Nepal Journal of Mathematical Sciences (NJMS)
ISSN: **2738-9928 (*online*), 2738-9812 (*print*)**
Vol. 3, No. 1, 2022 (February): 75-80
DOI: https://doi.org/10.3126/njmathsci.v3i1.44127

Research Article
Received Date: September 25, 2021
Accepted Date: February 17, 2022
Published Date: February 28, 2022

On Certain Series to Series Transformation and Analytic Continuations by Matrix Methods

Suresh Kumar Sahani[*1], Gyan Prasad Paudel[2], Jhavi Lal Ghimire[3] and Amarnath Kumar Thakur[4]

[*1]*Department of Mathematics, MIT campus, Janakpurdham- Nepal*
[2]*Central Department of Mathematics, Mid-Western University, Surkhet, Nepal*
[3]*Central Department of Mathematics, Tribhuvan University, Kritipur, Nepal*
[4]*Central Department of Mathematics, Dr. C.V.Raman University, Kota, Bilaspur (C.G) 495113, India*

Email*:* [*1]*sureshkumarsahani35@gmail.com* , [2]*gyan.math725114@gmail.com* ,
[3]jhavighimire@gmail.com *and* [4]*akthakur@cvru.ac.in*

Abstract: *In this research paper, we proved some general theorems on the absolute convergence transformation of matrix which is expressed in terms of preserving transformation under the very general conditions. This works is motivated by the works of* [3],[12] *and* [14].

Keywords: *Summability, Transformation, Convergence.*

AMS Mathematics Subject Classification: 40A05, 40C05, 40F05.

1. Introduction

Mathematical analysis is primarily concerned with the notion of limit of a sequence of real or complex numbers which forms the basis for the study of infinite series. One important branch of the field of infinite series is the study of summability of divergent sequence (series). This study is an attempt to attach (in some series) a generalized limit to those sequences which do not converge in the usual sense, realizing at the same time that when the generalized limit is applied to a convergent sequence then it must agree with the limit in the ordinary sense. This procedure of assigning a new limit in generalized sense to divergent sequences is called a summability method. A sequence space is a linear space whose elements are sequences chosen from another linear space. The summability theory deals with the study of linear transformations on sequence spaces. In the earliest stage, the idea of summability theory were perhaps contained in a letter written by Leibnitz to C. Wolf (1713). In 1880, Frobenius introduced the method of summability by arithmetic means, which was generalized by Ces`aro (1890) (see [9]) as the (C, k)-method of the summability. These types of summability can also be presented by the use of infinite matrix transformation. So, we now turn to the fact that how infinite matrix transformation can be used to define generalized limits. A very important application of matrices, namely to the theory of summability of divergent sequence and the series was initiate by Toeplitz [12] in 1911. Although, the concept of absolute summability was introduced as early as in 1911, by Fekete [4] in case of Cesàro [9]method, and the same for Reisz [9] and Abel [9] methods was defined by Obrechkoff [10] and Whittaker[14] in 1928 and 1932 respectively, for matrix transformation in general this was considered in 1937 by Mears [13].

Before proceeding with the main work, we use following German abbreviation in this paper.

FF for sequence -to- sequence (1)

RF for series - to – sequence (2)

RR for series – to- series. (3)

Let $P = (p_{ab})$, (a,b = 1, 2,…) be a given matrix and consider the transformation

$$u_n = \sum_{k=1}^{\infty} p_{ab} v_k \tag{4}$$

then the matrix P provides an *FF, RF* or *RR* transformation according as it transform a sequence $z = \{v_b\}$ into the sequence r $=\{u_a\}$, the series $\sum v_a$ into the series$\sum u_a$, provided that each of the series (4) is convergent. A corresponding to *FF* – transformations each be made applicable with obvious changes o RF and RR transformation.

If a transformation (4) the sequences $r = \{u_a\}$ belongs to the space of convergent sequences. We say that the sequences $x = \{v_k\}$ is summable by the matrix method P, or by the matrix method P, or by the matrix P or simply P- summable and we write either P- $\lim z = \lim r$. The class $[P]$ of all P - summable sequences is called the convergence field of P. The matrix P is said to be convergence preserving if it transfers every convergent sequence $r = \{u_a\}$, with $\lim$ not necessary is same as that of $\{v_a\}$. The matrix P is said to be permanent if it is transform every convergent sequences $z = \{v_a\}$ into a convergent sequence $r = \{u_a\}$ and

$$P\text{-}\lim x = \lim r. \tag{5}$$

Also, the matrix P is said to be absolutely convergence preserving if

$$\sum_{a=2}^{\infty} |v_a - v_{a-1}| < \infty \tag{6}$$

$$\Rightarrow \sum_{a=2}^{\infty} |u_a - u_{a-1}| < \infty$$

In an addition to (6) , (5) also, the matrix P is said to be absolutely permanent. The matrix P is *called* reversible if the equation $P(z) = r$, has exactly one solution z, convergent, or not for each value of r in space.

