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Half Cauchy-Modified Exponential Distribution: Properties and Applications

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Abstract: *A new distribution having three parameters using half Cauchy family of distribution named half Cauchy modified exponential distribution is deliberated and studied in this work. Its mathematical and statistical properties are examined. Model parameters of novel distribution are evaluated using least-square, maximum likelihood and Cramer-Von-Mises estimations approaches. R programming software is applied to carry out all of the calculations. To evaluate the new distribution's application and goodness-of-fit test, an actual data set is studied for illustration. The suggested new distribution is performed better as compared to some existing distributions.*

Keywords: *Half Cauchy distribution, Hazard function, Modified exponential distribution, Parameter estimation*.

Introduction

In this work, we have investigated the half-Cauchy model, which is made from the Cauchy model and generated through which the curve is folded on the origin to see non-zero positive numbers only. Shaw [17] used the heavy-tailed half-Cauchy distribution as an option to modeling spreading distances because it can foresee additional frequent long-distance spreading events in the future. Consider a half-Cauchy distribution on a non-negative random variable X such that $x > 0, \theta > 0$ and its CDF is

$$G(x;\theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right). \tag{1}$$

The associated PDF of (1) is

$$g(x;\theta) = \frac{2}{\pi}\left(\frac{\theta}{\theta^2 + x^2}\right), \quad x > 0, \theta > 0. \tag{2}$$

As a parent distribution, the half-Cauchy model has been employed by various academics during the last few decades. Cordeiro and Lemonte [5] reformed the half-Cauchy model named as a beta-half-Cauchy model. As a prior, the half-Cauchy model was employed for a universal scale parameter by (Polson& Scott, [14]) in Bayesian analysis. The extension of the half-Cauchy distribution had established by (Ghosh, [7]) named Kumaraswamy-half-Cauchy distribution. Cordeiro et al. [6] established a family of generalized odd half-Cauchy distributions based on the half-Cauchy distribution.

Therefore, we'd want to generate new distribution using half-Cauchy family of distribution. The generating family of distribution developed by (Zografas & Balakrishnan, [21]) and the CDF of a distribution family might be calculated as

$$F(x) = \int_{0}^{-\ln\{-G(x)\}} r(t)\, dt\,, \tag{3}$$

Here, any baseline distribution's CDF is $G(x)$, and any distribution's PDF is $r(t)$. The family of half-Cauchy distribution whose CDF can be defined by using $r(t)$ as PDF of half-Cauchy distribution with $\theta > 0$ *and* $x > 0$, defined in (1) as

$$F(x) = \int_{0}^{-\ln[1-G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\, dt = \frac{2}{\pi}\arctan\left(-\frac{1}{\theta}\ln\left[1-\ G(x\ )\ \right]\right) \tag{4}$$

The (4)'s PDF is

$$f(x) = \frac{2}{\pi\ \theta}.\frac{g(x)}{[1-\ G(x)]}\left[1+\left\{-\frac{1}{\theta}\ \log\left[\ 1-G(x)\ \right]\right\}^2\ \right]^{-1} \tag{5}$$

As a baseline distribution in this study, we have used a modified exponential distribution defined by (Rosaiah et al., [16]) which was previously used by many authors to generate the new distribution some of them are; Okoli et al., [13] have defined the modified exponential distribution. The three-parameter exponential distribution was introduced by (Afify & Mohamed, [1]). The half Logistic modified exponential distribution was created by integrating modified exponential distribution with the half Logistic family of distribution by (Chaudhary& Kumar, [4]). With the help of the Logistic distribution family, the Logistic modified exponential model produced by (Chaudhary & Kumar, [3]).

The residual parts of this study are managed in following manner. We define half Cauchy modified exponential distribution and some of their statistical and mathematical properties are also derived. The most widely used estimation techniques namely least-square (LSE), Cramer-Von-Mises (CVM) and maximum likelihood estimators (MLE) are applied to calculate the model parameters. The recommended model's application will be discussed. Finally, some concluding remarks are made.

