV = S_1 {}_m q_x e^{-10\delta} + S_2 ({}_{m/m} q_x e^{-20\delta}) \quad (160)$$

$$APV = e^{-10\delta} S_1 \times \left(\int_0^m P_x \mu_{x+s} ds \right) + e^{-20\delta} S_2 \times \left(\int_m^{2m} P_x \mu_{x+s} ds \right) \quad (161)$$

$$APV = e^{-10\delta} S_1 \times \left(\int_0^m f_{T(x)}(s) ds \right) + e^{-20\delta} S_2 \times \left(\int_m^{2m} f_{T(x)}(s) ds \right) \quad (162)$$

$$f_{T(x)}(s) = \left(\frac{99310.24193(0.999078419)^{1.069931341^{x+s}}}{99310.24193(0.999078419)^{1.069931341^x}} (0.00006232251093) e^{0.067594479(x+s)} \right) \quad (163)$$

defines the male density function for the random lifetime $T(x)$

If $\mu(x) = ae^{bx}$ defines the force of mortality, then the ageing rate is given as

$$K(x) = \frac{d}{dx} \mu(x) = bae^{bx} = b\mu(x) \quad (164)$$

Then the male ageing rate is given as

$$K_{MALE}(x) = (0.030698069)(0.000401227)e^{(0.030698069)x} \quad (165)$$

While the female ageing rate is given as

$$K_{FEMALE}(x) = (0.067594479)(0.00006232251093)e^{(0.067594479)x} \quad (166)$$

5. Conclusion

This paper investigates the numerical behavior of the parsimonious Gompertz function. It emphasizes parametric evaluation of mortality applicable to the development of parameterized mortality projections. The parametric evaluations can ease out comparisons of mortality schemes across demographic locations. This proves useful in setting assumptions for mortality forecasts. Explicit numerical computations from the indirect method for the continuous mortality functions were obtained for the ageing rate. It becomes clear that the parsimonious representation of Gompertz's law elicit optimal actuarial characteristics of the parameters. As a subset of mortality table evaluation of assured lives, the parsimonious parameters have been estimated from Gompertz's function to obtain a model for both the survival function and the force of mortality in respect of male and female gender.

The function $\mu(x)$ is an increasing concave function of age x .

For a small interval $[x, x + \Delta]$, the quantity $\frac{l_{x+\Delta} - l_x}{\Delta}$ is taken as a numerical estimate of $l'_x = -\mu(x)l_x$.

The maximum and minimum points of the curve of death $l_{x+s}\mu(x+s)$ in theorem 4 above will correspond to the points of inflexion on the survival curve. The survival function l_x decreases with increasing age and asymptotically tends to zero at limit of life $\omega_T = \inf \{t \geq 0; P_x = 0\}$.

The method used in this paper is a different technique that is computationally superior to many traditional methods. Although all functions used in estimating mortality analysis are meant for modelling using numerical optimization, however it is very rare to satisfactorily obtain optimally estimated values for all different parameters. The method used here resulted in parameter estimation in its simplest form.

Many techniques which model mortality seem too complex to estimate the parameters since there are numerous parameters to deal with simultaneously. In order to gain a better understanding of the gender differential in assured mortality, we have estimated the model parameters separately for male and female

and show distinctions between male and female. Ageing and gender are serious contributing factors in determining the technique in which life offices price mortality risk so to avoid an adjustment which the insured will have to bear. The accuracy of age-specific mortality is influenced by the accuracy of the age-sex distribution of population and death.

This paper has demonstrated that Gompertz model could be applied efficiently to model mortality for both sexes. The Gompertz model is not limited to the forecast of death probabilities but also the forecast of all life table parameters and provided that the functional structure of the model can be expressed in close form representations, we can always obtain the survival and death probabilities from the force of mortality architecture. Because the forecasts of the parameters by numerical scheme adopted above have a clear indication of actuarial precision. The parsimonious Gompertz easily permits an expanded evaluation of mortality rates for various age to be functionally constructed on few parameters from age zero to the limit of life, consequently, the Gompertz's analytic form though intractable can be employed within a complex mathematical mortality framework. It is very important to observe that when the number of parameters estimated progressively increases, then it is apparent that the actuarial estimation to mortality data improves substantially but the actuarial stability which usually depends on both the cardinality of parameters and the chosen loss functions will definitely be eroded progressively.

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