Received Date:3rd September, 2024
Revision Date: 15th September, 2024
Accepted Date:24th September, 2024

A Numerical Technique of Modelling Mortality Rate Intensity for Life Insurance Implementation

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Abstract -- Numerical estimation technique of mortality intensities is integrated into actuarial modelling to ensure cost effective approximations and ascertain benchmarks against which accuracy of mortality table production is verified for applications in life insurance product development. However, some of the governing functional mortality laws analytically formulated as an integral part of underwriting process in life table constructions may not readily have closed form solutions because they are completely intractable. This accounts for why many extant actuarial literatures avoid investigating their asymptotic properties at extreme ages. In this study, the method of successive differencing was deployed to even out errors associated with source vital statistics in order to track the level of any momentous change. The objectives of this study are to produce the instantaneous mortality rate intensities using the Generalized Makeham's mortality law, to confirm the age at which the probability of death will be 1 and to compute the curve of death. Computational evidence from our result shows that the probability of death for a life aged x approaches 1 at age 120 hence the model supports longevity up to 120 as the omega age. The mortality rate computed is based on the age attained by the insured such that the rate for any given policy at a defined age is independent of the year an insurance policy is incepted

Keywords -- Generalised Makeham, curve of deaths, functional mortality, longevity, omega age

Introduction

Mortality dynamics deals with human mortality to address issues on life contingencies from birth till death and provides insight for human population over demographic variables such as age, sex and year. The main source of mortality data is usually obtained from population census or vital statistics. Since the data obviously shows the mortality and longevity trends of cohorts concerning underlying information, the information on human mortality data is provided in life table. The mortality table of a defined population over a specific year is normally specified by the decreasing function l_x specifying the number of lives surviving to age x where $x = 0,1,2,3,\ldots\Omega$ and Ω is a natural upper bound in the life table and l_x satisfying the conditions that $l_x > 0$ and $l_{x+\Omega} = 0$.

Mortality is always conducted on a closed group whose birth dates are the same. The cohort of individuals born in year ξ is subsequently longitudinally observed such that the number of lives surviving to year ξ is computed through the following years for the same cohort to obtain the function l_x. However, life table could be considered in form of period life table where a period represents a time interval. In the time interval, mortality table assumes that a hypothetical cohort of individuals experiencing contingency events is observed only for a chosen period of time. Because of the very long period that either approach usually takes, most developing economies such as Nigeria do not have a mortality table of their own because it is very difficult to longitudinally observe the chosen number of lives in a defined cohort.

The Nigerian domestic insurance industry especially life insurance as well as pension subsectors operates in an extreme, volatile, uncertain, complex and ambiguous business environment (EVUCA). Part of the problem till today is the inappropriate use and interpretation of mortality table to generate underwriting results and lack of actuarial audit process.

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The actuarial audit should is usually conducted to review the accuracy of the financial statements associated to life insurance schemes including the balance sheet and income statement, ensure strict adherence to generally accepted accounting principles or international financial reporting standards for annuity and pension plan accounting, enforce compliance with applicable life and pension plan regulations such as those set by the regulators NICOM and PENCOM, assess the adequacy of the pension plan's funding level to meet its future obligations, evaluate the reasonableness of the actuarial assumptions applied in the pension plan's calculations such as discount rates, salary projections and mortality rates.

The dearth of deep and requisite knowledge in the area of mortality modelling, analysis and interpretation has become a big issue plaguing the industry. Currently exotic life tables which are not the true reflection of the operating environment dominate the Nigerian industry without the knowledge of formulation and interpretation. The consequence is that the underwriting results obtained from such tables are apparently invalid. Consequently, the system therefore seems isolated from the advanced economies which apply feasible sophisticated models to generate life tables for their insurance industry. In this study, we generate constrained mortality intensities graduated in section three. We are restricted by the chosen mortality graduation technique given the need to address the problem of life table generation from the available data.

The generation of mortality rates through analytical technique is pursued because of the need to develop functional form parsimoniously for use in survival probability, life insurance pricing, policy values computations and to address some underwriting problems associated with EVUCA. One of the main advantages of generating mortality rate by mathematical formula is that the resulting graduation will be smooth. Parsimonious functions are forces of mortality which describe age pattern of death rates parametrically and are important for smoothing the predicted rates over time.

Extant study such as [1], confirms that age specific functional mortality intensities have been analytically developed expressing age pattern of mortality in continuous form. The essence is to address issues connected with the risk of parameter estimation which are inadvertently ignored in mortality analysis because life office managers are unaware of the exposure to these risk of mortality mis-estimation and hence the need for robust estimation and modelling technique of influential human mortality intensities.

Consequently, numerical estimation technique of mortality intensities is integrated into actuarial modelling to ensure cost effective approximations and ascertain benchmarks against which numerical accuracy of mortality table production is verified for applications in life insurance product development. However, some of the governing functional mortality laws analytically formulated as an integral part of underwriting process in life table constructions may not readily have closed form solutions because they are completely intractable. This accounts for the reason why many actuarial literatures avoid investigating their asymptotic properties at extreme ages for instance, the Heligman-Pollard forth law studied in [2].

The intensity of the governing mortality laws changes at each instant of time. In order to track the level of this momentous change, numerical techniques are adopted to examine the underlying life table functions which seem analytically intractable to address inadequacies in mortality and interest rate intensity modelling.

The numerical procedure of modelling begins with the evolution of an influential underlying mortality model specifying the relevant assumptions of the underwriting framework under series of functionally constructed estimations to obtain meaningfully interpretable mortality trend. Consequently, the entire study seeks numerical techniques including mathematical special functions to address computational problems commonly plaguing life insurance product pricing quantification. The challenge of deploying the appropriate numerical techniques to evaluate some of these functional actuarial laws marks the beginning of the motivation for the evolution of this study.

Classical life contingency domain assumes that the insurance market is functionally deterministic such that all insurance asset will experience same level of dynamics that describe the forces which influence the pricing of life insurance products. Premiums are estimated such that income and loses balance up in the mean through the classical law of actuarial equivalence where the actuarial present value of benefits Φ_{ξ} at time ξ equals the actuarial present value of premiums π_{ξ}, that is

$$\int_0^{\Omega} \Phi_{\xi} e^{-\delta\xi} ({}_{\xi}p_x) \mu_{x+\xi} d\xi = \int_0^{\Omega} \pi_{\xi} e^{-\delta\xi} ({}_{\xi}p_x) d\xi \tag{1}$$

Following definitions in [3,4], life assurance policies can be mathematically defined to mean the vector $(\pi_{\xi}, \Phi_{\xi})_{\xi \in \{0,1,2,2,\dots,T\}}$ of pairs $\left(\pi_{\xi}, \Phi_{\xi}\right)$ of ξ life insurance portfolios. For any $\xi \in \{0,1,2,\ldots,T\}$, $\pi_\xi \in \boldsymbol{R}^+$ is the premium payable to the life office while $\Phi_\xi \in \boldsymbol{R}^+$ is the claim against the life office at some definite time ξ. This is to mitigate against both the unfavourable economic effect of random contingencies and financial loss resulting from sudden death of the scheme holder. The death benefits of a specified value in consideration of fully paid premium are usually provided to a named beneficiary who will be shielded from the economic impact resulting from the sudden death of the assured life.

From the foregoing, a life insurance scheme is completely specified by a set of functions $\{T_x; F_{T_x}(s); \zeta; X; \Phi_X; \pi_s\}$ where $F_{T_x}(s)$ is the distribution function of the random life time $T_x: \Omega \to (0,\infty)$; $\zeta \in (0, s_{max}())$ is the terminal time of the insurance policy where $s\{\zeta \in \boldsymbol{R}^+ | F_{T_x}(\zeta) \le 1\}_{max}$; $X = \min(T_x, \zeta)$ is the random time of benefit; Φ_X is the benefits payable to the insured at time X and π_s is the premium function.

In practice, the life office cannot accurately predict the time of death of the scheme holder and thus, following [5,6,7], the function $T_x = X - x$ with continuous distribution within the interval $[0,\infty)$ is used to denote the time until death called complete future life time of the insured aged x where X is the age at death of a new born baby. It is then assumed that the insured will die at some unknown age $T_x + x$. In [8], it is observed that that the distribution of $T: \mho \to (0,\infty)$ will be continuous provided that the death density is not centered at a single age where $\mho$ is a set of contingencies such as death, sickness or disability.