Half Cauchy Modified Exponential (HCME) Distribution

We've defined a novel model known as the half Cauchy modified exponential distribution in this section. To generate HCME distribution, we have utilized the modified exponential distribution as a parent distribution which has presented by (Rosaiah et al., [16]). The CDF of the modified exponential distribution having two shape parameters $(\beta,\lambda) > 0$ is

$$G(\ x;\ \lambda,\beta\ ) = 1\text{-}\exp\left\{\ \text{-}\beta\ x\ e^{\lambda x}\right\}\ ;x>0 \tag{6}$$

The (6)'s PDF is

$$g(\ x;\ \lambda\ ,\beta\ )\ =\ \beta\ \left(1+\lambda x\right)\ \exp\left\{\lambda\ x-\beta\ x\ e^{\lambda x}\right\}\quad ;\ x>0 \tag{7}$$

Substituting equations (6) and (7) in (4) and (5), with $(\beta,\ \lambda,\ \theta) > 0$ the CDF and PDF of $HCME(\beta,\lambda,\theta)$ model correspondingly are

$$F(x\ )\ =\ \frac{2}{\pi}\ \arctan\left(\frac{\lambda}{\theta}x\ e^{\beta x}\right)\ ;x>0\ \ \text{and} \tag{8}$$

$$f(x) = \frac{2\lambda}{\pi\theta}(1+\beta x)\, e^{\beta x}\left\{1+\left(\frac{\lambda\, x\, e^{\beta x}}{\theta}\right)^2\right\}^{-1} ; x>0, (\beta\lambda\theta)>0 . \tag{9}$$

Survival function

The survival function of $HCME(\beta,\lambda,\theta)$ model with $x>0,$ is

$$R(x) = 1-\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}xe^{\beta x}\right); . \tag{10}$$

Hazard rate function (HRF)

The HRF for $HCME(\beta,\lambda,\theta)$ is

$$h(x) = \frac{2}{\pi}\frac{\lambda}{\theta}(1+\beta x)\, e^{\beta x}\left\{1+\left(\frac{\lambda x e^{\beta x}}{\theta}\right)^2\right\}^{-1}\left\{1-\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}xe^{\beta x}\right)\right\}^{-1} ; x>0, (\beta\lambda\theta)>0 . \tag{11}$$

Reversed hazard rate (RHR) function

The RHR function is

$$r(x) = \frac{2}{\pi}\frac{\lambda}{\theta}(1+\beta x)e^{\beta x}\left\{1+\left(\frac{\lambda x e^{\beta x}}{\theta}\right)^2\right\}^{-1}\left\{\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}xe^{\beta x}\right)\right\}^{-1} ;\; x>0. \tag{12}$$

We have plotted the graphs of PDF and HRF for numerous parameters values of HCME distribution in Fig. 1.

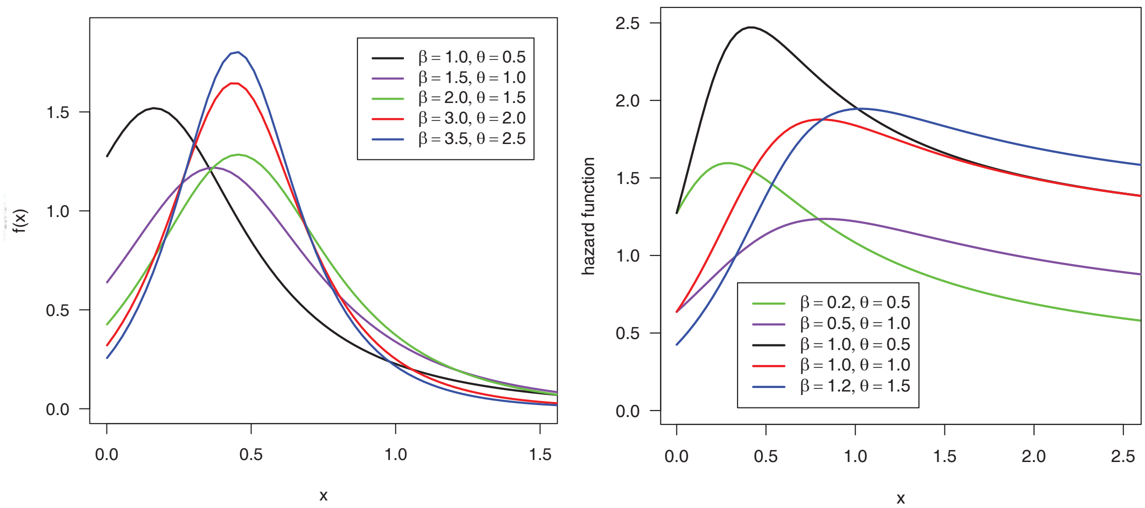


Fig. 1: HRF (right-hand section) and PDF (left-hand section) for numerous values of θ and β keeping λ as constant.

