Methods of Solving System of Linear Equations

Aasis Baral[1], Sandip Karki[2], Hom Nath Baral[3], Moham Bikram Malla[4], Nirak Kumar Shahi[5]
[1] Graduate School of Engineering, Mid-West University, Surkhet, Nepal
[2] Graduate School of Science & Technology, Mid-West University, Surkhet, Nepal
[3] Graduate School of Science & Technology, Mid-West University, Surkhet, Nepal
[4] Graduate School of Engineering, Mid-West University, Surkhet, Nepal
[5] Graduate School of Science & Technology, Mid-West University, Surkhet, Nepal
Correspondence Email: *aasis.baral@mu.edu.np; hombaral277@gmail.com*

Keywords:
Linear equation, Solution of equations, Augmented matrix.

Received: 2 November 2024
Revised: 25 November 2024
Accepted:28 December 2024

ISSN: 3102-0763 (Print)
3102-0771 (Online)

Abstract

This paper explores the various methods used to solve systems of linear equations, a fundamental topic in mathematics with wide-ranging applications in physics chemistry, biology, engineering, and economics. The methods examined include graphical solutions, substitution, elimination, and matrix-based techniques such as Gaussian elimination and Cramer's Rule. The project aims to provide a clear understanding of each method's principles, effectiveness, and appropriate contexts for use. A comparative analysis is conducted to highlight the strengths and limitations of each approach, guiding the selection of the most suitable method for different types of problems.

Introduction

The system of linear equations has its roots in ancient mathematics, notably in the works of the Chinese mathematician Liu Hui [1] around 200 AD and later by the Indian mathematician Brahma Gupta [2] in the 7th century. However, the systematic study and methods for solving these equations, such as Gaussian elimination, are attributed to Carl Friedrich Gauss [3] in the 19th century.

Systems of linear equations are essential because they model many real-world problems across various fields, such as physics, engineering, economics, and computer science. They provide a way to describe relationships between different quantities and find solutions to problems involving multiple variables, enabling precise and efficient problem-solving.

A linear equation is an equation where the highest power of the variable is one, and it forms a straight line when plotted on a graph.

Example: $ax + by = c$

Definition [4]; A finite set of linear equations is called a system of linear equation or linear system. The variables in a linear system are called unknowns.

For example: $x_1 - 2x_2 + 3x_3 = 10$

$x_1 + 3x_2 - 2x_3 = 10$

is a linear system of two equations in three unknown x_1, x_2 and x_3.

Methods for Solving System of Linear Equations

To solve this system, you can use various methods such as graphical, substitution, elimination, or matrix methods. Here is a brief overview of each method:

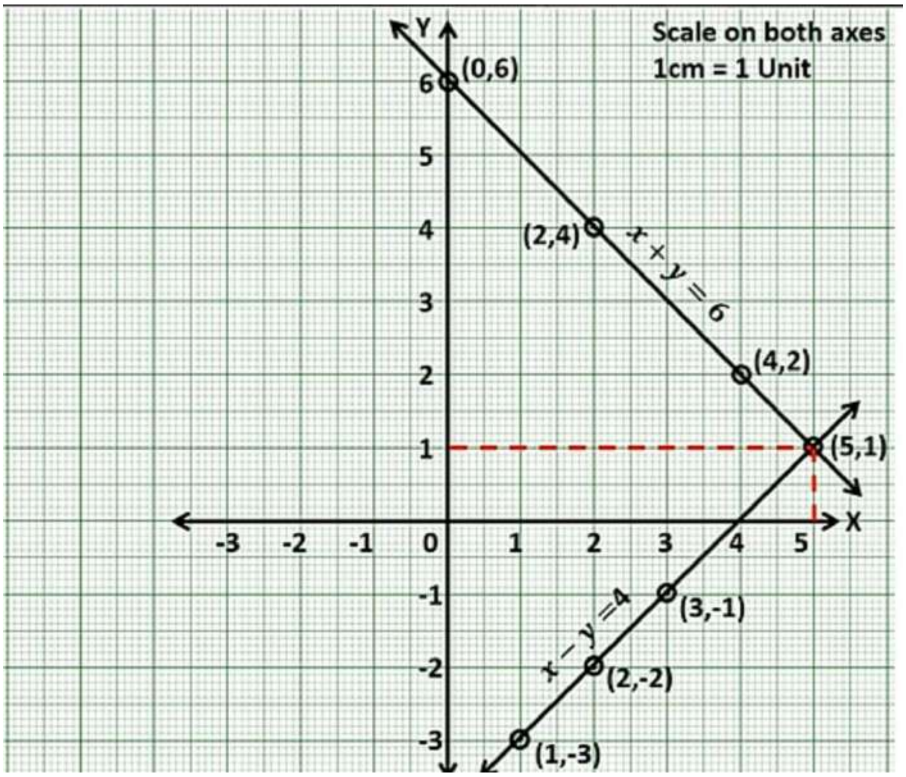
1. **Graphical Method [5]** It originated in the 17th century with René Descartes' work in analytic geometry, introduced in [5], to visually interpret algebraic equations. The graphical method for solving systems of linear equations involves plotting equations as lines on a graph, where

