# Relation of Pythagorean and Isosceles Orthogonality with Best approximation in Normed Linear Space

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## Abstract

*In an arbitrary normed space, though the norm not necessarily coming from the inner product space, the notion of orthogonality may be introduced in various ways as suggested by the mathematicians like R.C. James, B. D. Roberts, G. Birkhoff and S.O. Carlsson. We aim to explore the application of orthogonality in normed linear spaces in the best approximation. Hence it has already been proved that Birkhoff orthogonality implies best approximation and best approximation implies Birkhoff orthogonality. Additionally, it has also been proved that in the case of $\varepsilon$ −orthogonality, $\varepsilon$ −best approximation implies $\varepsilon$ −orthogonality and vice-versa. In this article we established relation between Pythagorean orthogonality and best approximation as well as isosceles orthogonality and $\varepsilon$ −best approximation in norned space.*

**Key words:** Best approximation, Birkhoff orthogonality, Pythagorean orthogonality, $\varepsilon$ −best approximation, Isosceles orthogonality.

## Introduction

For any non-empty subset M of X, where X is a normed space, an element $m_0 \in$ M is called best best approximation to x ∈ X from M if ∀ m ∈ M, $\|x-m_0\| \leq \|x-m\|$. The collection of all such elements $m_0 \in$ M which are best approximation to x∈ X is denoted by $P_M$(x). If $P_M$(x) contains at least one element, then the subset M is called a proximal set. If for each x ∈ X has a unique best approximation in M, in that case the set M is called Chebychev set of X. In another word the set M is called Chebychev if $P_M$(x) is singleton (Akramm, 2010).

**Theorem 1.1**. Let M be a subspace of a normed space X,
(i) If x ∈ M, then $P_M$(x) ={x} ( Singer 1974) (ii) If x ∈ cl(M)\ M, then $P_M$(x) = ∅( Akramm 2010)

*Proof. (i) Let* x ∈ M, then d(x, x) = 0 which implies that d(x, M) = 0.
Therefore $P_M$(x) = { x ∈ M : ‖x-y‖= d(x, M)} = { x ∈ M : ‖x-y‖= 0}={x}

(ii) Let x ∈ cl(M)\ M. Then∃ $(x_n)$∈ $M$: $\lim_{n\to\infty} ||x_n - x|| = 0$ implying that d(x, M) = 0.

Therefore $P_M$(x) = { y ∈ M : ‖ x-y‖ = d(x, M)} = { y ∈ M : ‖ x-y‖ = 0} = { y ∈ M : x = y} = ∅. QED.

**Theorem 1.2**. Let X be a normed linear space and M be a subspace of X. Then $m_0 \in P_M$(x) if and only if x- $m_0 \perp_B$ M (Mazaheri & Maalek, 2006).

*Proof*. Let us suppose that $m_0 \in P_M$(x). Put $m_1 = m_0 - \alpha$ m for any fixed m ∈ M and m ∈ **R.** Since $m_0 \in P_M$(x) and $m_1 \in$ M, ‖ x- $m_0|| \leq ||(x - m_1||$ and so ‖ x- $m_0|| \leq ||x - (m_0 - \alpha\, m)||$. Then $||(x - m_0) + \alpha\, m)||$ ,which shows that x- $m_0 \perp_B$ M.

Conversely suppose that, x- $m_0 \perp_B$ M. Then for all $\alpha \in R$ and $m_1 \in$ M,

‖ x- $m_0|| \leq ||x - m_0 + \alpha\, m_1)||$. Let m ∈ M be arbitrary and fixed. Let us take $m_1 = m_0$ – m with $\alpha = 1$, we get ‖ x- $m_0|| \leq ||(x - m||$. Therefore $m_0 \in P_M$(x). QED

**Theorem 1.3.** Let M be a non-empty subset of a normed space X. Then (i) $P_{M+y}$(x+y) = $P_M$(x) +y for every x, y ∈ $R$. (ii) $P_{\alpha M}(\alpha x) = \alpha P_M$(x) for every $x \in X$ and $\alpha \in R$(Holmes 1972) ( Akramm, 2010).