Cumulative hazard function (CHRF)

The CHRF of the $HCME(\beta,\lambda,\theta)$ is

$$H(x) = \int_{-\infty}^{x} h(y)dy = -\log\left[\,1-F(x)\,\right] = -\log\left[1-\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}xe^{\beta x}\right)\right] \tag{13}$$

The Quantile function of HCME distribution

The inverse function of CDF defined in (8) is used to calculate the quantile function as

$$Q(v) = F^{-1}(v)$$

Hence, it is obtained as,

$$-\lambda x e^{\beta x} + \theta \tan\left(\frac{\pi v}{2}\right) = 0 \quad ; 0 < v < 1 \tag{14}$$

where v is the uniformly distributed random variable which follows U (0,1).
The random deviates can be generated for HCME distribution using (14) as

$$-\lambda x e^{\beta x} + \theta \tan(\pi u / 2) = 0 \quad ; \; 0 < u < 1 .$$

HCME distribution's kurtosis and skewness

The coefficient of skewness is

$$S(Bowley's) = \frac{Q\left(3/4\right) + Q\left(1/4\right) - 2\,Q\left(1/2\right)}{Q\left(0.75\right) - Q\left(1/4\right)} .$$

HCME distribution's Kurtosis

Moors [11] coefficient of kurtosis is

$$K_u\left(M\right) = \frac{Q\left(0.875\right) - Q\left(0.125\right) - Q\left(0.625\right) + Q\left(0.375\right)}{Q\left(0.75\right) - Q\left(0.25\right)} .$$

Parameter estimation methods
MLE method

The ML estimators for $HCME(\beta,\lambda,\theta)$ are assessed by the MLE method. If $HCME(\beta,\lambda,\theta)$ is used to create a random sample $\underline{x} = \left(x_1, \ldots, x_n\right)$ having size 'n', log likelihood function is

$$\ell(\lambda,\beta,\theta \mid \underline{x}) = n\ln\left(\frac{2}{\pi}\right) + n\ln\lambda + n\ln\theta - \sum_{i=1}^{n}\ln\left\{\theta^2 + \left(\lambda + e^{\beta x_i}\right)^2\right\} + \sum_{i=1}^{n}\ln\left(1+\beta x_i\right) + \beta\sum_{i=1}^{n} x_i \tag{15}$$

After differentiating (15) with regard to λ, β and θ, we have

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - 2\sum_{i=1}^{n}\left\{\left(\lambda + e^{\beta x_i}\right)^2 + \theta^2\right\}^{-1}\left(\lambda + e^{\beta x_i}\right)^2$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n}\left(1+\beta x_i\right)^{-1} x_i + \sum_{i=1}^{n} x_i - 2\beta \sum_{i=1}^{n}\left[\left\{\theta^2 + \left(\lambda + e^{\beta x_i}\right)^2\right\}^{-1} e^{\beta x_i}\left(\lambda + e^{\beta x_i}\right)\right]$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - 2\theta \sum_{i=1}^{n}\left\{\theta^2 + \left(\lambda + e^{\beta x_i}\right)^2\right\}^{-1}$$

On solving $\frac{\partial \ell}{\partial \beta} = \frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \theta} = 0$ at the same time for the β, λ and θ, the corresponding $HCME(\beta,\lambda,\theta)$ model's ML estimators are obtained. However, solving such non-linear equations like the ones above is usually impossible for normally. So, with the help of appropriate computer software, these can be readily solved. If the parameter vector of $HCME(\beta,\lambda,\theta)$ be $\underline{\Theta} = (\beta,\lambda,\theta)$ & the associated MLE for $\underline{\Theta}$ is $\underline{\Theta} = (\hat{\beta},\hat{\lambda},\hat{\theta})$, then

$\left(\underline{\Theta}-\underline{\Theta}\right)\rightarrow N_3\left[0,\left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$ is produced as a result of asymptotic normality, where $I\left(\underline{\Theta}\right)$ stands for Fisher's information matrix (FIM) which is

$$I\left(\underline{\Theta}\right)= -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta\,\partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \beta\,\partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta\,\partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda\,\partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta\,\partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda\,\partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}$$

