Some characterization of Atkinson & Fredholm operators

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Abstract

In this paper, we study the Atkinson operator and Fredholm operator. We showed that every Atkinson operator with a nonpositive index could be represented as a finite-dimensional perturbation of a left-invertible Atkinson operator. Similar theorems can then be proved for Atkinson operators with non-negative index and Fredholm operators with zero and arbitrary index. With the help of these theorems, we gave a unique characterization of the index except for a rational factor.

Key words: Atkinson operator, Fredholm operator, index

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Introduction

In 1999, Christoph Schmoeger proved that T is a generalized Fredholm operator if and only if $T = T_1 \oplus T_2$, where T_1 is a Fredholm operator with jump $j(T_1) = 0$ and T_2 is a finite-dimensional nilpotent operator[8]. In 2000, A.G. Ramm defined the Fredholm operator T as one that satisfies the following conditions:

Index stability: The index of T, defined as dim N(T) - dim N(T*), remains constant under small perturbations of T.

Finite-rank perturbations: T can be modified by a compact operator of finite rank without changing its Fredholm property[7].

These conditions imply that the set of Fredholm operators is open in the Banach space L(X, Y) of bounded linear operators, equipped with the operator norm, and the index is locally constant [8].

Some of the tasks in Murphy's C-algebras and operator theory that deal with compact operators and Fredholm theory were solved by A. Delfn in 2018[3]. J. Milošević and D. Cvetković-Ilić also provided necessary and sufficient conditions for the Fredholmness of a sum of two idempotents in 2018[1].

The Fredholm alternative theorem is a classical well-known result in functional analysis and can be found in most texts on the subject. Fredholm has proved in his work (19000, 1903) that the alternative theorem is valid for a certain class of linear integral equations,

$$x(s) - \int_a^b K(s,t)x(t)dt = y(s).$$

These equations are known as Fredholm integral equations of the second kind. where, K(s,t) is a continuous function on [a,b] $\times$ [a,b] and is called the kernel of the integral equation, y(t) is continuous on [a,b], and solutions are only allowed from the space $C[a,b]$.

Throughout this paper, we take X as a linear space over the field Φ of real or complex numbers, $\mathcal{R}$ a saturated and therefore also a normal algebra of operators on X, which contains I.

2. Definitions

2.1. Definition: T$\in \mathcal{R}$ is called *Atkinson Operator*(relative to $\mathcal{R}$), if $\hat{T}$ is left or right invertible.

2.2. Definition: $K \in L(X)$ is known as of *finite rank* or of finite dimensional if

dim B(K) $< \infty$[7].

Or

A linear map T: X $\longrightarrow$ Y is said to be of finite rank if it is continuous and its image is a finite dimensional space [5].

2.3. Definition: Let X and Y be Banach spaces. A linear operator T from X to Y is called a *Fredholm Operator* if **(i)** T is closed. **(ii)** The Domain of T is dense in X.

(iii) $\sigma(T)$, the dimension of the null space N(T) of T is finite.

(iv) The range of T is closed in y. (v)B(T), the co-dimension of R(T) in y is finite[9].

OR

T$\in \mathcal{R}$ is called Fredholm Operator or σ – transformation (relative to $\mathcal{R}$), if$\hat{T}$ is invertible[10].

2.4. Definition: If T $\in$ L(X) and if at least one of the defects $\alpha(T), \beta(T)$ is finite, then ind(T) = $\alpha(T) - \beta(T)$ is called the *index of T* . ind(T) is + ∞ if and only if $\alpha(T) = \infty$ and $\beta(T) < \infty$, ind(T) = - ∞ if and only if $\alpha(T) < \infty$ and $\beta(T) = \infty$. We then say that T possesses an index[4,8].

2.5. Definition: Let A be an algebra with a unit e. An element x of A is called regular if there is an element b of A such that xbx = x[2].