In practice, when a life insurance scheme is bought, the time which elapsed from policy inception to the death of the policyholder defines the future life time random variable T_x. The function $S_{T_x}(s)$; $F_{T_x}(s)$ and other related mortality distributions constitute the basis of life insurance workflow system. All functions associated with T_x assume values within the interval $[0, s_{\max})$. If the limiting age is known, then T_x takes values within $[0, \Omega - x]$ since the insured has survived age x after time 0. The integrally whole number of years lived by the insured is the curtate expectation of life defined as $K_x = [T_x]$, the integer part of $[T_x]$

Following [9,10,11,12], μ_{x+s} measures the intensity of death at an instant given that the insured survives up to age $x+s$ where μ_{x+s} represents a smooth, monotone increasing function of age and becomes more steeply increasing at senescent ages where μ_x is obtained from the Generalized Makeham's law. Consequently, the probability function of surviving to age $x+1$ is defined as

$$ {}_1p_{x+0} = \exp\left(-\int_0^1 \mu_{x+\zeta} d\zeta\right) \tag{2}$$

where $\int_0^1 \mu_{x+\zeta} d\zeta$ is the severity to die function. Further, we can let

$$\alpha(x,y) = \frac{\xi(y)}{\xi(x)} \tag{3}$$

where $\xi(.)$ is a differentiable function, $\alpha(x,y)$ can be re-expressed as

$$\frac{\xi(V)}{\xi(U)} = \frac{e^{\ln\xi(V)}}{e^{\ln\xi(U)}} = e^{\ln\xi(V) - \ln\xi(U)} = \exp\left\{\int_U^V d\ln\xi(t)\right\} = \exp\left\{\int_U^V \frac{\xi'(t)}{\xi(t)} dt\right\} = \alpha(U,V) \quad (1.26)\ (4)$$

Using the substitution $\xi(t) = l_{x+t}$ then (4) is transformed as

$$\alpha(t,x) = \exp\int_x^{x+t} \frac{l'_{x+\zeta}}{l_{x+\zeta}} d\zeta = \exp\left(-\int_x^{x+t} \mu_{x+\zeta}\right) = ({}_tp_x) \tag{5}$$

Underwriting commences when a life buys life insurance scheme at time S_0. The life office will pay the assured a cash sum at defined period contingent on death or survival. However, the benefit and the premium streams in life insurance are part of the underwriting cash flows by reason of the evolving mortality risk. When payment of benefit is contingent on survival, it is the survival benefit but where a sum assured is paid at the moment of death, it becomes death benefit. For death benefit contracts, the time of death is significant in ascertaining the payment of the sum assured. In [5] Bowers et al (1997), the death benefits of a continuous life insurance scheme issued on a life aged x can be theoretically summarized in the form

$$\Phi = \begin{cases} 0 \quad if \quad 0 \le T_x \le n \\ \alpha_1 + \alpha_2 e^{-\delta T} \quad if \quad n \le T_x \le n+h \\ \alpha_3 \quad if \quad T_x \ge n+h \end{cases} \tag{6}$$

where $\{\alpha_i;\ i = 1,2,3\}$ are real constants and $\{n;h\}$ are non-negative integers.

Death benefits is computed on an assumption that the sum assured will be paid at the end of the year of death, the assumption of which is to ascertain that death or survival probabilities up to the end of a policy year is reasonably calculated from mortality data. The death benefits of a scheme holder who purchases a life insurance scheme can be paid at the moment of death if the scheme has been incepted on a fully continuous term.

Let

$$h(x)=\int_0^x \mu_s ds \quad (7)$$

The definition of mortality rate intensity from (7) implies

$$\int_0^x \mu_s ds=-\int_0^x d\ln l_s \qquad (7a)$$

$$\int_0^x \mu_s ds=-\left(\ln l_x-\ln l_0\right) \qquad (8)$$

Assuming $l_0=1$

$$\int_0^x \mu_s ds=-\left(\ln l_x-\ln 1\right)=-\ln l_x \qquad (9)$$

The entropy of mortality function is given by

$$\mathbf{H}=\frac{\int_0^\infty h(x)l_x dx}{\int_0^\infty l_x dx} \qquad (10)$$

$$\mathbf{H}=\frac{\int_0^\infty\left(\int_0^x \mu_s ds\right)l_x dx}{\int_0^\infty l_x dx}=\frac{-\int_0^\infty\left(\ln l_x\right)l_x dx}{\int_0^\infty l_x dx} \qquad (11)$$

Squeeze theorem for survival function.

Let ${}_s p_x$ be differentiable for all $s>0$ and if $\frac{d}{ds}\left({}_s p_x\right)\le 0$ for all $s>0$. If $\mathbf{E}\left[\left(T_x\right)^k\right]$ exists then $\lim_{k\to\infty}\left\{s^k-\left({}_s q_x s^k\right)\right\}=0$ and $s^k{}_s p_x\to 0$ for some $k>0$ where $\mathbf{E}$ is the expectation operator.

Proof

Probabilities of death ${}_s q_x$ and survival ${}_s p_x$ sum up to one

$$f_{T(x)}(s)=\frac{d}{ds}\left(-{}_s p_x\right)=\left(1-{}_s q_x\right)\mu_{x+s}=\mu_{x+s}\left({}_s p_x\right) \qquad (12)$$

We recognize that

$$\left(s^k-{}_s q_x s^k\right)=s^k\left(1-{}_s q_x\right)=s^k\left({}_s p_x\right) \qquad (13)$$

By definition, the kth moment of T_x is given by

$$\mathbf{E}\left[\left(T_x\right)^k\right]=\int_0^\Omega s^k f_{T_x}(s)ds \qquad (14)$$

$$\mathbf{E}\left[\left(T_x\right)^k\right]=\int_0^\Omega s^k\left(\mu_{x+s}-{}_s q_x\mu_{x+s}\right)ds \qquad (15)$$

$$\mathbf{E}\left[\left(T_x\right)^k\right]=\int_0^s \zeta^k\left(\mu_{x+\zeta}-{}_\zeta q_x\mu_{x+\zeta}\right)d\zeta+\int_s^\Omega \zeta^k\left(\mu_{x+\zeta}-{}_\zeta q_x\mu_{x+\zeta}\right)d\zeta \quad (16)$$

$$\mathbf{E}\left[\left(T_x\right)^k\right]=\int_0^s \zeta^k\mu_{x+\zeta}\left({}_\zeta p_x\right)d\zeta+\int_s^\Omega \zeta^k\mu_{x+\zeta}\left({}_\zeta p_x\right)d\zeta \qquad (17a)$$

$$\Rightarrow \mathbf{E}\left[\left(T_x\right)^k\right]=\int_0^s \zeta^k f_{T_x}(\zeta)d\zeta+\int_s^\Omega \zeta^k f_{T_x}(\zeta)d\zeta \qquad (17b)$$

Consequently,

$$\mathbf{E}\left[\left(T_x\right)^k\right]-\int_0^s \zeta^k f_{T_x}(\zeta)d\zeta\ge\int_s^\Omega \zeta^k f_{T_x}(\zeta)d\zeta \qquad (18)$$

Evaluating the RHS of (12), we have (19)

Consequently, **LHS** of equation (19) implies that

$$\int_s^\Omega \zeta^k f_{T_x}(\zeta)d\zeta\ge\left({}_s p_x\right)\times\zeta^s \qquad (20)$$

Therefore, (20) becomes

$$\mathbf{E}\left[\left(T_x\right)^k\right]-\int_0^s \zeta^k\left(\mu_{x+\zeta}-{}_\zeta q_x\mu_{x+\zeta}\right)d\zeta\ge s^k{}_s p_x \qquad (21)$$

$$\mathbf{E}\left[\left(T_x\right)^k\right]-\int_0^s \zeta^k\mu_{x+\zeta}\left({}_\zeta p_x\right)d\zeta\ge s^k{}_s p_x \qquad (22)$$

Taking the limit of (22) as $s\to\Omega$, we have

$$\mathbf{E}\left[\left(T_x\right)^k\right]=\lim_{s\to\Omega}\int_0^s \zeta^k{}_\zeta p_x\mu_{x+\zeta}d\zeta=\int_0^\Omega \zeta^k{}_\zeta p_x\mu_{x+\zeta}d\zeta \qquad (23)$$

Using (23), the LHS of (22) becomes zero.

and consequently $0\ge s^k{}_s p_x$. However, since the time $s>0$, we must have that $s^k{}_s p_x\ge 0$.

The squeeze theorem states that if $f(\theta)\le g(\theta)\le h(\theta)$ for all θ and if at some point $\theta=s$, $f(\theta)=h(\theta)$, then $g(\theta)$ will also be equal to each of them. Therefore, $0\le s^k{}_s p_x\le 0$ and by squeeze's theorem $s^k{}_s p_x=0$. This completes the proof.

Although the Committee of Mortality Investigation as observed in [13] has recommended the Generalized Makeham's law to prepare life tables, most mortality rates have been computed based on based on $GM(0,2)$ and $GM(1,2)$. It is observed to the best of our knowledge that the Generalized Makeham's mortality law considered has not been optimally employed to compute instantaneous mortality rates and the method of successive differencing has not been applied in modelling the Generalized Makeham's law, hence this study intends to employ this new method of successive iterations involving logarithmic transformations to estimate its mortality parameters which will directly consider human age x and the survival function l_x at that age.