the intersection point(s) represent the solution(s). It provides a straightforward, visual way to find solutions to linear equations, especially useful before the advent of more advanced algebraic techniques.

Example [8]: Solve these two equation using graphical method.

$x + y = 6$ and $x - y = 4$, The solution is as follows

If we plot above point on graph we get



Solution of system of linear equations using graph. [8]

In this method to find the value of x and y we need to find where the two line are intersecting. Here in the above graph we can see two line are intersecting at (5, 1).

Therefore, $x = 5$ and $y = 1$.

Substitution Method [3]: The substitution method for solving systems of linear equations was developed over centuries, with its roots traceable to ancient mathematics. Key contributions came from mathematicians like Diophantus of Alexandria and later René Descartes [3] in the 17th century, who formalized algebraic methods. The substitution method simplifies solving systems by reducing them to a single variable equation, making it easier to find solutions.

Following are the procedure for substitution method to solve system of linear equation.

(i) Solve one of the equations for one variable in terms of the other.

(ii) Substitute this expression into the other equation to find the value of the second variable.
(iii) Substitute the value of the second variable back into the first equation to find the value of the first variable.

Example: Solve these two equation using substitution method.

$x + y = 5$ and $x - y = 1$

Solving this two equations we get $x = 3$ and $y = 2$.

Elimination Method [3]: The elimination method for solving systems of linear equations was first documented in the ancient Chinese text "The Nine Chapters on the Mathematical Art" (circa 200 BCE) [3]. This method, also known as Gaussian elimination, systematically reduces a system of linear equations to simpler forms, ultimately solving the system. It was developed as a practical and systematic approach to handle multiple equations simultaneously, making it easier to find consistent solutions. Here is procedure for this method to solve system of linear equation.

1. Multiply one or both of the equations by a constant so that the coefficients of one of the variables are opposites.
2. Add or subtract the equations to eliminate one of the variables.
3. Solve the resulting equation for the remaining variable.
4. Substitute this value into one of the original equations to find the value of the other variable.

Example: Solve these two equations using elimination methods.

$x + 3y = 5$ and $x - y = 1$

Solving this two equations we get $x = 2$ and $y = 1$.

(i) **Matrix Method**: In matrix method we can use different methods to solve system of linear equation. Some of them are as given below.

(ii) **Gaussian Elimination Method**: The method known today as Gaussian elimination, named after Carl Friedrich Gauss [3], was actually first documented in the ancient Chinese text "The Nine Chapters on the Mathematical Art" (circa 200 BCE). It was later independently rediscovered and popularized by Gauss in the early 19th century. This method systematically reduces systems of linear equations to row-echelon form to facilitate solving. It came about as a practical and systematic approach to simplify and solve complex linear systems efficiently. Here is a short and simple algorithm for Gaussian elimination:

1. Write the system of equations as an augmented matrix.
2. Use row operations to transform the matrix into row echelon form (upper triangular form).
3. Use back-substitution to solve the resulting upper triangular system of equations.

Example: Apply Gauss elimination method to find the values of *x*, *y* and *z*

$x + 4y - z = 5; x + y - 6z = -12; 3x - y - z = 4$

Solving this three equation by this method applying following relations

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$ and $R_3 \to R_3 - \frac{13}{3} R_2$, we get the solution

x = 1.6479, y = - 1.1408 and z = 2.0845.

(iii) **Gauss Jordan Elimination [5]**: The Gauss-Jordan elimination method was first introduced by Wilhelm Jordan [5], a German mathematician, in 1887. This method is an extension of Gaussian elimination and involves transforming a matrix into reduced row-echelon form, which directly provides the solutions to the system of linear equations. It was developed to further simplify the process of solving linear systems by eliminating the need for back-substitution, making the solution process more straightforward and efficient.