We don't know $\underline{\Theta}$ in practice, thus the MLE's asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$ remains inadequate. Hence, by plugging in expected model parameter values, the asymptotic variance may be approximated. The estimated information matrix $I\left(\underline{\Theta}\right)$ is centered on the observed FIM $O(\underline{\Theta})$ and is given by

$$O(\underline{\Theta}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \hat{\beta}^2} & \frac{\partial^2 l}{\partial \hat{\beta}\,\partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\beta}\,\partial \hat{\theta}} \\ \frac{\partial^2 l}{\partial \hat{\beta}\,\partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} & \frac{\partial^2 l}{\partial \hat{\theta}\,\partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\beta}\,\partial \hat{\theta}} & \frac{\partial^2 l}{\partial \hat{\theta}\,\partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\theta}^2} \end{pmatrix}_{\left|(\hat{\beta},\hat{\lambda},\hat{\theta})\right.} = -H\left(\underline{\Theta}\right)_{\left|(\underline{\Theta}=\underline{\Theta})\right.} = \textit{Hessian matrix}$$

The observed information matrix is created using the Newton-Raphson technique to optimize likelihood. As a result, the variance-covariance matrix has been transformed as

$$\left[-H\left(\underline{\Theta}\right)_{\left|(\underline{\Theta}=\underline{\Theta})\right.}\right]^{-1} = \begin{bmatrix} \mathrm{v}(\hat{\beta}) & \mathrm{cv}(\hat{\lambda},\ \hat{\beta}\ ) & \mathrm{cv}(\hat{\beta},\hat{\theta}\ ) \\ \mathrm{cv}(\hat{\lambda},\ \hat{\beta}\ ) & \mathrm{v}(\hat{\lambda}) & \mathrm{cv}(\hat{\lambda},\ \hat{\theta}\ ) \\ \mathrm{cv}(\hat{\beta},\ \hat{\theta}\ ) & \mathrm{cv}(\hat{\lambda},\ \hat{\theta}\ ) & \mathrm{v}(\hat{\theta}) \end{bmatrix} \tag{16}$$

Where v = variance and cv = covarince

As a result of MLEs' asymptotic normality, we may construct approximate (1- γ) *100 percent confidence intervals for λ, θ & β of $HCME(\beta,\lambda,\theta)$ as below:

$$Z_{\gamma/2} * S.E.(\hat{\lambda}) \ \pm\ \hat{\lambda}\ ,\ \hat{\beta}\ \pm\ S.E.(\hat{\beta}) * Z_{\gamma/2} \quad \text{and}\ S.E.(\hat{\theta}) *\ Z_{\gamma/2}\ \pm\ \hat{\theta}\ . \tag{17}$$

LSE method

Swain et al. [19] developed estimators of ordinary and weighted LS as a new way to estimating the parameters of Beta distributions. The similar procedure is applied here to estimate the suggested model parameters. Minimizing (18) with regard to unknown β, λ and θ yields the LS estimators for unknown model parameters.

$$B\left(X;\beta,\lambda,\theta\right)=\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2 \tag{18}$$

Assume that $F(X_i)$ stands for the CDF for ordered random variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ and that with sample size n, a random sample $\{X_1, X_2, \ldots, X_n\}$ has come from CDF. Through minimizing (19) with regard to β, λ and θ, respectively, LS estimators $\hat{\beta}, \hat{\lambda}$ *and* $\hat{\theta}$ of unknown model parameters might be found.

$$B(X;\beta,\ \lambda,\ \theta)=\sum_{i=1}^{n}\left[\ \frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}\ x_{(i)}\ \ e^{\beta x_{(i)}}\right)-\frac{i}{n+1}\right]^2 \tag{19}$$

Differentiating (19) with regard to β, λ and θ ,we have,

$$\frac{\partial B}{\partial \beta}=\frac{4\lambda}{\pi\theta}\sum_{i=1}^{n}x_{(i)}^2 e^{\beta x_{(i)}}\left[W\left(x_{(i)}\right)-\frac{i}{n+1}\right]\left[1+\left\{W\left(x_{(i)}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial B}{\partial \theta}=\frac{-4\lambda}{\pi\theta^2}\sum_{i=1}^{n}x_{(i)} e^{\beta x_{(i)}}\left[W\left(x_{(i)}\right)-\frac{i}{n+1}\right]\left[1+\left\{W\left(x_{(i)}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial B}{\partial \lambda}=\frac{4}{\pi\theta}\sum_{i=1}^{n}x_{(i)}^2 e^{\beta x_{(i)}}\left[W\left(x_{(i)}\right)-\frac{i}{n+1}\right]\left[1+\left\{W\left(x_{(i)}\right)\right\}^2\right]^{-1}$$

Here, $\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}x_{(i)}e^{\beta x_{(i)}}\right)=W\left(x_{(i)}\right)$.

On solving $\frac{\partial B}{\partial \beta}=0,\ \frac{\partial B}{\partial \lambda}=0\ and\ \frac{\partial B}{\partial \theta}=0$ for model parameters at the same time, we'll obtain the $HCME(\beta,\lambda,\theta)$'s CVM estimators.