Axiomatic foundation of mortality measures

Since T_x is a non-negative random variable at time ξ, the survival function $S_{T_x}(\xi)$ represents the probability that a survival time of a life is greater than the time ξ. Thus

$$S_{T_x}(\xi) = \mathbf{P}(T_x > \xi) = 1 - \mathbf{P}(T_x \le \xi) = 1 - F_{T_x}(\xi) \quad (23a)$$

where

$$\mathbf{P}(T_x \le \xi) = F_{T_x}(\xi) \quad (23b)$$

and

$$\mathbf{P}(T_x \le \xi) = \int_0^{\xi} f_{T_x}(t)\,dt \quad (23c)$$

Since the death density $f_{T_x}(\xi)$ is the derivative of the distribution function $F_{T_x}(\xi)$, then,

$$dF_{T_x}(\xi) = -dS_{T_x}(\xi) = f_{T_x}(\xi)\,d\xi \quad (23d)$$

Such that the following conditions are satisfied

$S_{T_x}(0) = 1$ and $S_{T_x}(\Omega) = 0$

Suppose that $\Delta\xi$ is an infinitesimally small period of time, then

$$\mu(\xi, x) = \lim_{\Delta\xi \to 0}\left[\frac{\mathbf{P}(X \le \xi + \Delta\xi \mid X > \xi)}{\Delta\xi}\right] \quad (23e)$$

$$\mu(\xi, x) = \lim_{\Delta\xi \to 0}\left[\frac{\mathbf{P}(T_x \le \xi)}{\xi}\right] \quad (23f)$$

The non-negative function μ_x is therefore characterized by the following properties. For all $x > 0$, $\mu_x \ge 0$ and that $\int_0^{\infty} \mu_x dx = \infty$

$$\mathbf{P}(T_x \le \xi) = \frac{\mathbf{P}(x < T_x \le x + \xi)}{\mathbf{P}(T_x > 0)} = \frac{F_{T_x}(\xi + x) - F_{T_x}(x)}{S_{T_x}(x)} \quad (23g)$$

$$\mu_x = \lim_{\xi \to 0}\frac{F_{T_x}(\xi + x) - F_{T_x}(x)}{\xi \times S_{T_x}(x)} = \lim_{\xi \to 0}\frac{1}{S_{T_x}(x)} \times \frac{F_{T_x}(\xi + x) - F_{T_x}(x)}{\xi} \quad (23h)$$

$$\mu_x = \frac{1}{S_{T_x}(x)} \times \frac{d}{dx}F_{T_x}(x) = \frac{-1}{S_{T_x}(x)} \times \frac{d}{dx}S_{T_x}(x) \quad (23i)$$

$$\mu_x = \frac{-\frac{d}{dx}S_{T_x}(x)}{S_{T_x}(x)} = -\frac{d}{dx}\log_e S_{T_x}(x) \quad (23j)$$

$$\int_0^x \mu_y dy = -S_{T_x}(x) \Rightarrow S_{T_x}(x) = \exp\left(-\int_0^x \mu_y dy\right) \quad (23k)$$

Therefore, for any $\xi > 0$,

$$\mu(\xi, x) = \frac{F_{T_x}(x)}{S_{T_x}(x)} = \frac{f_{T_x}(x)}{1 - F_{T_x}(x)} = \frac{-S_{T_x}(x)}{S_{T_x}(x)} \quad (23l)$$

and

$${}_{\xi}q_x = \int_0^{\xi} S_{T_x}(\xi)\,\mu_{x+\xi}\,d\xi \quad (23m)$$

$${}_{\xi}m_x = \frac{{}_{\xi}q_x}{\int_0^{\xi} S_{T_x}(\xi)\,d\xi} \quad (23n)$$

For small values , the probability that a newborn who survives to age x and dies within times Δx

and $\Delta x + x$ is defined as

$$F_X'(x) = f_X(x) = S_X(x)\,\mu_x \quad (23o)$$

$$F_X'(x)\Delta x = \mathbf{P}(x \le X \le x + \Delta x) \quad (23p)$$

$$F_X'(x)\Delta x = \mathbf{P}(X \le x + \Delta x \mid X \ge x)\,P(X \ge x) \quad (23q)$$

$$F_X'(x)\Delta x = \mathbf{P}(X \le x + \Delta x \mid X \ge x)\,S_X(x) \quad (23r)$$

$$\mu_x \Delta x = \mathbf{P}(X \le x + \Delta x \mid X \ge x)$$
$$\Rightarrow (\Delta x) \times (A + Hx + BC^x) = \mathbf{P}(X \le x + \Delta x \mid X \ge x) \quad (23s)$$

The vital statistics and population census consist a number of deaths stratified according to defined age, calendar year and sex. Intuitively, $M(\xi, x) = \frac{d_x}{l_x}$ can be applied to compute death rates from original data. However, to estimate mortality rate intensity from pure death rate, computational assumptions are specified to define the relative terms. In mortality domain, estimations are required to generate life tables for all actual ages. In [14,15], it was argued that as the force of mortality fluctuates with a low speed over time, the force of mortality is assumed closed, stationary and remains constant for each integer age and calendar year. Consequently, for all $0 \le \theta, u < 1$,

$$\mu\left(\xi+\theta, x+u\right)=\mu\left(\xi, x\right) \tag{23t}$$

This assumption defines the force of mortality as constant for every integer age and calendar year. The population is stationary as well as the size of the population for all age x remains constant over the calendar year. Therefore,

$$M\left(\xi, x\right)=\mu\left(\xi, x\right) \tag{23u}$$

can be applied in death rate computations or

$$q\left(\xi, x\right)=1-\exp\left[-\mu\left(\xi, x\right)\right]=1-\exp\left[-M\left(\xi, x\right)\right] \tag{23v}$$

can be deployed in mortality analysis

Materials and methods derivations in the generalised makeham's mortality function

The Generalized Makeham function IS deployed to summarise the empirical distribution of deaths or chronological series which present a long-term trend. The problem of estimating the main and the auxiliary parameters is the goal based on the method of successive differencing. The quality of modifying the Generalized Makeham's function is extrapolated towards the advanced ages. The transformations deployed on the Generalized Makeham's mortality law will be efficiently applied to describe the mortality trajectories without major difficulty to obtain the parameters. Throughout this study, x represents the age of a life, μ_x represents the mortality rate at an instant, l_x represents the number of lives surviving to age x. Following [16], the Generalized Makeham's mortality law is defined as:

$$\mu_x = GM\left(m,n\right)=\begin{cases}\sum_{k=1}^{m}\beta_k x^{k-1}+\exp\sum_{k=m+1}^{m+n}\beta_k x^{k-m-1} & m\ge 1, n\ge 1\\ \exp\sum_{k=1}^{n}\beta_k x^{k-1} & m=0\\ \sum_{k=1}^{m}\beta_k x^{k-1} & n=0\end{cases} \tag{24}$$

Method of successive differencing in estimating parameters

Parameter estimation is a big problem in mortality analysis. It is sufficient for our purpose to state that the method used here is the method of successive differencing. This method has the capacity to even out random errors associated with mortality data through a process called normalization. The uniqueness of this method is the novelty to estimate the parameters. Equation (24a) has five auxiliary parameters and hence there will be five equations and consequently, the logarithm of points on the observed mortality curve should be equal to the sum of logarithms of points on the graduated curve. The survival function for $GM\left(2,2\right)=A+Hx+BC^x$ is obtained

$$l_x=\frac{l_0}{g}s^x W^{x^2} g^{C^x}=Ks^x W^{x^2} g^{C^x}\;;\quad K=\frac{l_0}{g} \tag{24a}$$

Essentially, the logarithms of (24a) can be taken to have

$$\log_e l_x=\log_e K+\log_e s^x+\log_e W^{x^2}+\log_e g^{C^x} \tag{25}$$

$$\log_e l_x=\left(\log_e K\right)+x\left(\log_e s\right)+x^2\left(\log_e W\right)+C^x\left(\log_e g\right) \tag{26}$$

let

$$Y=\log_e l_x;\overline{A}=\log_e K;\overline{B}=\log_e s;\overline{E}=\log_e W;\overline{F}=\log_e g \tag{27}$$

Then

$$e^Y=l_x;e^{\overline{A}}=K;e^{\overline{B}}=S;e^{\overline{E}}=W;e^{\overline{F}}=g \tag{28}$$

$$l_x=e^{\overline{A}}e^{\overline{B}x}e^{\overline{E}x^2}e^{\overline{F}C^x}=e^{\overline{A}+\overline{B}x+\overline{E}x^2+\overline{F}C^x} \tag{29}$$

We need to obtain the values of $\phi=\left\{\overline{A};\overline{B};\overline{E};\overline{F}\right\}$ first, hence we use another exponential transformation as follows,

Substituting (27) in (26) and obtain

$$Y=\overline{A}+\overline{B}x+\overline{E}x^2+\overline{F}C^x \tag{30}$$

Imposing equal step length

$x_2-x_1=x_3-x_2=x_4-x_3=x_5-x_4=h$, we have

$$x_2=x_1+h;x_3=x_1+2h;x_4=x_1+3h;x_5=x_1+4h \tag{31}$$

substituting arbitrary five ages x_1, x_2, x_3, x_4, x_5 in (30)

$$Y_1=\overline{A}+\overline{B}x_1+\overline{E}x_1{}^2+\overline{F}C^{x_1} \tag{32}$$

$$Y_2=\overline{A}+\overline{B}x_2+\overline{E}x_2{}^2+\overline{F}C^{x_2} \tag{33}$$

$$Y_3=\overline{A}+\overline{B}x_3+\overline{E}x_3{}^2+\overline{F}C^{x_3} \tag{34}$$

$$Y_4=\overline{A}+\overline{B}x_4+\overline{E}x_4{}^2+\overline{F}C^{x_4} \tag{35}$$

$$Y_5=\overline{A}+\overline{B}x_5+\overline{E}x_5{}^2+\overline{F}C^{x_5} \tag{36}$$