Here's a concise and simple methodology for Gauss-Jordan elimination:

1. Form the Augmented Matrix: Write the system of linear equations as an augmented matrix.
2. Forward Elimination of unknowns: We subtract suitable
   - For each column, start from the top row.
   - Ensure the leading entry (pivot) in the current row is 1. If not, swap rows or divide the row by a nonzero constant.
   - Make all entries below the pivot zero by subtracting appropriate multiples of the pivot row from the rows below.
3. Backward Elimination:
   - Ensure the pivot is 1 if not already.
   - Make all entries above the pivot zero by subtracting appropriate multiples of the pivot row from the rows above.
4. Continue until the matrix is in reduced row-echelon form, where each pivot is 1, and all other entries in the pivot columns are zero.
5. Extract the Solutions: Read the solutions directly from the matrix.

Example: Apply Gauss-Jordan method to solve the system of equations.

$$x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$$

Solving this three equation by this method applying following operation

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$, $R_2 \rightarrow -\frac{1}{5}R_2$, $R_1 \rightarrow R_1 - R_2$; $R_3 \rightarrow R_3 - R_2$, $R_3 \rightarrow \frac{5}{12}R_3$; $R_1 \rightarrow R_1 - \frac{7}{5}R_1$ and $R_2 \rightarrow R_2 + \frac{2}{5}R_3$ we get $x = 1, y = 3$ and $z = 5$.

(iv) **Matrix Inversion Method [6]:** The method of matrix inversion for solving systems of linear equations can be traced back to Arthur Cayley [6], a British mathematician, who introduced the concept of matrix inverses in his paper "A Memoir on the Theory of Matrices" published in 1858. The matrix inversion method came about as a systematic way to solve linear equations by leveraging the properties of matrices and their inverses, providing a general solution to systems of linear equations.

The inverse matrix method is a technique used in linear algebra to find the solution to a system of linear equations. Given a system of linear equations, it can be represented in matrix form as:

$Ax = b$

where, A is the coefficient matrix, x is the column vector of variables and b is the column vector of constant.

To solve for x using inverse matrix method, follow these steps:

1. Compute the inverse of matrix A: The inverse of A is denoted as A^{-1}, is a matrix that when multiplied by A, result in the identity matrix I:
$$AA^{-1} = I$$
2. Multiply both side of the equation $Ax = b$ by A^{-1} :
$$A^{-1}Ax = A^{-1}b$$
3. Simply the left side:
Since $AA^{-1} = I$, where I is the identity matrix we get $Ix = A^{-1}b$
4. Solve for x:
The identity matrix I multiplied by any matrix x is just x, so:
$x = A^{-1}b$

This gives the solution of x.

Example: Solve the following system of equation by matrix inversion method.

$$x + y + z = 6, 3x + 3y + 4z = 20, 2x + y + 3z = 13.$$

Solving this three equation with $A^{-1} = \frac{1}{|A|} A^{T,}$ where $|A| = 1$ and the solution is

$$x = 3, y = 2 \text{ and } z = 1$$

(v) **Cramer's Rule [3]**: Cramer's Rule for solving systems of linear equations was introduced by Gabriel Cramer [3], a Swiss mathematician, in his 1750 publication "Introduction à l'analyse des lignes courbes algébriques". Cramer developed this method to provide a systematic solution for linear systems using determinants, allowing for a straightforward calculation of each variable in terms of determinants of matrices derived from the system.

Cramer's rule is a method used to solve a system of linear equations with as many equations as unknowns, using determinants. Here's a concise methodology:

1. Form the coefficient matrix A from the system of linear equations.
2. Calculate the determinant det(A). If det(A) = 0, the system has no unique solution.
3. Form matrices A_i for each variable $x_i$by replacing the i-th column of A with the column vector of constants from the right-hand side of the equations.
4. Calculate the determinants det$A_i$for each A_i.
5. Solve for each variable x_i using the formula:
$x_i = \frac{\det(A_i)}{\det(A)}$
6. Interpret the solutions: If det (A) does not equal to 0 the system has a unique solution given by x_i.

Example: Solve the given equation by Cramer's rule

$$x - 3y = -7 \text{ and } 2x - y = 8$$

Solving this three equation where D= $1, D_x = 31, D_y, = 22$and the solution is

$x = \frac{D_1}{D}, y = \frac{D_2}{D}$ Therefore, $x = 6.2$ and $y = 4.4$.