In the same way, weighted least square estimators are calculated through minimizing B with regard to λ , β and θ.

$$B(X;\ \beta,\ \lambda,\ \theta)\ =\ \sum_{i=1}^{n}w_i\left[F(X_{(i)})-\ \frac{i}{1+n}\ \right]^2$$

Here, $w_i = weights = \frac{1}{Var(X_{(i)})} = \frac{(n+1)^2(n+2)}{(n-i+1)i}$

Consequently, through minimizing (20) with regard to β, λ and θ, weighted LS estimators of model parameters might be produced.

$$B(X;\ \lambda,\ \beta,\ \theta)\ =\ \sum_{i=1}^{n}w_i\left[\frac{2}{\pi}\arctan\left\{\ \frac{\lambda}{\theta}x_{(i)}e^{\beta x_{(i)}}\right\}-\ \frac{i}{n+1}\ \right]^2 \tag{20}$$

CVME method

Minimizing function (21) yields Cramer-Von-Mises estimators for β, λ and θ .

$$A(X;\beta,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)}\,|\,\beta,\lambda,\theta\right)\ -\ \frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2}{\pi}\arctan\left\{\frac{\lambda}{\theta}\ x_{(i)}e^{\beta x_{(i)}}\right\}-\ \frac{2i-1}{2n}\right]^2 \tag{21}$$

Differentiating (21) with regard to β, λ and θ, we get,

$$\frac{\partial A}{\partial \beta}=\frac{4\lambda}{\pi\theta}\sum_{i=1}^{n}\left[W\left(x_{(i)}\right)-\frac{2i-1}{2n}\right]e^{\beta x_{(i)}}\left[1+\left\{W\left(x_{(i)}\right)\right\}^{2}\right]^{-1}x_{(i)}^{2}$$

$$\frac{\partial A}{\partial \lambda}=\frac{4}{\pi\theta}\sum_{i=1}^{n}x_{(i)}^{2}\left[W\left(x_{(i)}\right)-\frac{2i-1}{2n}\right]e^{\beta x_{(i)}}\left[1+\left\{W\left(x_{(i)}\right)\right\}^{2}\right]^{-1}$$

$$\frac{\partial A}{\partial \theta}=\frac{-4\lambda}{\pi\theta^{2}}\sum_{i=1}^{n}x_{(i)}\left[W\left(x_{(i)}\right)-\frac{2i-1}{2n}\right]\left[1+\left\{W\left(x_{(i)}\right)\right\}^{2}\right]^{-1}e^{\beta x_{(i)}}$$

Here, $\frac{2}{\pi}\arctan\left(\frac{\lambda}{\theta}e^{\beta x_{(i)}}x_{(i)}\right)=W\left(x_{(i)}\right)$.

On solving $\frac{\partial A}{\partial \lambda}=0,\ \frac{\partial A}{\partial \beta}=0\ and\ \frac{\partial A}{\partial \theta}=0$ at the same time, CVM estimators of $HCME(\beta,\lambda,\theta)$ are derived.

Application

To exhibit the applicability of the HCME distribution in this section, we have used an actual dataset from previous work.

Data Set

The following data comes from a 59-conductor accelerated life test (Nelson & Doganaksoy, [12]). Atomic movement in the circuit's conductors can create failures in microcircuits, which is known as electro-migration. There are no censored observations, and the failure times are in hours.

5.923, 4.288, 6.522, 4.137, 6.071, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.545, 10.491, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033,7.459, 9.289 , 6.958.
From Figure 2, it is perceived that the assessed model parameters values via MLE method are typical.