Subtracting (32) from (33)

$$\Delta Y_1 = Y_2 - Y_1 = \overline{A} + \overline{B}x_2 + \overline{E}x_2^{\,2} + \overline{F}C^{x_2} - \left(\overline{A} + \overline{B}x_1 + \overline{E}x_1^{\,2} + \overline{F}C^{x_1}\right) \tag{37}$$

$$\Delta Y = \overline{A} + \overline{B}x_2 + \overline{E}x_2^{\,2} + \overline{F}C^{x_2} - \overline{A} - \overline{B}x_1 - \overline{E}x_1^{\,2} - \overline{F}C^{x_1} \tag{38}$$

$$\Delta Y = \overline{B}\left(x_2 - x_1\right) + \overline{E}\left(x_2^{\,2} - x_1^{\,2}\right) + \overline{F}\left(C^{x_2} - C^{x_1}\right) \tag{39}$$

$$\Delta Y_1 = \overline{B}\left(x_2 - x_1\right) + \overline{E}\left(x_2 - x_1\right)\left(x_2 + x_1\right) + \overline{F}\left(C^{x_2} - C^{x_1}\right) \tag{40}$$

Substituting x_2 defined in (31) into (40) we have

$$Y_2 - Y_1 = h\overline{B} + h\overline{E}\left(2x_1 + h\right) + \overline{F}C^{x_1}\left(C^h - 1\right) \tag{41}$$

Taking second difference between (34) and (33)

$$\Delta Y_2 = Y_3 - Y_2 = \overline{A} + \overline{B}x_3 + \overline{E}x_3^{\,2} + \overline{F}C^{x_3} - \left(\overline{A} + \overline{B}x_2 + \overline{E}x_2^{\,2} + \overline{F}C^{x_2}\right) \tag{42}$$

$$\Delta Y_2 = \overline{B}\left(x_3 - x_2\right) + \overline{E}\left(x_3^{\,2} - x_2^{\,2}\right) + \overline{F}\left(C^{x_3} - C^{x_2}\right) \tag{43}$$

$$\Delta Y_2 = h\overline{B} + \overline{E}\left(x_3 - x_2\right)\left(x_3 + x_2\right) + \overline{F}\left(C^{x_3} - C^{x_2}\right) \tag{44}$$

Following same procedure as before and substitute for x_3 defined in (31)

$$\Delta Y_2 = h\overline{B} + h\overline{E}\left(x_1 + 2h + x_1 + h\right) + \overline{F}\left(C^{x_1+2h} - C^{x_1+h}\right) \tag{45}$$

$$Y_3 - Y_2 = h\overline{B} + h\overline{E}\left(2x_1 + 3h\right) + \overline{F}C^{x_1+h}\left(C^h - 1\right) \tag{46}$$

Taking third difference using (34) and (35) and substitute for x_4

$$\Delta Y_3 = Y_4 - Y_3 = \overline{A} + \overline{B}x_4 + \overline{E}x_4^{\,2} + \overline{F}C^{x_4} - \left(\overline{A} + \overline{B}x_3 + \overline{E}x_3^{\,2} + \overline{F}C^{x_3}\right) \tag{47}$$

$$\Delta Y_3 = h\overline{B} + h\overline{E}\left(x_1 + 3h + x_1 + 2h\right) + \overline{F}\left(C^{x_1+3h} - C^{x_1+2h}\right) \tag{48}$$

$$Y_4 - Y_3 = h\overline{B} + h\overline{E}\left(2x_1 + 5h\right) + \overline{F}C^{x_1+2h}\left(C^h - 1\right) \tag{49}$$

Taking fourth difference using (35) and (36) and substitute for x_5

$$\Delta Y_4 = Y_5 - Y_4 = \overline{A} + \overline{B}x_5 + \overline{E}x_5^{\,2} + \overline{F}C^{x_5} - \left(\overline{A} + \overline{B}x_4 + \overline{E}x_4^{\,2} + \overline{F}C^{x_4}\right) \tag{50}$$

$$\Delta Y_4 = \overline{B}\left(x_5 - x_4\right) + \overline{E}\left(x_5^{\,2} - x_4^{\,2}\right) + \overline{F}\left(C^{x_5} - C^{x_4}\right) \tag{51}$$

$$\Delta Y_4 = \overline{B}\left(x_5 - x_4\right) + \overline{E}\left(x_5 - x_4\right)\left(x_5 + x_4\right) + \overline{F}\left(C^{x_5} - C^{x_4}\right) \tag{52}$$

$$\Delta Y_4 = h\overline{B} + h\overline{E}\left(x_1 + 4h + x_1 + 3h\right) + \overline{F}\left(C^{x_1+4h} - C^{x_1+3h}\right) \tag{53}$$

$$Y_5 - Y_4 = h\overline{B} + h\overline{E}\left(2x_1 + 7h\right) + \overline{F}C^{x_1+3h}\left(C^h - 1\right) \tag{54}$$

subtracting (41) from (46) and obtain

$$Y_3 - Y_2 - \left(Y_2 - Y_1\right) = h\overline{B} + h\overline{E}\left(2x_1 + 3h\right) + \overline{F}C^{x_1+h}\left(C^h - 1\right) - \begin{pmatrix} h\overline{B} + h\overline{E}\left(2x_1 + h\right) \\ +\overline{F}C^{x_1}\left(C^h - 1\right) \end{pmatrix} \tag{55}$$

$$Y_3 - Y_2 - Y_2 + Y_1 = h\overline{B} + h\overline{E}\left(2x_1 + 3h\right) + \overline{F}C^{x_1+h}\left(C^h - 1\right) - h\overline{B} - h\overline{E}\left(2x_1 + h\right) - \overline{F}C^{x_1}\left(C^h - 1\right) \tag{56}$$

$$Y_3 - 2Y_2 + Y_1 = h\overline{B} - h\overline{B} + h\overline{E}\left(2x_1 + 3h\right) - h\overline{E}\left(2x_1 + h\right) + \overline{F}C^{x_1+h}\left(C^h - 1\right) - \overline{F}C^{x_1}\left(C^h - 1\right) \tag{57}$$

$$Y_3 - 2Y_2 + Y_1 = 2h^2\overline{E} + \overline{F}C^{x_1}\left(C^h - 1\right)^2 \tag{58}$$

Subtracting (46) from (49) and have

$$Y_4 - Y_3 - \left(Y_3 - Y_2\right) = h\overline{B} + h\overline{E}\left(2x_1 + 5h\right) + \overline{F}C^{x_1+2h}\left(C^h - 1\right) - \begin{bmatrix} h\overline{B} + h\overline{E}\left(2x_1 + 3h\right) \\ +\overline{F}C^{x_1+h}\left(C^h - 1\right) \end{bmatrix} \tag{59}$$

$$Y_4 - Y_3 - \left(Y_3 - Y_2\right) = h\overline{B} + h\overline{E}\left(2x_1 + 5h\right) + \overline{F}C^{x_1+2h}\left(C^h - 1\right) - h\overline{B} - h\overline{E}\left(2x_1 + 3h\right) - \overline{F}C^{x_1+h}\left(C^h - 1\right) \tag{60}$$

$$Y_4 - 2Y_3 + Y_2 = h\overline{B} - h\overline{B} + h\overline{E}\left(2x_1 + 5h\right) - h\overline{E}\left(2x_1 + 3h\right) + \overline{F}C^{x_1+2h}\left(C^h - 1\right) - \overline{F}C^{x_1+h}\left(C^h - 1\right) \tag{61}$$

$$Y_4 - 2Y_3 + Y_2 = 2h\overline{E}x_1 + 5h^2\overline{E} - 2h\overline{E}x_1 - 3h^2\overline{E} + \overline{F}C^{x_1+h}\left(C^h - 1\right)^2 \tag{62}$$

$$Y_4 - 2Y_3 + Y_2 = 2h^2\overline{E} + \overline{F}C^{x_1+h}\left(C^h - 1\right)^2 \tag{63}$$