(vi) **Row Echelon form [4]:** This technique was developed over centuries, with significant contributions from mathematicians like Carl Friedrich Gauss [4] in the 18th century. Gauss's work on Gaussian elimination laid the foundation for systematically reducing matrices. Row echelon form emerged as a practical method to streamline calculations, especially in fields like engineering, computer science, and physics, where solving large systems of equations is essential.

Definition: A rectangular matrix is in row echelon form or echelon form if it satisfies following properties.

(i) All zero rows have been moved to the bottom.

(ii) Each leading nonzero element of any row is to the right of the leading nonzero element in the row just above it.

(iii) In each column containing a leading nonzero element, the entries below that leading nonzero element are zeros.

A matrix in row echelon form looks like this:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

Example: Solve system of linear equation using REF

$2x + 3y - z = 1, 4x + 4y - 3z = -2$ and $2x - 3y + 2z = 3$.

Solving this three equation by following operation $R_1 \to \frac{1}{2} R_1$, $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_2$, $R_2 \to -\frac{1}{2} R_2$, $R_3 \to R_3 + 6R_2$ and $R_3 \to \frac{1}{6} R_3$: we get $x = \frac{5}{12}$, $y = \frac{5}{6}$ and $z = \frac{7}{3}$.

(vii) **Reduced Row Echelon Form [4]:** The reduced row echelon form (RREF) is a refined matrix structure used primarily in solving systems of linear equations. It was developed from the concept of Gaussian elimination, named after the German mathematician Carl Friedrich Gauss [4], who pioneered these methods in the early 19th century. Gauss introduced these techniques as part of his work in advancing numerical solutions to systems of equations, helping to streamline complex calculations in mathematics, physics, and engineering. Later, this evolved into RREF to provide an even more simplified matrix form that makes identifying solutions such as unique, infinite, or no solutions clear and systematic.

Definition: A matrix is in reduced echelon form (or reduced row echelon form) if

(i) All zero rows have been moved to the bottom of the matrix.

(ii) Each nonzero row has 1 as its leading nonzero entry, using left-to-right ordering. Each such leading 1 is called a pivot.

(iii) In each column containing a pivot, there are no other nonzero elements.

(iv) The pivot in any row is farther to the right than pivots in rows above.

A matrix in reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: Solve system of linear equation using REF

$2x + 3y - z = 1, 4x + 4y - 3z = -2$ and $2x - 3y + 2z = 3$

Solving this three equation by following operation in augmented matrix ; $R_1 \to \frac{1}{2} R_1$, $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 2R_2$, $R_2 \to -\frac{1}{2} R_2$, $R_3 \to R_3 + 6R_2$ and $R_3 \to \frac{1}{6} R_3$.

Here we can write from first row $x = \frac{5}{12}$ and similarly from second and third row $y = \frac{5}{6}$ and z = = $\frac{7}{3}$.

(viii) ***LU*- Factorization [4]**: *LU* factorization, introduced by mathematician **Alan Turing [4]** in the **1940s**, is a technique in linear algebra for decomposing a square matrix *A* into the product of a **lower triangular matrix** *L* and an **upper triangular matrix** *U*. This method simplifies solving systems of linear equations, inverting matrices, and computing determinants by breaking complex problems into manageable steps. It was developed to make computational processes more efficient, especially for large systems, and remains widely used in numerical methods and engineering.

Steps to Solve Using LU Factorization

1. **Start with the matrix *A*:**
   Ensure the matrix is square (i.e., has equal rows and columns).
2. **Decompose *A*:**
   Find two matrices, *L* (lower triangular) and *U* (upper triangular), such that $A = LU$.
   - *L*: has ones on its diagonal, with non-zero entries below the diagonal.
   - *U*: is upper triangular with non-zero entries above the diagonal.
3. **Forward Substitution:**
   Solve *Ly=b b* is the vector of constants from the system *Ax=b*.
   (Here, *y* is an intermediate vector.)
4. **Backward Substitution:**
   Solve *U x=y* to find the solution vector *x*.
5. **Verify Results:**
   Check if *A=LU* and if the solution *x* satisfies the original equation *Ax=b* .

Example: Solve the following system of equations, making use of the LU-factorization of the preceding coefficient matrix.

$2x_1 + 2x_2 + 3x_3 = 13, 6x_1 + 7x_2 + 14x_3 = 60,\ 14x_1 + 10x_2 - 5x_3 = -23$

Solution: There are two systems to be solved, each having a triangular coefficient matrix. First we have

$$ly = b \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 60 \\ -23 \end{bmatrix}$$

We solve this using forward substitution, getting $y = [13,21,-30]^T$ then we solve second system

$$Ux = y \begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \\ -30 \end{bmatrix}$$

The solution of this system is $x = [3,-4,5]^T$, which we obtained by using back substitution.