*Proof.* (i) $y_0 \in P_{M+y}(x + y)$ if and only if $y_0 \in M + y$ and $||x + y - y_0|| \leq ||x + y - (m + y)||$

For all m+y∈ $M + y$ if and only if $y_0 - y \in M$ and ‖ $x - (y_0 - y)$ ‖≤ $||x - m||$

For all m∈ $M$ if and only if $y - y_0 \in P_M(x)$ if and only if $y_0 \in P_m(x) + y$.

Therefore $P_{M+y}$(x+y) = $P_M$(x) +y

(ii) It is obvious for $\alpha = 0$, since $P_{\alpha M}(\alpha x) = P_{\{0\}}(0)$ =0 because 0∈ {0} and $\alpha P_M$(x) =0 $P_M$(x) = 0.

So, let us assume that $\alpha \neq 0$. Now $y_0 \in P_{\alpha M}(\alpha x)$ if and only if $m_0 \in \alpha M$ and $\|\alpha x - y_0\| \leq \|\alpha x - \alpha m\|$ $\forall\, m \in M,$ if and only if $|\alpha| \left\|x - \frac{1}{\alpha} y_0\right\| \leq |\alpha| \|x - m\|$.

$\forall\, m \in M,$ if and only if $\frac{1}{\alpha} y_0 \in M$ and $\left\|x - \frac{1}{\alpha} y_0\right\| \leq \|x - m\|$ if and only if $\frac{1}{\alpha} y_0 \in P_m(x)$ if and only if $y_0 \in P_m(x)$. Therefore $P_{\alpha M}(\alpha x) = \alpha P_M$(x). QED

**Existence of best approximation**

**Theorem 1.4.** Let M be a non-empty subset of a normed space X. Then

(i) M is proximal if and only if M+y is proximinal for any given y∈ $X$.

(ii) M is proximinal if and only if $\alpha M$ is proximinal for any given $\alpha \in R - \{0\}$(Holmes, 1972).

**Theorem 1.5.** Let M be a proximal subset of a normed space X. Then M is closed. But the converse may not be true. ( Akramm, 2010&Holmes, 1972).

*Proof.* Suppose that M is a proximinal set and let $(x_n) \subseteq M$ be such that $x_n \to x$. To show M is closed it is sufficient to show that x$\in M$. Since M is proximinal set, so $P_M$(x)$\neq \phi$ . then $\forall\, n \exists\, m_0 \in P_M$(x): $\|x - m_0\| \leq \|x - x_n\|$. But $\lim_{n\to\infty} x_n = x$ implies that $\|x_n - x\| = 0$ and $x = m_0 \in M$. This shows that M is closed. The following example shows that the converse of above theorem may not be true. For this let us take

M= $\{y \in C_2[-1,1]: \int_0^1 y(t)dt = 0\}$. Then M is closed subspace of$C_2[-1,1]$, which is not proximinal.

By linearity of integration M is subspace of $C_2[-1,1]$. For closeness of M let $(y_n) \subset M$ such that$y_n \to y$. We shall show that $y \in M$. By Schwarz's inequality,

$|\int_0^1 y(t)dt| = |\int_0^1 [y(t) - y_n(t)]dt| \leq \int_{-1}^1 |y(t) - y_n(t)|\, dt \leq \left[\int_{-1}^1 |y(t) - y_n(t)|^2\right]^{\frac{1}{2}} [\int_{-1}^1 dt]^{\frac{1}{2}} = \sqrt{2}||y - y_n|| \to 0$. Therefore $\int_0^1 y(t)dt = 0$ implying that $y \in M$ and M is closed.

Now define x on [-1, 1] by x(t) = 1 for all t. Then $x \in C_2[-1,1]$ and for each $y \in M$,

$$||x - y||^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt = \int_{-1}^0 |1 - y(t)|^2 dt + \int_0^1 |1 - y(t)|^2 dt$$

$$= \int_{-1}^0 |1 - y(t)|^2 dt + \int_0^1 [1 - 2y(t) + y^2(t)]dt$$

$$= \int_{-1}^0 |1 - y(t)|^2 dt + 1 + \int_0^1 y^2(t)dt \geq 1$$ and equality holds

if and only if $y(t) = \begin{cases} 1 & for\ -1 \leq t \leq 0 \\ 0 & for\ 0 \leq t \leq 1 \end{cases}$. But y is continuous, so $||x - y||^2 > 1$ and So$||x - y|| > 1$.