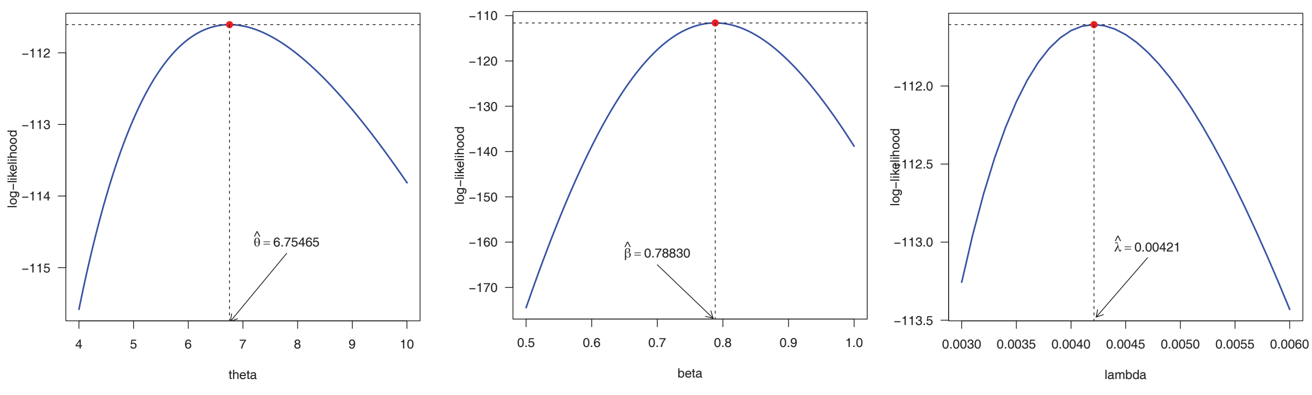


Fig. 2: Profile log-likelihood functions for β, θ and λ.

The ML estimates might be derived directly from R programming software developed by (R Core Team, [15]) and (Ming Hui, [10]). The Log-Likelihood value obtained is l = -111.6071. The MLEs for β, λ and θ are listed in Table 1 along with their standard errors (S.E.).

Table 1

S.E. & MLE for β, λ and θ

Parameters	**MLE**	**S.E.**
beta	0.788304	0.110453
lambda	0.004211	0.004031
theta	6.754650	3.683513

We have presented the predicted parameters values, AIC, log-likelihood, BIC, HQIC and AICC for the MLE, CVE and LSE approaches in Table 2. It is found that the MLEs are quite better among these three estimation methods.

Table 2

$\hat{\beta}, \hat{\lambda}$ *and* $\hat{\theta}$ values, AIC, log-likelihood, AICC, HQIC & BIC

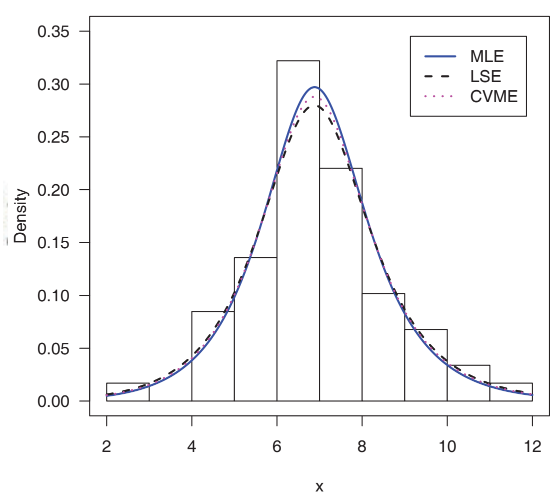
Methods	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	**-LL**	**AIC**	**BIC**	**HQIC**	**AICC**
LSE	0.73403	0.00233	2.56771	-111.7337	229.4675	235.7001	231.9004	229.9038
MLE	0.78830	0.00421	6.75465	-111.6071	229.2141	235.4467	231.6471	229.6505
CVE	0.75960	0.00672	8.84587	-111.6418	229.2836	235.5162	231.7166	229.7200

The goodness-of-fit of the LSE, MLE, and CVME methods are observed by the test statistics values and their p-values for CVM (Cramer-Von Mises), KS (Kolmogorov-Simnorov), and AD (Anderson-Darling) which are displayed in Table 3.

Table 3

Statistics values and their associated p-values for the goodness-of-fit

Methods	***KS. (p-value)***	***CVM. (p-value)***	***AD. (p-value)***
LSE	0.0509(0.9961)	0.1541(0.9983)	0.0222(0.9948)
MLE	0.0566(0.9861)	0.1543(0.9983)	0.0224(0.9946)
CVE.	0.0538(0.9921)	0.1439 (0.9990)	0.0210(0.9962)



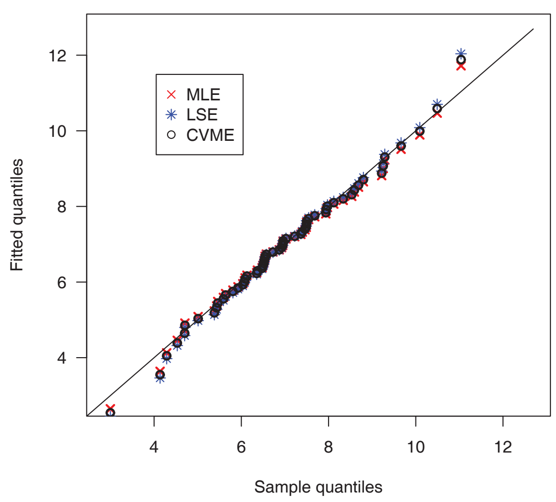


Fig. 3: The Q-Q plot (right section) and fitted distributions' histogram and the density function (left section) having MLE, CVM and LSE of HCME distribution.