Subtract (49) from (54) and have

$$Y_5 - Y_4 - \left(Y_4 - Y_3\right) = h\overline{B} + h\overline{E}\left(2x_1 + 7h\right) + \overline{F}C^{x_1+3h}\left(C^h - 1\right) - \begin{bmatrix} h\overline{B} + h\overline{E}\left(2x_1 + 5h\right) \\ +\overline{F}C^{x_1+2h}\left(C^h - 1\right) \end{bmatrix} \tag{64}$$

$$Y_5 - Y_4 - \left(Y_4 - Y_3\right) = h\overline{B} + h\overline{E}\left(2x_1 + 7h\right) + \overline{F}C^{x_1+3h}\left(C^h - 1\right) - h\overline{B} - h\overline{E}\left(2x_1 + 5h\right) - \overline{F}C^{x_1+2h}\left(C^h - 1\right) \tag{65}$$

$$Y_5 - Y_4 - \left(Y_4 - Y_3\right) = h\overline{B} - h\overline{B} + h\overline{E}\left(2x_1 + 7h\right) - h\overline{E}\left(2x_1 + 5h\right) + \overline{F}C^{x_1+3h}\left(C^h - 1\right) - \overline{F}C^{x_1+2h}\left(C^h - 1\right) \tag{66}$$

$$Y_5 - 2Y_4 + Y_3 = 2h\overline{E}x_1 + 7h^2\overline{E} - 2h\overline{E}x_1 - 5h^2\overline{E} + \overline{F}C^{x_1+2h}\left(C^h - 1\right)^2 \tag{67}$$

$$Y_5 - 2Y_4 + Y_3 = 2h^2\overline{E} + \overline{F}C^{x_1+2h}\left(C^h - 1\right)^2 \tag{68}$$

Subtract (58) from (63) and have

$$Y_4 - 2Y_3 + Y_2 - \left(Y_3 - 2Y_2 + Y_1\right) = 2h^2\overline{E} + \overline{F}C^{x_1+h}\left(C^h - 1\right)^2 - \left(2h^2\overline{E} + \overline{F}C^{x_1}\left(C^h - 1\right)^2\right) \tag{69}$$

$$Y_4 - 2Y_3 + Y_2 - Y_3 + 2Y_2 - Y_1 = 2h^2\overline{E} + \overline{F}C^{ü_1+}\left(C^h - 1\right)^2 - 2h^2\overline{E} - \overline{F}C^{\ _1}\left(C^h - 1\right)^2 \tag{70}$$

$$Y_4 - 3Y_3 + 3Y_2 - Y_1 = \overline{F}C^{x_1}\left(C^h - 1\right)^3 \tag{71}$$

Subtract (63) from (68) and obtain

$$Y_5-2Y_4+Y_3-\left(Y_4-2Y_3+Y_2\right)=2h^2\overline{E}+\overline{F}C^{x_1+2h}\left(C^h-1\right)^2-\left[2h^2\overline{E}+\overline{F}C^{x_1+h}\left(C^h-1\right)^2\right] \quad (72)$$

$$Y_5-2Y_4+Y_3-\left(Y_4-2Y_3+Y_2\right)=2h^2\overline{E}+\overline{F}C^{x_1+2h}\left(C^h-1\right)^2-2h^2\overline{E}-\overline{F}C^{x_1+h}\left(C^h-1\right)^2 \quad (73)$$

$$Y_5-2Y_4+Y_3-Y_4+2Y_3-Y_2=2h^2\overline{E}+\overline{F}C^{x_1+2h}\left(C^h-1\right)^2-2h^2\overline{E}-\overline{F}C^{x_1+h}\left(C^h-1\right)^2 \quad (74)$$

$$Y_5-3Y_4+3Y_3-Y_2=\overline{F}C^{x_1+h}\left(C^h-1\right)^3 \quad (75)$$

combining equations (71) and (75) together again and have

$$\begin{cases} Y_4-3Y_3+3Y_2-Y_1 &= \overline{F}C^{x_1}\left(C^h-1\right)^3 \\ Y_5-3Y_4+3Y_3-Y_2 &= \overline{F}C^{x_1+h}\left(C^h-1\right)^3 \end{cases} \quad (76)$$

Dividing the lower equation by the upper in system (76) to obtain the ageing parameter C

$$C^h=\frac{Y_5-3Y_4+3Y_3-Y_2}{Y_4-3Y_3+3Y_2-Y_1}\Rightarrow C=\left(\frac{Y_5-3Y_4+3Y_3-Y_2}{Y_4-3Y_3+3Y_2-Y_1}\right)^{\frac{1}{h}} \quad (77)$$

Setting (77) in (75) and have

$$\overline{F}\left[\left(\frac{Y_5-3Y_4+3Y_3-Y_2}{Y_4-3Y_3+3Y_2-Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5-3Y_4+3Y_3-Y_2-Y_4+3Y_3-3Y_2+Y_1}{Y_4-3Y_3+3Y_2-Y_1}\right]^3 \quad (78)$$

$$=Y_5-3Y_4+3Y_3-Y_2 \quad (79)$$

$$\overline{F}\left[\left(\frac{Y_5-3Y_4+3Y_3-Y_2}{Y_4-3Y_3+3Y_2-Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5-4Y_4+6Y_3-4Y_2+Y_1}{Y_4-3Y_3+3Y_2-Y_1}\right]^3=Y_5-3Y_4+3Y_3-Y_2 \quad (80)$$

$$\overline{F}=\frac{Y_5-3Y_4+3Y_3-Y_2}{\left[\left(\frac{Y_5-3Y_4+3Y_3-Y_2}{Y_4-3Y_3+3Y_2-Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5-4Y_4+6Y_3-4Y_2+Y_1}{Y_4-3Y_3+3Y_2-Y_1}\right]^3} \quad (81)$$

Using the first equation in (58), we obtain

$$\overline{E}=\frac{Y_3-2Y_2+Y_1-\overline{F}C^{x_1}\left(C^h-1\right)^2}{2h^2} \quad (82)$$

Recall from (41)

$$\overline{B}=\frac{Y_2-Y_1-h\overline{E}\left(2x_1+h\right)-\overline{F}C^{x_1}\left(C^h-1\right)}{h} \quad (83)$$

Using equation (32),

$$\overline{A} = Y_1 - \overline{B}x_1 - \overline{E}x_1^{\ 2} - \overline{F}C^{x_1} \tag{84}$$

where $\overline{F}$ and C are given by (81) and (77)

The following auxiliary parameters $K; S; W; C; g$ can now be found from $\{\overline{A}; \overline{H}; \overline{B}, C, \overline{F}\}$ using the transformation $e^Y = l_x; e^{\overline{A}} = K; e^{\overline{B}} = S; e^{\overline{E}} = W; e^{\overline{F}} = g$. However, we can now compute the original mortality parameters $A; H; B$ using the transformations exponential transformations above.

Substituting the values into the force of mortality and the parameters of $GM(2,2)$ are completely estimated.

b) THE MODAL AGE AT DEATH

In orer to track the age where individuals die most, it is necessary to locate the point where the survival curve inflexes known as the age with the highest mortality.

THEOREM

If $\lim\limits_{\Delta \to 0} \frac{l_{x+\Delta} - l_x}{\Delta}$ exists, then the point at which l_x inflexes is the modal age at death

$$x_M = \left[\frac{-\left\{ \begin{array}{l} 2\left(2\alpha B^2 + 2BH\right)\left(2B^2 + 2AB - B\alpha\right) \\ +4B^2\alpha H \end{array} \right\} + \sqrt{ \left\{ \begin{array}{l} 2\left(2\alpha B^2 + 2BH\right)\left(2B^2 + 2AB - B\alpha\right) \\ +4B^2\alpha H \end{array} \right\}^2 - 4\left(2\alpha B^2 + 2BH\right)^2 \left\{ \begin{array}{l} \left(2B^2 + 2AB - B\alpha\right)^2 \\ +4AB^2\alpha - B^2\alpha^2 - 4HB^2 \end{array} \right\} } }{2\left(2\alpha B^2 + 2BH\right)^2} \right] \tag{85}$$

Proof

$$-\frac{1}{l_x}\frac{dl_x}{dx} = \mu_x \Rightarrow \frac{dl_x}{dx} = -l_x \mu_x \tag{86}$$

Taking the second derivative, we have

$$\frac{d^2 l_x}{dx^2} = -\frac{d}{dx}\left(l_x \mu_x\right) = -l_x' \mu_x - l_x \mu_x' \tag{87}$$

$$\frac{d^2 l_x}{dx^2} = l_x \mu_x^2 - l_x \mu_x' = 0 \tag{88}$$

$$\mu_x^2 - \mu_x' = 0 \Rightarrow \frac{\mu_x'}{\mu_x} - \mu_x = 0 \tag{89}$$

Recall

$$\mu_x = A + Hx + BC^x \tag{90}$$

Where,

$$e^\alpha = C; \quad \alpha \in \mathbf{R}^+ \tag{91}$$

$$\mu_x = A + Hx + Be^{\alpha x} \tag{92}$$

$$\frac{d\mu_x}{dx} = H + B\alpha e^{\alpha x} \quad (93)$$

$$\frac{H + B\alpha e^{\alpha x}}{\left(A + Hx + Be^{\alpha x}\right)} - \left(A + Hx + Be^{\alpha x}\right) = 0 \quad (94)$$

$$H + B\alpha e^{\alpha x} = \left(A + Hx + Be^{\alpha x}\right)\left(A + Hx + Be^{\alpha x}\right) \quad (95)$$

$$H + B\alpha e^{\alpha x} = A\left(A + Hx + Be^{\alpha x}\right) + Hx\left(A + Hx + Be^{\alpha x}\right) + Be^{\alpha x}\left(A + Hx + Be^{\alpha x}\right) \quad (96)$$

$$H + B\alpha e^{\alpha x} = A^2 + AHx + ABe^{\alpha x} + AHx + H^2x^2 + BHxe^{\alpha x} + ABe^{\alpha x} + BHxe^{\alpha x} + B^2e^{2\alpha x} \quad (97)$$

$$B^2e^{2\alpha x} + \left(2AB - B\alpha + 2BHx\right)e^{\alpha x} + \left(A^2 - H + AHx + AHx + H^2x^2\right) = 0 \quad (98)$$