3. Lemmas/Theorems

3.1. Lemma: If $\mathcal{R}$ is an operator algebra on X containing I and T $\in \mathcal{R}$ is relatively $\mathcal{R}$ - regular, then there exist S $\in \mathcal{R}$ such that T S is a projection on B(T). I-ST is a projection on N(T). If S is so chosen that STS = S then E = B(T)$\oplus$N(S), where TS is the projection of X onto B(T) along N(S).

Proof: Since T ∈ $\mathcal{R}$ is relatively - regular, then there exist S ∈ $\mathcal{R}$ such that T ST = T; and so T S and S T are projections. Also, N(T) ⊂ N(ST) ⊂ N(TST) = N(T)

and B(T) ⊃ B(TS) ⊃ B(TST) = B(T). So, N(T) = N(ST) and B(T) = B(TS) hence TS is a projection of X onto B(T) and I – ST is a projection of X onto N(T).

Hence, X = B(TS)⨁N(TS) = B(T)⨁N(TS), ……………………… (A)

Moreover, if STS = S, then from N(S) ⊂ N(TS) ⊂ N(STS) = N(S) and from (A), we have,

X = B(T)⨁N(S).

3.2. Lemma: If T ∈ $\mathcal{R}$ is relatively regular, therefore TST = T for some S ∈ $\mathcal{R}$, then TS and ST are idempotent elements of $\mathcal{R}$. If $\mathcal{R}$ possesses a unit element e, then ST = eP_1 - and TS = e - P_2 holds with the idempotent elements P_1, P_2, given by P_1 = e – ST, P_2 = e – TS.

Proof: From the relative regularity of T it follows that: $(TS)^2$ = T S T S = T S and $(ST)^2$ = S T S T = S T ; from which it follows that P_1, P_2 are idempotent.

3.3. Theorem:

(a) $a_{\alpha,} a_\beta$ and $\sum$ are semi-groups.

(b) If S T ∈ $a_{\alpha,}$, then T ∈ $a_{\alpha,}$; in particular, if T^n ∈ $a_{\alpha,}$ for n∈ N then T∈ $a_{\alpha,}$.

(c) If S T ∈ a_β, then S ∈ a_β; in particular, if T^n ∈ a_β for n∈ N then T∈ a_β.

(d) If S T ∈ $a_{\alpha,}$ and T∈ $\sum$, then T ∈ $a_{\alpha,}$.

(e) If S T ∈ a_β and S ∈ $\sum$, then T∈ a_β.

(f) If S T ∈ $\sum$ and T∈ $\sum$, then S ∈ $\sum$.

(g) If S T ∈ $\sum$ and S ∈ $\sum$, then T ∈ $\sum$.

(h) If T ∈ $a_\alpha - a_\beta$ resp, $a_\beta - a_\alpha$ resp. $\sum$ and K ∈ $\mathfrak{J}(\mathcal{R})$, then T +K lies in $a_\alpha - a_\beta$resp, $a_\beta - a_\alpha$ resp. $\sum$.

The proofs follow from the definitions (2.1 & 2.3)[10]

3.4. Theorem (Index theorem): If S,T ∈ L(X) are operators with finite index, then ST has also a finite index and we have, ind(ST) = ind (S) + ind (T).

Proof: For S,T∈ $\sum(L(X)$, we have ST∈ $\sum(L(X)$, we prove the index equation.

If N(ST) = N(T)⊕A, then the operator T is injective on A and A ≅ T(A) =T(A⊕ N(T) = T(N(ST). We then prove that T(N(ST)) = T(X)∩ N(S), in fact if y∈ T(N(S T)), then y = Tx, x ∈ N(ST) or Tx ∈ N(S) and hence y ∈ T(X)∩N(S).

Conversely, if y ∈ T(X)∩N(S), then y = Tx, Tx∈N(S) and so y ∈ T(N(ST)). We then obtain

$\alpha(ST) = \dim N(ST) = \dim A + \dim N(T) = \dim(T(X) \cap N(S)) + \alpha(T)$. ....(i)

We can write the vector space X as the direct sum

$X = X_1 \oplus T(X) \cap N(S) \oplus X_2 \oplus X_3$, ...(ii)

where $T(X) = X_1 \oplus T(X) \cap N(S)$ and $N(S) = S(X) \cap N(S) \oplus X_2$.