Therefore $d(x, M) = \inf\{d(x, y): y \in M\} = \inf\{\|x - y\|: y \in M\} \geq 1$ due to the reason that $\|x - y\| > 1$ for all $y \in M$. Hence $d(x, M) \geq 1$ and $||\, x - y||>1$ for all $y \in M$.

Next given any $0 < \mu < 1$, define $y_\mu$ on [-1, 1] by

$$y_\mu(t) = \begin{cases} 1 & for \quad -1 \le t \le -\mu \\ -\mu^{-1}t & for \quad -\mu < t < 0 \\ 0 & for \quad 0 \le t \le 1 \end{cases}.$$

Clearly $y_\mu$ is continuous on [-1, 1], $y_\mu \in M$ and

$$||x - y_\mu||^2 = \int_{-1}^{1} |x(t) - y_\mu(t)|^2 dt$$

$$= \int_{-1}^{\mu} |1-1|^2\, dt + \int_{-\mu}^{0} |1 + \mu^{-1}t|^2 dt + \int_{0}^{1} |1-0|^2 dt = 1 + \frac{\mu}{3}.$$

Which shows that $d(x, M) \le 1$ and it follows that $d(x, M) = 1 < ||x - y||$ for all $y \in M$. Hence x has no best approximation in M.

*Definition.* (Convex Sets) A subset M of a normed space X is said to be strictly convex, if $\forall\, m_1, m_2 \in M,\ m_1 \ne m_2$, the points $\{\alpha m_1(1-\alpha)m_2 : 0 < \alpha < 1\}$ are interior points of M.

**Theorem1.6.** Let M be a subspace of a normed space X and $x \in X$. Then $P_M(x)$ is convex( Mazaheri&Modarres 2005).

*Proof.* Let $m_1, m_2 \in P_M(x)$ so that $||x - m_1|| \le ||x - m||$ for all $m \in M$
and $||x - m_2|| \le ||x - m||$ for all $m \in M$

Now for 0$\le \mu \le 1$ and any $\in M$ , we have

$$\begin{aligned} ||x - [\mu m_1 + (1-\mu)m_2]|| &= ||x - \mu m_1 - m_2 + \mu m_2|| \\ &= ||x - \mu m_1 - m_2 + \mu m_2 - \mu x + \mu x|| \\ &= ||\mu(x - m_1) + (1-\mu)(x - m_2)|| \\ &\le \mu||x - m_1|| + (1-\mu)||x - m_2|| \\ &\le \mu||x - m|| + (1-\mu)||x - m|| = ||x - m|| \end{aligned}$$

So we have $||x - [\mu m_1 + (1-\mu)m_2]|| \le ||x - m||$, which shows that $x - [\mu m_1 + (1-\mu)m_2] \in P_m(x)$ and hence $P_m(x)$ is convex.

**Theorem1.7.** Let M be a subspace of normed space X. Then $P_m(x)$ is bounded set(Mazaheri 2008 & Akramm 2010)

*Proof.* Let $m_1, m_2 \in P_M(x)$. Then $\forall\, m \in M$, we have $||x - m_1|| \le ||x - m||$ and $||x - m_2|| \le ||x - m||$ . Now fix $m_0 \in M$. Then $||m_1 - m_2|| = ||m_1 - x + x - m_2|| \le ||m_1 - x|| + ||x - m_2|| = 2||x - m_0||$= k for some k = $2||x - m_0||$ > 0. Therefore $P_m(x)$ is bounded.

**Uniqueness of Best Approximation**

**Theorem1.8.** Let M be a subspace of normed space X. Then each $x \in X$ has at most one best approximation in M. In particular, every convex proximinal set is Chebyshev (Akramm 2010).