Let

$$y = e^{\alpha x} \quad (99)$$

$$A^2 + AHx + AHx - H + H^2x^2 + B^2y^2 + BHxy + ABy + BHxy + ABy - B\alpha y = 0 \quad (100)$$

$$B^2y^2 + \left(2BHx + 2AB - B\alpha\right)y + \left(A^2 + 2AHx - H + H^2x^2\right) = 0 \quad (101)$$

$$y = \frac{-\left(2BHx + 2AB - B\alpha\right) \pm \sqrt{\left(2BHx + 2AB - B\alpha\right)^2 - 4B^2\left(A^2 + 2AHx - H + H^2x^2\right)}}{2B^2} \quad (102)$$

$$\left(2BHx + 2AB - B\alpha\right)^2 = 2BHx\left(2BHx + 2AB - B\alpha\right) + 2AB\left(2BHx + 2AB - B\alpha\right) - B\alpha\left(2BHx + 2AB - B\alpha\right) \quad (103)$$

$$\left(2BHx + 2AB - B\alpha\right)^2 = 4B^2H^2x^2 + 4B^2AHx - 2B^2\alpha Hx + 4AB^2Hx + 4A^2B^2 - 2AB^2\alpha - 2B^2\alpha Hx - 2B^2\alpha A + B^2\alpha^2 \quad (104)$$

$$4B^2\left(A^2 + 2AHx - H + H^2x^2\right) = 4A^2B^2 + 8AB^2Hx - 4HB^2 + 4B^2H^2x^2 \quad (105)$$

$$\left(2BHx + 2AB - B\alpha\right)^2 - 4B^2\left(A^2 + 2AHx - H + H^2x^2\right) = 4B^2H^2x^2 + 4B^2AHx - 2B^2\alpha Hx + 4AB^2Hx + 4A^2B^2 - 2AB^2\alpha - 2B^2\alpha Hx - 2B^2\alpha A + B^2\alpha^2 - \left(4A^2B^2 + 8AB^2Hx - 4HB^2 + 4B^2H^2x^2\right) \quad (106)$$

$$\left(2BHx+2AB-B\alpha\right)^2-4B^2\left(A^2+2AHx-H+H^2x^2\right)=4B^2H^2x^2+4B^2AHx-2B^2\alpha Hx$$
$$+4AB^2Hx+4A^2B^2-2AB^2\alpha-2B^2\alpha Hx-2B^2\alpha A+B^2\alpha^2-4A^2B^2-8AB^2Hx+4HB^2 \quad (107)$$
$$-4B^2H^2x^2 \quad (108)$$
$$\left(2BHx+2AB-B\alpha\right)^2-4B^2\left(A^2+2AHx-H+H^2x^2\right)=$$
$$-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2 \quad (109)$$
$$y=\frac{-\left(2BHx+2AB-B\alpha\right)\pm\sqrt{-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2}}{2B^2} \quad (110)$$
$$e^{\alpha x}=\frac{-\left(2BHx+2AB-B\alpha\right)\pm\sqrt{-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2}}{2B^2} \quad (111)$$
$$2e^{\alpha x}B^2+\left(2BHx+2AB-B\alpha\right)=\sqrt{-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2} \quad (112)$$
$$\left[2e^{\alpha x}B^2+\left(2BHx+2AB-B\alpha\right)\right]^2=-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2$$
$$e^{\alpha x}=1+\sum_{n=1}^{\infty}\frac{(\alpha x)^n}{n!} \quad (113)$$

Since α is small, we are permitted to express the exponential term as an approximation

$$e^{\alpha x}=1+\alpha x \quad (114)$$
$$\left[2B^2\left(1+\alpha x\right)+2BHx+2AB-B\alpha\right]^2=-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2 \quad (115)$$
$$\left[\left(2\alpha B^2+2BH\right)x+\left(2B^2+2AB-B\alpha\right)\right]^2=-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2 \quad (116)$$
$$\left(2\alpha B^2+2BH\right)^2x^2+2\left(2\alpha B^2+2BH\right)\left(2B^2+2AB-B\alpha\right)x+\left(2B^2+2AB-B\alpha\right)^2$$
$$=-4B^2\alpha Hx-4AB^2\alpha+B^2\alpha^2+4HB^2 \quad (117)$$
$$\left(2\alpha B^2+2BH\right)^2x^2+2\left(2\alpha B^2+2BH\right)\left(2B^2+2AB-B\alpha\right)x+4B^2\alpha Hx$$
$$+\left(2B^2+2AB-B\alpha\right)^2+4AB^2\alpha-B^2\alpha^2-4HB^2=0 \quad (118)$$
$$\left(2\alpha B^2+2BH\right)^2x^2+\left\{2\left(2\alpha B^2+2BH\right)\left(2B^2+2AB-B\alpha\right)+4B^2\alpha H\right\}x$$
$$+\left\{\left(2B^2+2AB-B\alpha\right)^2+4AB^2\alpha-B^2\alpha^2-4HB^2\right\}=0 \quad (119)$$

Solving the above equation for x, the result follows.

This completes the proof

In order to unify all $GM\left(m,n\right)$ laws based on the arguments above and unequivocally state and claim the following proposition without proof.

PROPOSITION 1: THE UNIVERSAL LAW OF MORTALITY (CLAIM)

A life after birth continues to be in a state of vitality or continuous longevity unless otherwise acted against his survival by an external force of morbidity or mortality due to a source.

MORTALITY EXPERIMENTS ON THE GENERALIZED MAKEHAM'S LAW FOR MORTALITY INTENSITIES.

Within the factors determining the evolution of mortality intensities, the human ageing characterizes the most pervasive effect in influencing the vulnerability of many causes of death. Consequently, age changes constitute the highest demographic risk factor in all age-related sicknesses. It is on this argument that this study is based on age dependent mortality intensities satisfying Occam's razor and moreover, life insurance schemes issued for protection and pension purposes are defined on long- or short-term durations.

The data set employed in this study is a single year of age population data set l_x taken from the published German female DAV 2008 survival data. German data is considered because the country has proven mortality data collection records. Moreover, Germany has death rate pattern similar to that of Nigeria. The ages considered in the analysis are in single year within the age range $0-120$. Our goal in this report is to compute the values of instantaneous mortality for a life insurance firm to underwrite life insurance contracts and their variants offered to a life aged x

THE GENERALISED MAKEHAM'S MORTALITY TABLE

In the Table which follows, x represents the age, l_x is the survival's function, μ_x is the mortality rate intensity, $l_x\mu_x$ is the curve of death, p_x is the probability that a life aged x survives to the next age $x+1$, q_x is the probability that a life aged x will die before reaching age $x+1$. The table below was generated from mortality functions of the previous section. The parameters of the mortality function $\mu_x = A + Hx + BC^x$ have been numerically developed and based on these numerical estimates, the following mortality table for life insurance application is presented in Table 1.

Table 1:
$GM(2,2)$ Female mortality table

x	l_x	μ_x	$l_x\mu_x$	q_x	p_x
0	1000000	0.00010001	100	0.00010400	0.99989600
1	999896	0.00010751	107	0.00011101	0.99988899
2	999785	0.00011500	115	0.00011903	0.99988097
3	999666	0.00012249	122	0.00012604	0.99987396
4	999540	0.00012999	130	0.00013406	0.99986594
5	999406	0.00013748	137	0.00014108	0.99985892
6	999265	0.00014497	145	0.00014811	0.99985189
7	999117	0.00015247	152	0.00015614	0.99984386
8	998961	0.00015997	160	0.00016417	0.99983583
9	998797	0.00016746	167	0.00017121	0.99982879
10	998626	0.00017496	175	0.00017824	0.99982176
11	998448	0.00018246	182	0.00018629	0.99981371