Some remarkable notations are:

(1) θ_P – matrix (*RF* absolute convergence preserving)
(2) σ_P – matrix (*RR* absolute convergence preserving)

Kojima, in 1917 began the work in this direction. He proved the result for *FF*- transformation by lower semi- matrices. His result was generalized by Schur, who proved that an *FF*- transform matrix gives convergence preserving transformation iff it is a K- matrix, 1931, Basanquet[3] proved that a matrix of an RF- transformation is convergence preserving iff it is a θ- matrix (*RF* convergence preserving). Vermes further studied the θ- matrix (*RF* convergence preserving) and obtained the result that necessary and sufficient condition for a matrix to give a convergence preserving *RR*-transformation .Several researchers like Sariö l [4], Gökce and Sarigöl [4],[6] , Dawson [7] , Borsik et al [8], Borsik [9] and Vermes [14] have studied in the same directions but there results are almost different.Our utmost effort goes on extending P. Vermes [14] works:

2 Main Theorems

In this section, we shall investigate some theorems that will be proved in section 4.

Theorem 2.1

If $h_{ab} = q_{1b} + q_{2b} + \cdots + q_{ab}$, ($a,b \geq 1$) (7)

then $H = (h_{ab})$ is a θ_P – matrix iff $Q = (q_{ab})$ is a σ_P- matrix.

Theorem 2.2

The product HQ of θ_p – matrix H and σ_P- matrix Q exits and is a θ_P – matrix.

Theorem 2.3

Every finite linear combination of σ_P – matrices (or θ_P – matrices) is a σ_P – matrices (or θ_P – matrix).

Theorem 2.4 The product of a θ_P- matrix H and σ_P-matrix Q is not commutative.

Theorem 2.5 The product matrix $R = HQ$ exists and is a θ_p- matrix for every θ_p- matrix H iff Q is a σ_p- matrix.

For the proof of our theorems, following lemmas are required:

Lemma 2.6

In order that RF- transformation given by the matrix $H = (h_{ab})$ be absolute convergence preserving, it is necessary and sufficient that the conditions

$$\sum_{a=2}^{\infty} |h_{ab} - h_{a-1,b}| < L(H) \quad (8)$$

$$|h_{ab}| < R_a(H) \quad (9)$$

be satisfied, where the absolute constants $L(H)$ and $R_a(H)$ are independent of b.

Lemma 2.7

The RR transformation given by the matrix $Q = (h_{ab})$ is absolute convergence preserving iff there exists constant L and R_a, both independent of R , such that the conditions

$$\sum_{a=1}^{\infty} |q_{ab}| < L(Q) \quad (10)$$

$$|q_{ab}| < R_a(Q) \quad (11)$$

are satisfied.

3. Proof of the theorems

In this section , we shall prove the theorems with the help of Lemma 2.6 and 2.7

3.1 Proof of theorem 2.1

Suppose that the matrix $Q = (q_{ab})$ in (7) is a σ_P- matrix. Then by **lemma 2.6,**

$| h_{ab} | = | q_{1b} + q_{2b} + \cdots + q_{ab}|$

$\leq | q_{1b} |+| q_{2b} |+...+| q_{ab}|$

$< aR_a(Q)$

$\leq R_a(Q)$ say (12)

where the constant R_a is independent of *b*. Also, by definition (7), we can write

$h_a - h_{a-1,b} = h_{ab}$, $(a> 1, b \geq 1)$

$h_{1b} = q_{1b}$.

Therefore

$\sum_{a=2}^{\infty}\left|h_{ab} - h_{a-1,b}\right| = \sum_{a=2}^{\infty}|q_{ab}| < L(Q)$ (13)

By condition (11). Thus, the inequalities (12) and (13) show that

$H = (h_{ab})$ in (7) is a θ_P- matrix.

Conversely, let $H = (h_{ab})$ in (7) by $a\ \theta_P$- matrix then it is easy to verify that $Q = (q_{ab})$ so defined satisfies the condition of Lemma 2.6

This completes the prof of the theorem.

3.2 Proof of theorem 2.2:

We write

$F = HQ = (f_{ab})$, so that

$f_{ab} = \sum_{l=1}^{\infty} h_{al} q_{lb}$. (14)

Now, from (9), we have

$| \sum_{l=1}^{\infty} h_{al} q_{lb}| \leq \sum_{l=1}^{\infty}|h_{al}| \ |q_{lb}| < R_a(H) \sum_{l=1}^{\infty}|q_{lb}|$

$< R_a(H).L(Q) < L(F)$ (15)

independent of *b*.

Therefore, the matrix $F = (f_{ab})$ in (14) exists for all a and b. Also, we have from (15),

$|f_{ab}| < R_a(F)$, $\forall\ a$ (16)

where R_a is independent of *b*.

Also, as in (15), we can prove that,

$\sum_{a=2}^{\infty}\left|f_{ab} - f_{a-1,b}\right| < L(F)$, (17)

which is Independent of *b*.

Thus, the result follows from (16) and (17) by virtual of lemma 2.6.