Since $S(N) = S(X_1) \oplus S(X_3)$, X can also be written as

$X = S(X_1) \oplus S(X_3) \oplus X_4$...(iii)

Since by (ii) $S(X_1) = ST(X)$, we have $\beta(ST) = \dim S(X_3) + \dim X_4$.

From (ii) we obtain further: $S(X_3) \cong X_3$, so $\dim S(X_3) + \dim X_3$.

Since by (iii), $\dim X_4$ = co-dim $S(X) = \beta(S)$. $\beta(ST) = \dim X_3 + \beta(S)$.

From (ii) it follows that

$\dim(X_2 + X_3) = \dim X_2 + \dim X_3$ = co-dim $T(X) = \beta(T)$, so $\dim X_3 = \beta(T) - \dim X_2$ and

$\beta(ST) = \beta(S) + \beta(T) - \dim X_2$. ....(iv)

From (ii) it follows that $\dim (T(X) \cap N(S)) = \dim N(S) - \dim X_2$ and by (i)

$\alpha(ST) = \alpha(S) + \alpha(T) - \dim X_2$. Then we have

ind $ST = \alpha(ST) - \beta(ST) = \alpha(S) - \beta(S) + \alpha(T) - \beta(T) = \text{ind } S + \text{ind } T$.

We now wish to generalize the index theorem to Atkinson operators.

3.5. Theorem (Generalized index theorem): If $S, T \in L(X)$ are operators with finite null defects resp. finite image defects then ind (S T) = ind (S) + ind (T)[10].

3.6. Theorem: If $T \in L(X)$ has an index and $K \in \sum(X)$ then ind(T +K) = ind(T).

Proof: If $T \in L(X)$ has a finite index, then there exist $S \in L(X)$ and $F \in \mathfrak{I}(X)$ such that $ST = I - F$. We then have ind(S) = - ind(T). Since $S(T + K) = ST + SK = I - F + SK = I - F_1$, $F_1 \in \sum(X)$ and $T + K \in \sum(L(X))$(b), we have, ind(S) + ind(T + K) = ind(I - F_1) = 0,

∴ ind(T + K) = - ind(S) = ind(T).

3.7. Theorems: $T \in \mathcal{R}$ is an Atkinson operator with $\alpha(T) \leq \beta(T)$, i.e. ind(T)≤ 0 if $T = R + K$, where $R \in \mathcal{R}$ possesses left inverse $S \in \mathcal{R}$ and $K \in \mathfrak{I}(\mathfrak{R})$.

Proof: By (Lemma-3.1) for $T \in a(\mathcal{R})$ there exist projections $P \in \mathcal{R}$ onto N(T) and $Q \in \mathcal{R}$ onto a complement space to B(T). Since $\alpha(T) \leq \beta(T)$ we have $\alpha(T) < \infty$, so $P \in \mathfrak{I}(\mathfrak{R})$. If $\alpha(T) = 0$, then we can choose R = T and K = 0. Then R is relatively $\mathcal{R}$ - regular and hence

R possesses a left inverse in $\mathcal{R}$. SO, we can let $\alpha(T) > 0$. If $\{x_1 x_2, \dots, x_n\}$ is a basis of N(T), then since $\alpha(T) \leq \beta(T)$, there exist a linearly independent set, $\{y_1 y_2, \dots, y_n\}$ in B(Q). Since $\mathcal{R}$ is saturated, there exist S$\in \mathcal{R}$ with S$x_i = y_i, i = 1,2, \dots, n.$