*Proof.* Let $x \in X$ and $m_1, m_2 \in P_m(x)$. Then $\frac{m_1 + m_2}{2} \in M$ by convexiety of M, and $\frac{1}{2}(m_1 + m_2) \in P_M(x)$. Since $P_M(x)$ is convex,

$$d(x, M) = ||x - \tfrac{1}{2}(m_1 + m_2)|| = ||\tfrac{1}{2}(x - m_1) + \tfrac{1}{2}(x - m_2)||$$

$$\leq \tfrac{1}{2}||(x - m_1)|| + \tfrac{1}{2}||(x - m_2)|| = d(x, m). \text{ So,}$$

$||\frac{1}{2}(x - m_1) + \frac{1}{2}(x - m_2)|| = \frac{1}{2}||(x - m_1)|| + \frac{1}{2}||(x - m_2)||$. Hence equality must hold

throught these inequalities. By the condition of equality in the triangle inequality, $x - m_1 = \rho(x - m_2)$ for some$\rho > 0$. But $||x - m_1|| = d(x, m) = ||x - m_2||$ implies that $\rho = 1$ and hence$m_1 = m_2$. Therefore M is a Chebyshev set.

*Definition.*($\varepsilon$ −orthogonality) For any $\varepsilon > 0$. A vector x is said to be $\varepsilon$ −orthogonal to y if and only if $||x + \lambda y|| + \varepsilon \geq ||x||$ for all scalar $\lambda$ with$|\lambda| \leq 1$. It is denoted by $x \perp_\varepsilon y$. For$M_1$, $M_2 \subseteq X$,

$M_1 \perp_\varepsilon M_2$ if and only if , $m_1 \perp_\varepsilon m_2$ for all $m_1 \in M_1$ and $m_2 \in M_2$.(Akramm, 2010)

*Definition.* Let X be a normed linear space and M be a subset of X and $\epsilon > 0$. A point $m_0 \in M$ is said to be $\epsilon$ −best approximation for $x \in X$ if$\forall m \in M$, $||x - m_0|| \leq ||x - m|| + \varepsilon$. The set of all $\varepsilon$ −best approximation of x in M is denoted by$P_M(x, \varepsilon)$ (Akramm 2010).

**Theorem1.9.** Let M be a subspace of a normed space X. Then $P_M(x, \varepsilon)$ is bounded(Vaezpour, Hassani, Mazaheri 2007).

*Proof.* Let$m_1, m_2 \in P_M(x, \varepsilon)$. Then$\forall m \in M$,

$||x - m_1|| \leq ||x - m|| + \varepsilon$ and $||x - m_2|| \leq ||x - m|| + \varepsilon$. Now

$$||m_1 - m_2|| = ||m_1 - x + x - m_2|| \leq ||m_1 - x|| + ||x - m_2||$$
$$\leq ||x - m|| + \varepsilon + ||x - m|| + \varepsilon$$
$$\leq 2||x - m|| + 2\varepsilon = \mathrm{k}$$

So we have, $||m_1 - m_2|| \leq k$, where k $= 2||x - m|| + 2\varepsilon$, which shows that $P_M(x, \varepsilon)$ is bounded.

*Theorem.* Let M be a subspace of normed linear space X , and $x \in X$.Then $P_M(x, \varepsilon)$.

*Proof.* $m_1, m_2 \in P_M(x, \varepsilon)$ and $0 \leq \mu \leq 1$, then $\forall m \in M$, $\| x - m_1 \| \leq ||x - m|| + \varepsilon$ and $\| x - m_1 \| \leq ||x - m|| + \varepsilon$. Now,

$$||x - [\mu m_1 + (1 - \mu)m_2]|| = ||x - \mu m_1 - m_2 + \mu m_2||$$

$$= ||x - \mu m_1 - m_2 + \mu m_2 - \mu x + \mu x||$$
$$= ||\mu(x - m_1) + (1 - \mu)(x - m_2)||$$
$$\leq \mu\big||x - m_1|\big| + (1 - \mu)||x - m_2||$$
$$\leq \mu\big(\big||x - m|\big| + \varepsilon\big) + (1 - \mu)\big(\big||x - m|\big| + \varepsilon\big) = \big||x - m|\big| + \varepsilon$$

Thus, $\mu m_1 + (1 - \mu)m_2 \in P_M(x, \varepsilon)$. Hence $P_M(x, \varepsilon)$ is convex.