12	998262	0.00018996	190	0.00019434	0.99980566
13	998068	0.00019747	197	0.00020039	0.99979961
14	997868	0.00020497	205	0.00020945	0.99979055
15	997659	0.00021248	212	0.00021550	0.99978450
16	997444	0.00021998	219	0.00022457	0.99977543
17	997220	0.00022749	227	0.00023064	0.99976936
18	996990	0.00023501	234	0.00023872	0.99976128
19	996752	0.00024252	242	0.00024680	0.99975320
20	996506	0.00025004	249	0.00025389	0.99974611
21	996253	0.00025756	257	0.00026098	0.99973902
22	995993	0.00026509	264	0.00026908	0.99973092
23	995725	0.00027262	271	0.00027618	0.99972382
24	995450	0.00028015	279	0.00028329	0.99971671
25	995168	0.00028769	286	0.00029141	0.99970859
26	994878	0.00029524	294	0.00029953	0.99970047
27	994580	0.00030279	301	0.00030666	0.99969334
28	994275	0.00031035	309	0.00031380	0.99968620
29	993963	0.00031792	316	0.00032194	0.99967806
30	993643	0.00032549	323	0.00032909	0.99967091
31	993316	0.00033307	331	0.00033625	0.99966375
32	992982	0.00034067	338	0.00034442	0.99965558
33	992640	0.00034828	346	0.00035260	0.99964740
34	992290	0.00035590	353	0.00035977	0.99964023
35	991933	0.00036353	361	0.00036696	0.99963304
36	991569	0.00037118	368	0.00037516	0.99962484
37	991197	0.00037885	376	0.00038237	0.99961763
38	990818	0.00038654	383	0.00039059	0.99960941
39	990431	0.00039425	390	0.00039781	0.99960219
40	990037	0.00040198	398	0.00040605	0.99959395
41	989635	0.00040974	405	0.00041328	0.99958672
42	989226	0.00041754	413	0.00042154	0.99957846
43	988809	0.00042536	421	0.00042880	0.99957120
44	988385	0.00043322	428	0.00043809	0.99956191
45	987952	0.00044113	436	0.00044435	0.99955565
46	987513	0.00044907	443	0.00045265	0.99954735
47	987066	0.00045707	451	0.00046198	0.99953802
48	986610	0.00046513	459	0.00046827	0.99953173

49	986148	0.00047324	467	0.00047762	0.99952238
50	985677	0.00048143	475	0.00048495	0.99951505
51	985199	0.00048969	482	0.00049432	0.99950568
52	984712	0.00049803	490	0.00050167	0.99949833
53	984218	0.00050647	498	0.00051107	0.99948893
54	983715	0.00051502	507	0.00051946	0.99948054
55	983204	0.00052368	515	0.00052787	0.99947213
56	982685	0.00053247	523	0.00053629	0.99946371
57	982158	0.00054140	532	0.00054574	0.99945426
58	981622	0.00055048	540	0.00055520	0.99944480
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60	980523	0.00056920	558	0.00057418	0.99942582
61	979960	0.00057888	567	0.00058370	0.99941630
62	979388	0.00058879	577	0.00059323	0.99940677
63	978807	0.00059897	586	0.00060380	0.99939620
64	978216	0.00060944	596	0.00061541	0.99938459
65	977614	0.00062024	606	0.00062499	0.99937501
66	977003	0.00063140	617	0.00063766	0.99936234
67	976380	0.00064296	628	0.00064831	0.99935169
68	975747	0.00065497	639	0.00066103	0.99933897
69	975102	0.00066747	651	0.00067378	0.99932622
70	974445	0.00068052	663	0.00068757	0.99931243
71	973775	0.00069418	676	0.00070037	0.99929963
72	973093	0.00070852	689	0.00071627	0.99928373
73	972396	0.00072360	704	0.00073118	0.99926882
74	971685	0.00073952	719	0.00074818	0.99925182
75	970958	0.00075636	734	0.00076522	0.99923478
76	970215	0.00077422	751	0.00078333	0.99921667
77	969455	0.00079322	769	0.00080251	0.99919749
78	968677	0.00081348	788	0.00082380	0.99917620
79	967879	0.00083513	808	0.00084721	0.99915279
80	967059	0.00085833	830	0.00086965	0.99913035
81	966218	0.00088326	853	0.00089628	0.99910372
82	965352	0.00091009	879	0.00092505	0.99907495
83	964459	0.00093904	906	0.00095390	0.99904610
84	963539	0.00097034	935	0.00098699	0.99901301
85	962588	0.00100424	967	0.00102224	0.99897776

86	961604	0.00104103	1001	0.00106073	0.99893927
87	960584	0.00108103	1038	0.00110141	0.99889859
88	959526	0.00112459	1079	0.00114848	0.99885152
89	958424	0.00117209	1123	0.00119676	0.99880324
90	957277	0.00122397	1172	0.00125251	0.99874749
91	956078	0.00128072	1224	0.00131056	0.99868944
92	954825	0.00134284	1282	0.00137617	0.99862383
93	953511	0.00141095	1345	0.00144728	0.99855272
94	952131	0.00148570	1415	0.00152605	0.99847395
95	950678	0.00156780	1490	0.00161148	0.99838852
96	949146	0.00165806	1574	0.00170680	0.99829320
97	947526	0.00175739	1665	0.00180998	0.99819002
98	945811	0.00186676	1766	0.00192533	0.99807467
99	943990	0.00198728	1876	0.00205193	0.99794807
100	942053	0.00212017	1997	0.00219096	0.99780904
101	939989	0.00226678	2131	0.00234577	0.99765423
102	937784	0.00242862	2278	0.00251444	0.99748556
103	935426	0.00260735	2439	0.00270251	0.99729749
104	932898	0.00280481	2617	0.00291029	0.99708971
105	930183	0.00302306	2812	0.00313917	0.99686083
106	927263	0.00326437	3027	0.00339278	0.99660722
107	924117	0.00353127	3263	0.00367161	0.99632839
108	920724	0.00382655	3523	0.00398274	0.99601726
109	917057	0.00415333	3809	0.00432579	0.99567421
110	913090	0.00451504	4123	0.00470381	0.99529619
111	908795	0.00491551	4467	0.00512437	0.99487563
112	904138	0.00535899	4845	0.00558985	0.99441015
113	899084	0.00585017	5260	0.00610399	0.99389601
114	893596	0.00639428	5714	0.00667416	0.99332584
115	887632	0.00699711	6211	0.00730483	0.99269517
116	881148	0.00766508	6754	0.00800433	0.99199567
117	874095	0.00840532	7347	0.00877708	0.99122292
118	866423	0.00922575	7993	0.00963502	0.99036498
119	858075	0.01013513	8697	0.01058416	0.98941584
120	848993	0.01114320	9460	1.00000000	0.00000000

Source: Authors' Computation, 2024

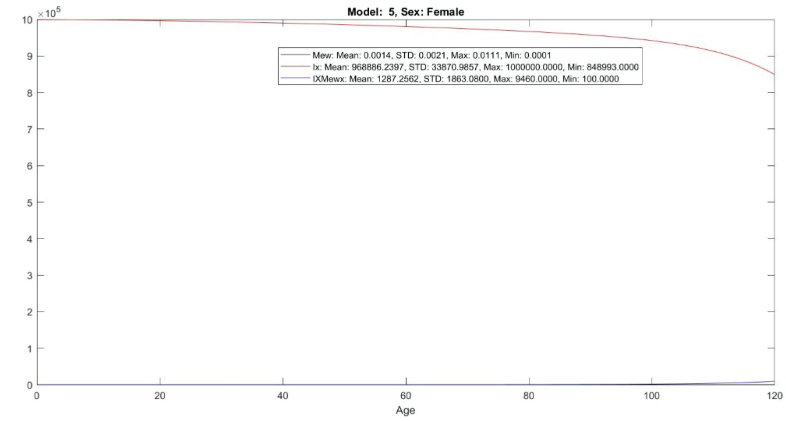

Fig. 1 $GM(2,2)$ Female's Survival and Curve of Death Functions

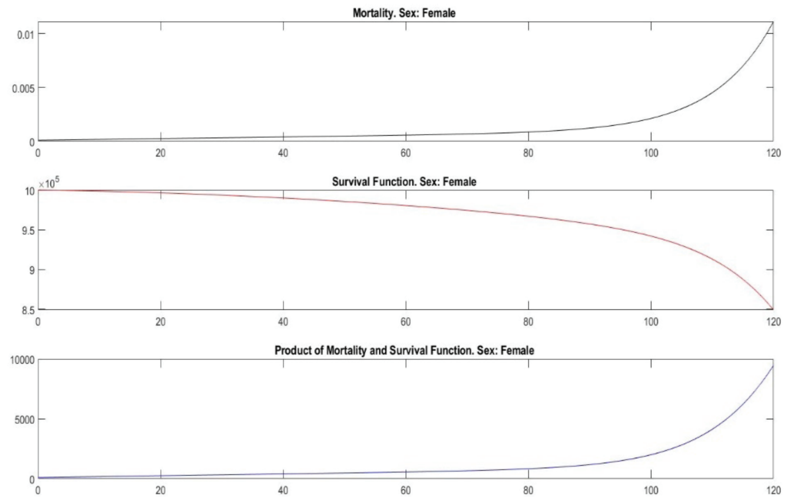

Fig. 2 $GM(2,2)$ Female's Mortality, Survival and Curve of Death Functions

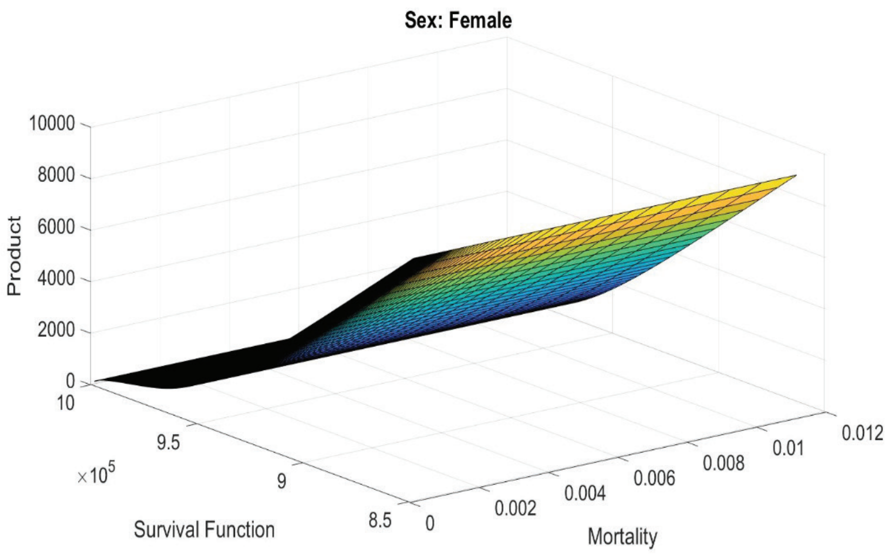

Fig. 3 $GM(2,2)$ Surface Plot for Female's Mortality Survival, Mortality and Curve of Death Functions

Results and discussion

In Table 1, the neonatal and infant mortality from birth to the first birthday is high but the mortality rate intensity falls rapidly within $0 \le x \le 7$ and becomes relatively stable within $8 \le x \le 9$ before attaining a minimum risk around age 10. The relative stability in this interval can be due to the predictable trend's gradual reduction in mortality over time or it can be as a result of another idiosyncratic condition which is neither due to shocks nor trends.