Application of System of Linear Equation

Systems of linear equations using the matrix method can be applied to various real-life problems. Some of its application are as follows;

1. In Economic**;** In the field of economics, where businesses need to determine production levels to meet market demands while minimizing costs.

For example: Production planning

A company produces two products, Product A and Product B. Each product requires different amounts of two resources, Resource 1 and Resource 2. The company wants to determine how many units of each product to produce to maximize resource utilization.

- Each unit of Product A requires 2 units of Resource 1 and 1 unit of Resource 2.
- Each unit of Product B requires 1 unit of Resource 1 and 2 units of Resource 2.
- The company has a total of 100 units of Resource 1 and 80 units of Resource 2.
  Find the number of units of Product A (let's call it x) and Product B (let's call it y) to produce, given the constraints on the resources.

Solution; Formulating the System of Equations

Based on the problem, we can set up the following system of linear equations:

1. $2x + y = 100$ (Resource 1 constraint)
2. $x + 2y = 80$ (Resource 2 constraint)

Solving this equation by using inverse matrix method**.** The company should produce 40 units of Product A and 20 units of Product B to utilize the resources optimally.

This example demonstrates how the matrix method can be applied to solve real-life problems involving systems of linear equations.

2. In Physic**;** Another real-life application of systems of linear equations using the matrix method in physics is in solving problems related to mechanical equilibrium. Consider a situation involving forces and torques acting on a rigid body. Let's look at an example involving a static equilibrium of a bridge.

Example [7]: Static Equilibrium of a Bridge

Imagine a simple bridge that is supported by two pillars, A and B, with three forces acting on it due to weights at three different points. The goal is to determine the reactions at the supper

- ***Length of the bridge (L)****: 12 meters*
- **Forces acting on the bridge:**
    - $F_1 = 10kN$ at $x_1 =$ 2 meters from the left end (A)
    - $F2 = 15kN$ at $x_2 =$ 5 meters from the left end (A)
    - $F3 = 20kN$ at $x_3 =$ 8 meters from the left end (A)
      (1 kilo newton (k N) is equal to 1,000 newton (N))
- **Reactions at supports:**
    - Reaction at support A: R_A
    - Reaction at support B: R_B

Apply the Conditions for Static Equilibrium

For the bridge to be in static equilibrium:

1. The sum of all vertical forces must be zero.

2. The sum of all torques (moments) about any point must be zero.

Sum of Vertical Forces: $R_A + R_B - F_1 - F_2 - F_3 = 0$

$R_A + R_B - 10 - 15 - 20 = 0$

$R_A + R_B = 45$ -------(i)

Sum of Torques about A: $-F_1.x - F_2.x - F_3.x + R_B L = 0$

Or, $-10 \times 2 - 15 \times 5 - 20 \times 8 + R_B .12 = 0$

Or, $12R_B = 255$

$R_B = 21.25$

Here we use substitution method

Putting the value of R_B in equation (i) then we get $R_A = 23.75$

Therefore, $R_A = 23.75$ and $R_B = 21.25$

Interpretation: Each support must provide an upward force of 25 kN to counteract the downward force of the uniform load. This ensures the bridge remains stationary and balanced. Thus, 25 kN is the required force at each support to maintain static equilibrium.

This example demonstrates how systems of linear equations can be used to analyze the static equilibrium of a bridge using the matrix method. The same principles can be applied to more complex structures and mechanical systems in engineering and physics.

3. In Chemistry**;** In chemistry, systems of linear equations using the matrix method are often applied to balance chemical equations, analyze reaction networks, and solve problems related to chemical equilibria. Let's explore an example involving the balancing of a chemical reaction using matrix methods.

Example: Balancing a Chemical Equation

Consider the chemical equation

$aH_2 + bO_2 \rightarrow cH_2O$--------(*)

Solution; Here we need to find the coefficient *a*, *b* and *x* that balance the equation

For hydrogen (H); $2a = 2c \Rightarrow 2a - 2c = 0$------(i)

For oxygen (O); $2b = c \Rightarrow 2b - c = 0$-----------(ii)

Here we use substitution methods

If we put *c* = 1 then we get from the equation (i)

$$2a = 2c \Longrightarrow 2a = 2 \Longrightarrow a = 1$$

And then from equation (ii)

$$2b = c \Longrightarrow b = \frac{c}{2} \Longrightarrow b = \frac{1}{2}$$

Now putting their respective value in equation (*) then we get balance chemical equation