We then set K = QSP and R =T – K, then since Q$\in \mathfrak{J}(\mathfrak{R})$ we have K $\in \mathfrak{J}(\mathfrak{R})$. It follows that R$\in a(\mathcal{R})$, therefore R is relatively $\mathcal{R}$ - regular. If we also show that $\beta(\mathcal{R}) = 0$, then R possesses a right inverse in $\mathcal{R}$. If y$\in x$, then

y = $\sum_{i=1}^{n} \alpha_i y_i + T x_o, x_o \in X.$

Then there exist U$\in \mathcal{R}$ with U $y_i = x_i$, 1$\leq i \leq$ n. If we put z = UKx_o, then since K$x_o \in B(Q)$ we have K$x_o = \beta_1 y_1 + \cdots + \beta_n y_n.$

So, z = UKx_o= $\sum_{i=1}^{n} \beta_i x_i \in N(T)$ and Kz = QSP($\beta_1 x_1 + \cdots + \beta_n x_n$) = $\sum_{i=1}^{n} \beta_i y_i = K x_i,$

If we put u = $x_o - \sum_{i=1}^{n} \alpha_i x_i - z$, then we have

Ru = Tu – Ku = T$x_o - K x_o + \sum_{i=1}^{n} \alpha_i K x_i + Kz$ = Tx_o +$\sum_{i=1}^{n} \alpha_i y_i$= y.

Thus, $\beta(T) = 0$.

In the following theorem we show that index is uniquely determined except for rational factor.

3.8. Theorem: Let d be an integral valued function on $\sum$ with the following properties:

(a) d(ST) = d(S) + d(T)

(b) d(S + T) = d(T), provided K$\in \mathfrak{J}(\mathcal{R})$

(c) d(T) = 0 if T possesses an inverse in $\mathcal{R}$.

Then there exists a rational number r such that d(T) = r ind(T) for all T$\in \sum$.

Proof: If $\sum_n$ = {T$\in \sum$: ind(T) = n}, then for T$\in \sum_n$, we have T = T_n R + K [10] and by the Index theorem and the above index properties (i), (ii), we also have d(T) = d(T_n); hence d is constant on $\sum_n$. We wish to denote by d resp. $\widetilde{ind}$ maps of the set $\{\sum_n\}$, defined as follows:

$\tilde{d}(E_n)$ = d(T_n), $T_n \in \sum_n$

$\widetilde{ind}(\sum_n)$ = ind(T_n) = n, $T_n \in \sum_n$:

The $\widetilde{ind}$ex is one-one. On account of the index theorem we have $T_n T_m \in \sum_{n+m}$, from which it follows that h = $\tilde{d}$ o $\widetilde{ind}^{-1}$ is a homomorphism of the subgroup

$\mathcal{U} = \{$ ind(T) : T$\in \sum$ } of the additive group Z of integers into $\mathbb{Z}$. In fact if n, m $\in \mathcal{U}$ and hence

$h(n+m) = \tilde{d}(\widetilde{ind}^{-1}(n+m)) = \tilde{d}(\sum_{n+m})$ = d($T_n T_m$) = $d(T_n) + d(T_m)$ = $\tilde{d}(\sum_n) + \tilde{d}(\sum_m)$

$= \tilde{d}(\widetilde{ind}^{-1}(n)) + \tilde{d}(\widetilde{ind}^{-1}(m)) = h(n) + h(m).$

In $\mathcal{U}$ there exists a smallest positive number n_o such that u = {kn_o: k ∈ $\mathbb{Z}$}. For n ∈ $\mathcal{U}$, we have n = k(n) n_o. If T lies in $\sum$, then T∈ $\sum$, for an n = K(n) n_o ∈ $\mathcal{U}$ and it follows that

d(T) = $\tilde{d}(E_n)$ = (h o $\widetilde{ind}$)($\sum_n$) = h($\widetilde{ind}$)($\sum_n$) =h(n) = h(K(n)(n_o)) = K(n) h(n_o) = $\frac{h(n_o)}{n_o}$ n $=\frac{h(n)}{n_o}$ ind(T) = r ind(T) , where r = $\frac{h(n)}{n_o}$.

If $\mathcal{U} = \mathbb{Z}$, therefore $n_o = 1$, then d is an integral multiple of ind.

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