When utilizing the maximum likelihood technique to derive the parameters, the Kolmogorov-Smirnov (KS) is calculated to ensure that the model fits well. Also, we have provided Q-Q plot and the KS plot in Fig. 4, which displays that the proposed distribution delivers excellent match to the given data. The recommended HCME's goodness-of-fit is visualized using the CDF plot of the model having the lack-of-fit at the tails, as well as the P-P chart, which also shows the lack-of-fit. The HCME distribution fits the data nicely, as can be seen in Fig. 5.

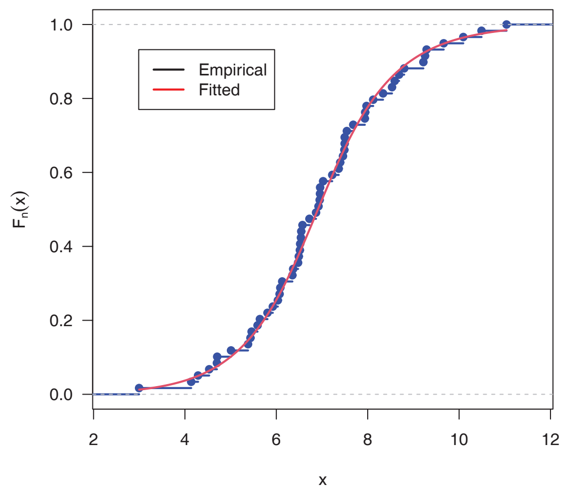


Fig.4: Empirical distribution function and the fitted CDF of HCME model.

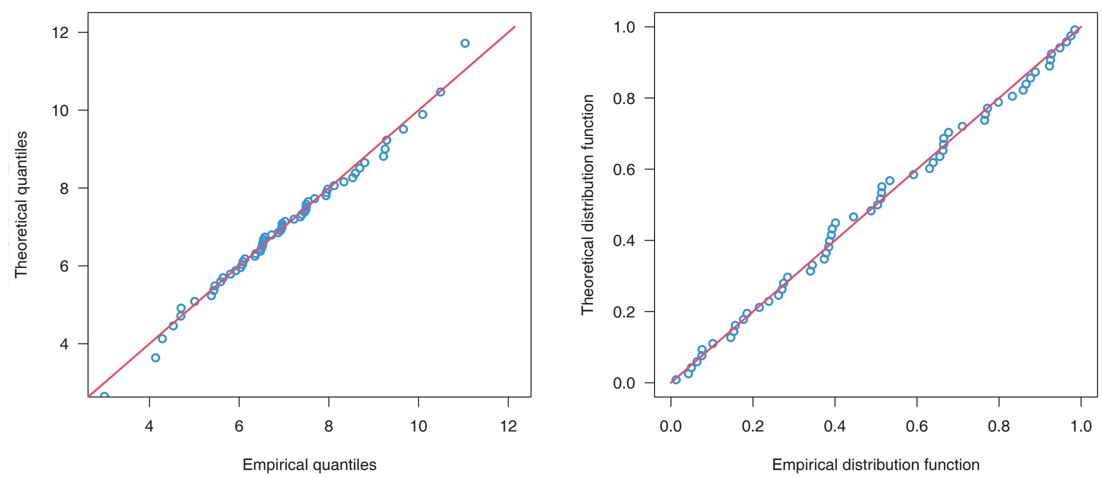


Fig. 5. The P-P chart (right-hand sections) and Q-Q chart (left-hand section) for HCME model.

We compared the recommended model's potentiality to that of Lindley-Exponential (LE) model (Bhati, [2]), generalized exponential model (Gupta & Kundu, [8]), modified Weibull (MW) model (Lai et al., [9]), exponential power (EP) model (Smith & Bain, [18]), and Weibull extension (WE) model (Tang et al., [20]).

To analyze the potentiality of the suggested distribution the Hannan-Quinn information (HQIC), Akaike information (AIC), Corrected Akaike information(CAIC), and Bayesian information criterions (BIC) are calculated, with the results given in Table 4.The HCME distribution contains the least CAIC, AIC, BIC, HQIC , and log likelihood values.