$H_2 + \frac{1}{2}O_2 \rightarrow H_2O$

4. In Biology [4]; System of linear system has also wide application in the field of biology.

Example: Feeding Bacteria

A bacteriologist has placed three types bacteria, labeled B_1, B_2 and B_3 in a culture dish, along with the certain quantities of three nutrients, labeled N_1, N_2 and N_3. She knows the amounts of each nutrient that can be consumed by each bacterium in a 24-hour period. These data are collected in a table;

	B_1	B_2	B_3
N_1	4	2	6
N_2	3	1	2
N_3	7	5	2

This table tells us, for example, that bacterium B_1 in one day can consume 4 units of N_1, 3 units of N_2, and 7 units of N_3. How many bacteria of each type can be supported daily by 4200 units of N_1, 1900 units of N_2 and 4700 units of N_3?

Solution: Let x_1, x_2 and x_3 denote the number of bacteria of each type represented in culture. Then the other equation governing the nutrients N_1, N_2 and N_3 are

$$4x_1 + 2x_2 + 6x_3 = 4200$$
$$3x_1 + x_2 + 2x_3 = 1900$$
$$7x_1 + 5x_2 + 2x_3 = 4700$$

The augmented matrix can solve the following operation; $R_1 \to R_1 - R_2$, $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$, $R_2 \to -\frac{1}{2}R_2$, $R_3 \to -\frac{1}{2}R_3$ and $R_3 \to R_3 - R_2$.

Hence the solution is $x(x_1, x_2, x_3) = (200, 500, 400)$.

5. **Statistics and Data Analysis:**

Least Squares Regression: Fitting a linear model to a set of data points to minimize the sum of the squares of the differences between observed and predicted values. This is the most popular and widely used mathematical method of measuring trend. This is the frequently used for the future prediction.

Example: Fit the regression equation of the type $y = a + bx$ for the data given below. Estimate the likely demand when the price is Rs 20.

Price (Rs)(x)	10	12	13	12	16	15
Amount demand (y)	40	38	43	45	37	43

Solution; Let price (Rs) = x

Amount demand = y

Then the regression line is y on x, whose equation is $y = a + bx$

For the values of a and b the normal equation are

$\sum y = na + b\sum x$ -------(i)

$\sum xy = a\sum x + b\sum x^2$------(ii)

Computation of regression :$\sum x = 78$, $\sum y = 246$, $\sum xy = 3192$ and $\sum x^2 = 1038$.

Then putting their respective value in equation (i) and (ii) we get the solution

$x = 20$ and $y = 39.25$.

Conclusion

Matrix methods for solving system of linear equations are essential tools in both theoretical and applied mathematics. Understanding these methods allows for efficient and accurate solution to complex problems across various disciplines. In the field of economics, we can use system of linear equation where businesses need to determine production levels to meet market demands while minimizing costs and so on. In physic system of linear equation is used to analyze complex electrical circuits by applying Kirchhoff's laws and so on. In the field of chemistry, we use system of linear equation for different purpose such as balancing chemical equations, modeling the rates of chemical reactions, predicting the dispersion of pollutants and so on. In biology systems of linear equations model the interactions between different species in an ecosystem, predicting changes in population or bacteria sizes over time and so on. In statistics, multiple regression analysis involves solving systems of linear equations to determine the relationship between multiple independent variables and a dependent variable.

References

Shen, K., Crossley, J. N., & Lun, A. W. C. (1999). *The nine chapters on the mathematical art: Companion and commentary*. Oxford University Press, USA.

Pingree, D. (1983). Brahmagupta, Balabhadra, Pṛthūdaka and Al-Bīrūnī. *Journal of the American Oriental Society*, 353-360.

Boyer, C. B., & Merzbach, U. C. (2011). *A history of mathematics*. John Wiley & Sons.

Ward C. and David K., *Linear Algebra Theory and Application*, Jones and Bartlett Learning, 2014.

Strang, G. (2000). *Linear algebra and its applications*.

Cayley, A. (1858). II. A memoir on the theory of matrices. *Philosophical transactions of the Royal society of London*, (148), 17-37.

Abdulla, S., Nafees, M., El-Borgi, S., & Srinivasa, A. (2023). An integrated e-learning approach for teaching statics to undergraduate engineering students. *International Journal of Mechanical Engineering Education*, 03064190231209332.

https://youtu.be/k-_Z67HFK_A?si=ksZAxiNkl9ymWMDv.