3.3 Proof of theorem 2.3

Let *C* and *H* be a θ_P-matrices corresponding to σ_P – matrices *P* and *Q* respectively i.e.

$$h_{ab} = q_{1b} + q_{2b} + \cdots + q_{ab} \quad (a, b \geq 1)\ , \tag{18}$$

$$c_{ab} = p_{1b} + p_{2b} + \cdots + p_{ab,} \quad (a, b \geq 1)\ .$$

Also, let *z* and *r* be any two complex number and write

$$C = z^C + r^H \tag{19}$$

then from (9), we have

$$|f_{nk}| \leq |z||c_{ab}| + |r||h_{ab}|$$

$$\leq |z||R_a(c)| + |r||R_a(H)| < R_a(C), \tag{20}$$

where the constant R_a is independent of *b*.

By (18), we have

$$f_{1b} = z.\ c_{1b} + r\ h_{1b} \qquad (b \geq 1)$$

$$f_{ab} - f_{a-1,b} = z\left(c_{ab} - c_{a-1,b}\right) + r\left(h_{ab} - h_{a-1,b}\right), \quad (a > 1, b \geq 1).$$

Hence,

$$\sum_{a=2}^{\infty}\left|f_{ab} - f_{a-1,b}\right| \leq |z||c_{1b}| + \sum_{a=2}^{\infty}|z|.\left|c_{ab} - c_{a-1,b}\right| + |r|.|h_{1b}| + \sum_{a=2}^{\infty}|r|.\left|h_{ab} - h_{a-1,b}\right|$$

$$< |z|.L(C) + |r|L(H)$$

$$< L(F) \tag{21}$$

which is independent of *b*.

The condition (20) and (21) are precisely, the condition of lemma3.1, and hence, the matrix $(F) = (f_{ab})$ as defined in (19) is a θ_P- matrix, this proved the result for θ_p- matrices.

Now, $f_{1b} = z\ c_{1b} + r\ h_{1b}$, $\qquad (b \geq 1)$

$$f_{ab} - f_{a-1,b} = z\ c_{ab} + r\ h_{ab}, \quad (a > 1, b \geq 1)$$

Therefore, the matrix $D = (d_{ab})$ defined as

$$d_{1b} = f_{1b} \quad (b \geq 1)$$

$$d_{ab} = f_{ab} - f_{a-1,b,} \quad (a > 1, b \geq 1)$$

which corresponds to the θ_P- matrix it is a σ_P- matrix, and

$$D = z^P + r^Q\ .$$

This completes the proof of the theorem 2.3.

Similarly, we can show that remaining theorems.

4. Conclusion

In this paper, we have proved some general theorems on the absolute convergence transformation of matrix which is expressed in terms of preserving transformation under the very general conditions. In fact, these results can be used for further study in many practical problems in science and engineering.

References

[1] Borsik, J. **(2008)**. Funetsons preserving some types of series. *Journal of Applied Analysis*, 14(2) :149-163.

[2] Bosanquet , L.S.**(1950)**. *Absolute summability of series with constant transformation - Mathematical Reviews*, 11, 654.

[3] Dawson, David F. **(1963)**. On certain sequence to sequence transformations which preserve convergence. *Proceedings of the Americas Mathematical Society*, 14(4): 542-545,.

[4] Fekete, M. **(1911)**. Zur theorie der divergenten Riehen. *Mat. Es Term. Ert* (Budhapesst) 29: 719-729.

[5] Gökse, F and Sarigöl, M. A. **(2019)**. Generalization of the absolute Cesàro space and some matrix transformations. *Numerical functional analysis and optimazion*, 40(9): 1039-1052.

[6] Gökse, F and Sarigöl, M. A.**(2020)**. Series spaces derived from absolute Fibonacci summability and matrix transformation. *Boll Unime Mat ital* , 13: 29-38.

[7] Jàn, J , et al .**(1995)**. Tibor Salat, Remarks on functions preserving convergence of infinite series. *Real Analysis Exchange*, 21(2) :725-731.

[8] Mears F.M. **(1935)**. Some multiplication therems for the Nörlund means. *Bull of American Math. Soc.* , 41: 875-880.

[9] Nigam, H. K.**(2013)**. Birth and growth of summability and approximation theory. *Indian International Journal of pure and Applied Mathematics*, 83(5): 639-641.

[10] Obreckkoff J N. **(1934)**. Sur la sommation des series trigonometrique de Fourier per les moyennes aithmetiques. *Bull de la soc. Math. De France*, 62: 84-109.

[11] Sazigre, M. A.**(2011)**. Matrix tronsformations on fields of absolute weighted mean summability. *Studia Scientiarum Mathematicarum Hungarica*, 48 (13): 331-342.

[12] Toeplitz O. **(1992)**. über allgemeine lineare Mittelbildungen. *prace math.* , 22: 113-119.

[13] Vermes, P. **(1949)**. Series to series transformation and analytic contribution by matrix methods. *American Journal of Mathematics*, 71(3): 541-562

[14] Whittaker J. M. **(1930)**. The absolute summability of Fourier series. *Proceedings of Edinburgh Mathematical Soc.* , 2(2): 1-5.