Table 4

BIC, HQIC, CAIC, AIC and Log-likelihood (LL)

Models	-LL	AIC	BIC	CAIC	HQIC
HCME	-111.6071	229.2141	235.4467	229.6505	231.6471
MW	-112.5218	231.0435	237.2761	231.4799	233.4765
WE	-113.6745	233.3491	239.5817	233.7855	235.7821
GE	-114.9473	233.8946	238.0497	234.1089	235.5166
LE	-114.9528	233.9055	238.0606	234.1198	235.5275
EP	-116.5015	237.0029	241.1580	237.2098	238.6249

Figure 6 displays the fitted distributions' histogram and density function, as well as the HCME model's an empirical distribution along estimated distribution functions and a few sample models.

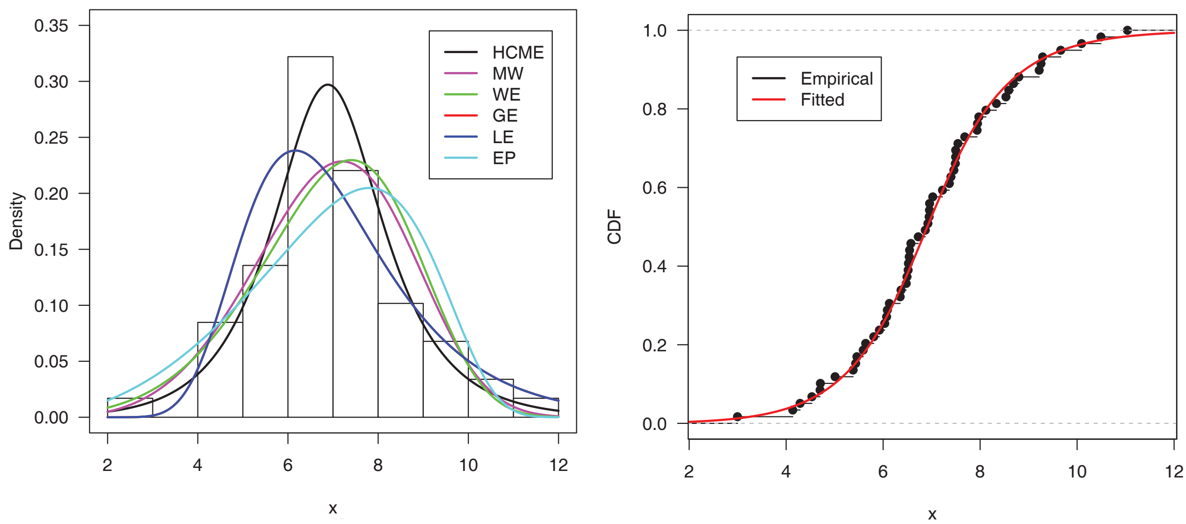


Fig. 6: Fitted distributions' histogram and the density function (left-hand section) and fitted CDF (right-hand panel).

Table 5 compares the goodness-of-fit for HCME model to those of other competing distributions using KS, AD, and CVM statistics. Because HCME distribution has the least test statistic value and the highest p-value, we can deduce that it has a considerably better fit and more consistent and trustworthy findings than the other distributions studied.

Table 5
The statistics values and their related p-values for goodness-of-fit

Models	*KS.(p-value)*	*AD.(p-value)*	*CVM.(p-value)*
HCME	0.0566(0.9861)	0.0224(0.9946)	0.1543(0.9983)v
MW	0.0914(0.6738)	0.0821(0.6816)	0.4839(0.7626)
WE	0.1067(0.4796)	0.1154(0.5160)	0.6800(0.5751)
EP	0.1365(0.2021)	0.2398(0.2021)	1.3735(0.2098)
LE	0.1042(0.5099)	0.1173(0.5077)	0.7373(0.5279)
GE	0.1042(0.5103)	0.1173(0.5079)	0.7368(0.5282)

Concluding Remarks

The three-parameter half Cauchy modified exponential (HCME) distribution is discussed in this study. The PDF, CDF, and hazard function shapes are provided for our research. The HCME distribution's PDF is unimodal and positively skewed, even if the failure rate function is growing. The purposed model fits the real dataset much better, as seen by the P-P and Q-Q charts. Finally, we use an actual data set to explore the MLEs for parameters and their associated confidence intervals. We also consider LSE and CVM estimation approaches to predict the model parameters. A goodness of fit is applied to compare the recommended model's potentiality to that of other distributions, and found that the HCME distribution fits considerably better than the others.

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