**Theorem1.10.** Let X be a normed linear space and M be a subspace of X, and $\varepsilon > 0$. Then for all $x \in X$, $m_0 \in P_M(x, \varepsilon)$ if and only if $x - m_0 \perp_\varepsilon M$(Vaezpour, Hassani, Mazaheri 2007)

*Proof.* Suppose $m_0 \in P_M(x, \varepsilon)$. Put $m_1 = m_0 - m$ for all $m \in M$ and $|\lambda| \leq 1$. Since $m_0 \in P_M(x, \varepsilon)$ and $m_1 \in M$ , so $\| x - m_0\| \leq \| x - m_0 + \lambda m\| + \varepsilon$ . Therefore $x - m_0 \perp_\varepsilon M$.

Conversely suppose that $x - m_0 \perp_\varepsilon M$. Then for all $\lambda$ with $|\lambda| \leq 1$ and $m_1 \in M$ we have, $\| x - m_0\| \leq \| x - m_0 + \lambda m_1|| + \varepsilon$. For any $m \in M$ and setting $m_1 = m_0 - m$ and $\lambda = 1$, we get $\|x - m_0|| \leq \big||x - m|\big| + \varepsilon$. Therefore $m_0 \in P_M(x, \varepsilon)$.QED

**Main Result**

Theorem 1.2 shows that best approximation implies Birkhoff orthogonality and Birkhoff orthogonality implies best approximation. In the same way Theorem 1.10 gives the same result i.e. $\varepsilon$ −orthogonality implies best $\varepsilon$ −best approximation and conversely.

*Definition.* (Pythagorean Orthogonality) A vector x is said to Pythagorean orthogonal to y if and only if $||x - y||^2 = ||x||^2 + ||y||^2$ (Alonso, Martini, Wu 2012).

The following theorem shows that the Pythagorean orthogonality implies best approximation.

*Theorem2.1* Let X be a normed linear space. If $\forall\, x \in X, \exists m_0 \in M: x - m_0 \perp_P y$, then $m_0 \in P_M(x)$.

*Proof.* Suppose $x - m_0 \perp_P y$ . Then $||x - m_0 - y||^2 = ||x - m_0||^2 + ||y||^2$ and from this equality we can conclude that $||x - m_0||^2 \leq ||x - m_0 - y||^2$ due to the fac that $||y||^2$ is a non-negative quantity and then we obtain$||x - m_0|| \leq ||x - m_0 - y||$.

Now setting $m = m_0 + y$, we get $||x - m_0|| \leq ||x - m||$, which shows that $m_0$ is best approximation to x, i.e. $m_0 \in P_M(x)$.

*Definition.* (Isosceles Orthogonality). A vector x is said to isosceles orthogonal to y if and only if $\| x - y \| = \| x + y \|$ (Alonso, Martini, Wu 2012).

Theorem 2.2 Let M be a subspace of real normed space X. Then $m_0 \in P_M(x, \varepsilon)$ if and only if x is isosceles orthogonal to $m_0$.

*Proof.* Let $\varepsilon > 0$ be arbitrary and$m_0 \in P_M(x, \varepsilon)$.

Then$\forall\, m \in M$, $||x - m_0|| \leq ||x - m|| + \varepsilon$. Setting $m = -m_0$, we get $||x - m_0|| \leq ||x + m_0|| + \varepsilon$ . Since the norm of any vector is a non-negative real number, and

using the property (if $\forall\, \varepsilon > 0, a \le b + \varepsilon, then\ a = b$), we obtain $\left\|x - m_0\right\| = ||x + m_0||$, which shows that x is isosceles orthogonal to $m_0$.
Conversely suppose that $x \perp_I m_0$ and let $\varepsilon > 0$. Then$\left\|x - m_0\right\| = ||x + m_0||$. Since $\varepsilon > 0$ be arbitrary, we can write$\left\|x - m_0\right\| = ||x + m_0|| +\ \varepsilon$. Setting$m = -m_0$, we get
$||x - m_0|| \le ||x - m|| +\ \varepsilon$. Therefore $m_0 \in P_M(x, \varepsilon)$.

## Conclusion

Theorem 2.1 shows that Pythagorean orthogonality implies best approximation and Theorem 2.2 indicates that Isosceles orthogonality implies $\epsilon -$best approximation, and conversely.

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