By the method of successive differencing employed to model $GM(2,2)$, the female ageing parameter values 1.109423265 fall within the globally accepted interval $1.08 \le C \le 1.12$ $GM(2,2)$ family. This method is superior to the method of maximum likelihood estimation adopted in [17] where the ageing C parameter is estimated as 1.024738. The authors' method violated the permissible interval.

The mortality rate intensity further declines within $11 \le x \le 33$ and finally increases in the interval $34 \le x \le 120$. The possible medical intervention in the public health system to manage young adulthood mortality could be observed as the male's mortality rates have reduced. The decline in mortality rates within the former interval has significant impact on life annuities and on life insurers. However, the increase in mortality rates with age in the latter interval could be associated with various health risk factors. In men specifically, the risk of prostate cancer increases with age around age 65 and beyond until it is peaked between ages 78 and 79. Irrespective of sex, the probability of contracting a chronic sickness or disability increases with age while immunity reduces thus exposing lives more vulnerable to health risk and consequently increasing the risk of mortality.

The survival l_x function in Figures 1 and 2 representing the expected number of lives surviving to age x out of an initial group of 1000000 lives clearly forms asymptote on the age axis. The observed pattern of μ_x completely differs from its usual behaviour in Table 1 for female. Since it is not feasible to observe any age interval where the survival function $l_x \to 0$ in Table 1, then it can be concluded that female has a longer lifespan up to the senescence. The parameters of $GM(2,2)$ is

$$(A,B,H,C)_{FEMALE} = (9.99752 \times 10^{-05}, 3.93158 \times 10^{-08}, 7.48763 \times 10^{-06}, 1.109423265) \quad (120)$$

The probability density function of the distribution of deaths represents an important mortality statistic since it is an immediate indication of key longevity measures describing how long a population will live on the average and the extent of variability of ages at death. In Table 1, the curve of death function $l_x\mu_x$ describes the expected density of deaths at age x with respect to the population of lives surviving to age x and there is a local minimum of $l_x\mu_x$ around 10 years. From our observation, the local extreme points of $l_x\mu_x$ corresponds to points of inflexion of l_x following the investigations below

$$\frac{d}{dx} l_x\mu_x = \mu_x \frac{d}{dx} l_x + l_x \frac{d}{dx}\mu_x . \quad (121)$$

Observing that

$$\frac{1}{l_x}\frac{dl_x}{dl_x} = -\mu_x \quad (122)$$

The point at which l_x inflexes becomes

$$\frac{d}{dx} l_x\mu_x = \mu_x\left(-\mu_x l_x\right) + l_x\left(\mu_x'\right) = 0 \quad (123)$$

$$\left(-\frac{1}{l_x}\frac{dl_x}{dl_x}\right)^2 = \frac{1}{l_x}\left(\frac{d}{dx} l_x\mu_x - \mu_x \frac{d}{dx} l_x\right) \quad (123a)$$

So that $\mu_x^2 = \mu_x'$

Since the functions $\{l_x, \mu_x, (l_x\mu_x)\}$ are functionally related, their behaviours are jointly examined through mortality surface as displayed in Figure 3.

Notably, the function $\mu_x = GM(2,2) > 0$ is continuous for all $x \ge 0$ and satisfies the boundary condition $\int_0^{\infty} \mu_x dx = \mu(0)$ and $\mu'(10) = 0$

In Table 1, the probability of death satisfies the condition $0 < q_x \le 1$ but increases with age and does not exceed 1. However, in same Table, the mortality rate intensity similarly satisfies the $0 < \mu_x \le 1$ throughout the interval $0 \le x \le 120$.

The highest age in the mortality table is defined by $\Omega = Sup\left\{\zeta \in \mathbf{R}^+ \middle| F_{T_x}(\zeta) \le 1\right\}$. Consequently, the female's Omega age obtained as $\Omega_x(F) = 120$ is numerically determined from the estimated l_x to be the first age where $q_x = 1$. To study the behaviour of mortality pattern beyond age 90, the female's intensity is extrapolated to age 120 where the mortality rates at extreme age still seems to exhibit exponential increase. This exponential increase may not hold true in practice.

For female, it is observed that within the interval, $0 \le x \le 120$, $\mu_x < q_x$ throughout.

Observe that the probability of death is defined as $q_x = \int_0^1 \left({}_{\xi}p_x\right)\mu_{x+\xi} d\xi$. If $l_{x+\xi}\mu_{x+\xi}$ were increasing, then $\frac{d}{d\xi} l_{x+\xi}\mu_{x+\xi} > 0$ and at the beginning of the interval $0 < \xi < 1$, $l_x q_x > l_x \mu_x \Rightarrow q_x > \mu_x$ provided the curve of death $l_x \mu_x$ is increasing. By definition, $-\frac{d}{dx} l_x = l_x \mu_x$, it then follows that if $l_x \mu_x$ is increasing then clearly $\frac{d}{dx} l_x$ is decreasing and the gradient of the tangent to the curve l_x will be decreasing. The survival function l_x will then be concave to the age axis. Consequently, if the survival function l_x is concave to the age axis, the condition $q_x > \mu_x$ is satisfied. However, if the survival function l_x is convex to the age axis, then $q_x < \mu_x$.

Most published works in extant mortality literatures do not account for μ_x based on the governing mortality intensities because of the computational intractability associated with their estimations. The ageing parameter for female in our model in shown to satisfy the globally accepted ageing interval $1.08 < C < 1.12$. An interesting point in the modelling of $GM(2,2)$ is the novelty of the estimation technique through successive differencing applied. Putra, Fitriyati, & Mahmudi (2019) deployed the MLE technique to estimate the ageing parameter of $GM(1,2)$ but the authors' ageing parameter violates the globally accepted interval of validity $1.08 < C < 1.12$. Comparatively, the implication is that in terms of computational accuracy and adequacy, the estimation technique for the ageing parameter is better than the maximum likelihood method (MLE) commonly applied in most mortality estimations.

Conclusions

In this study, the analytic behaviour of the Generalised Makeham's age dependent mortality functions has been investigated to generate mortality with higher precision mortality rate intensities. Critical issues relating to computational problems plaguing life insurance underwriting operations call for immediate attention to employ numerical techniques of estimation since the problem of mortality estimation and other related issues of life insurance are conveniently expressed as problems of numerical analysis. This study adopts evidence based computational technique in constructing mortality modelling and estimations. The goal is to hypothesize new analytical techniques in the field of mortality modelling by constructing age specific life table models. Mortality which is associated with the risk of death occurring has complex functional structure and continuously varies across ages. This study recommends that the table be deployed for Nigerian life insurance operations.

As data connected with mortality rates of the insured population often changes, the term of life insurance underwriting becomes very much critical. Because of the analytically intractable nature of the underlying non-linear functional mortality structure especially $GM(2,2)$, a novel method of successive differencing was adopted for estimating the mortality intensities to even out possible errors at source. Hence our numerical computations have improvised potential solutions to the hydra-headed problems plaguing mortality analysis in connection with life table generation, life insurance and pension valuation, annuity and premium computations.

Since life insurance underwriting complexity may likely increase, the number of the applicable underlying functional techniques will correspondingly increase in terms of computational strategy, consequently it is necessary to deepen solutions to such problem areas of core underwriting and mortality analysis. It is therefore expected that the selection of appropriate analytical functions will be significant as the difficulty level and dimensionality of underwriting problems grow. This study therefore reconciles the interests of life office actuaries and regulators saddled with the responsibilities of the actuarial professional to supervise the design of life insurance schemes. The computations generated from the deployment of successive differencing technique is of demonstrable importance to the life office actuaries to undertake their monitoring and gatekeeping operations in insurance industry.

Acknowledgement

We would like to express our sincere gratitude to the editors and the anonymous reviewers who have contributed. immensely to the successful completion of this paper.

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