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A Study on Flett's Mean Value Theorem

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Abstract : *The aim of this study is to evaluate the parallels between the results of Flett's Mean Value Theorem, Lagrange's Mean Value Theorem, and Rolle's Theorem, as well as their geometrical significance. It also covers Thomas M. Flett's 1958 Mean Value Theorem of Differential and Integral Calculus and its various extensions. We intend to present a thorough analysis of various (existing as well as new) adequate conditions for this theorem's validity because it is a topic of interest in many areas of mathematics, including functional equations.*

Keywords: Lagrange's mean value theorem, differentiable, continuous, geometrical meaning

Introduction

A very basic yet incredibly powerful tool for mathematical analysis, the mean value theorems of differential and integral calculus provide the framework for addressing a wide range of mathematical problems. All math students are familiar with the Lagrange's mean value theorem, which was first presented in Lagrange's book in 1797 and is an extension of Roll's from 1691. More precisely, Lagrange's theorem asserts that

Theorem1: If f is a real valued function which is continuous on closed interval [a, b] and is differentiable on open interval (a, b) there exists a point c ∈ (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

According to Lagrange's mean value theorem, there exists a point c in (a, b) such that the tangent at the point [c, f(c)] which is parallel to the given line L. For which the given line L, joining two points on the graph of a differentiable function f, such as [a, f(a)] and [b, f(b)].

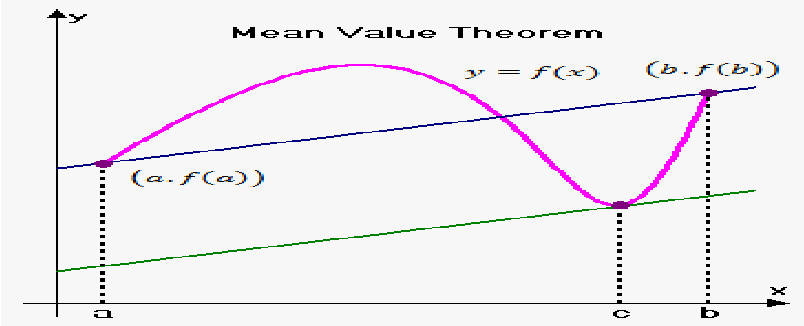

Figure (i)

The Rolle's theorem can be derived from the Lagrange's mean value theorem if f(a)=(b). The following queries might come up in relation to these well-known facts.

*Are there any modifications to Rolle's theorem when higher-order derivatives are considered in the hypothesis?
*Is the Lagrange's theorem analogous in any way?
*What noteworthy aspects of both can be seen in terms of their geometrical significance?
The following notations will be used throughout this study:
C(M): Domain of the continuous function .
$D^n(M)$: Real-valued functions on a set t $M \subseteq \mathbb{R}$, which are n times differentiable.
In most of the cases, we use the compact set of real lines, i.e. with a ,b, M ∈ [a, b].
Usually we use the compact set of real line, i.e. M= [a, b] with a < b. Similarly we use the notation of differentiability on a closed interval for the functions f, g on an interval [a, b] the expressions of the form $\frac{f^n(b)-f^n(a)}{g^n(b)-g^n(a)}$, n ∈ N ∪ $\{\mathbf{0}\}$,
Which is denoted by the symbol $T_a^b(f^{(n)}, g^{(n)})$ for the denominator which is equal to (b- a), be denoted by only $(T)_a^b(f^{(n)})$. So that the Lagrange's mean value theorem can be restated as:
If f ∈ C [a, b]∩D(a , b) then there exists c ∈ (a, b) such that f'(c)= $T_a^b(f)$, now the original result of Flett as well as its generalization due to Ridel and Sahoo (2004) by restricting the boundary conditions.

2. Flett's Mean Value Theorem

Let us begin with the following simple observation from (2009) as:
Theorem 2 (Gologan, 2009) **:** f, g ∈ C [a, b], then from the integral mean value theorem there exists c ∈ (a, b) such that

$$g(c) = \frac{1}{b-a}\int_a^b g(t)dt.$$

Moreover, if we consider the function g ∈ [a, b] with the properties

$$g(a)=0, \int_a^b g(t)dt = 0, \qquad (1)$$

and let us define a function

$$\emptyset(x) = \begin{cases} \dfrac{1}{x-a}\displaystyle\int_a^b g(t)dt, & x \in (a,b], \\ 0 , & x = a, \end{cases}$$

Then ∅ ∈ C [a, b] ∩ D (a , b) and $\emptyset(a) = 0 = \emptyset(b)$. So by Rolle's theorem there exists c ∈ (a, b) such that $\emptyset'(c) = 0$, i .e.,

$$\frac{g(c)}{c-a} - \frac{1}{(c-a)^2}\int_a^c g(t)dt = o \Leftrightarrow g(c) = \frac{1}{c-a}\int_a^c g(t)dt.$$

With formally substituting b by c, the second expression resembles the integral mean value theorem formula. It is common knowledge that the integral mean value, often known as the mean value, of a function g on the interval [a, x], typically exhibits less unpredictable behavior than the function g itself. When creating the function g. i.e. with the assumption that g(b)=0, the second condition in (1) may be swapped out for any other straightforward condition. We shall further show that it is plausible and the result in this more general form due to Darboux's intermediate value theorem (Lupu, 2009). For the time being, if we define a function as
f(x)= $\int_a^x g(t)dt$, x $\in [a, b]$,
Our consideration leads to an equivalent type of result, which is the topic of this work (1999). This result is an observation made in 1958 by THOMAS MUIRHEAD FLETT (1923–1976),

which was reported in his work (2009). It is specifically a variant of Rolle's Theorem in which f'(a)=f'(b) is used instead of f(a)=f (b). In other words, it is a Rolle's types condition on a mean value theorem of the Lagrange type.

Theorem 3 (Flett, 2001): If $f \in D[a, b]$ and $f'(a)=f'(b)$, then there exists a point exists $c \in (a, b)$ such that $f'(c)= T_a^c(f)$.

Proof : Without loss of generality let us assume that by $f'(a)=f'(b)=0$. If this is not possible we may assume $h(x)=f(x)- x f'(a)$ for some $x \in [a, b]$. Define

$$g(x)=\begin{cases} {}_a^xT(f), & x \in (a,b] \\ f'(a), & x = a \end{cases}$$

Obviously, $g \in C[a, b] \cap D(a, b)$ and

$$G'(x) = -\frac{{}_a^xT(f) - f'(x)}{x-a} = {}_a^xT(g) + {}_a^xT(f'), \quad x \in (a,b].$$

Now we will show that there exists $c \in (a ,b)$ such that $g'(c)= 0$.

But by the definition of g we have $g(a)=0$. If $g(b)=0$, then by Rolle's theorem we can find a point $c \in (a ,b)$ such that $g'(c)=0$.

If possible let $g(b)\neq 0$ and assume that $g(b)>0$ (by similar way we can say that if $g(b)<0$).

Then

$$g'(b) = -{}_a^bT(g) = -\frac{g(b)}{b-a} < 0.$$

Since $g \in C[a, b]$ and $g'(b)<0$, i.e. g is strictly decreasing in b, then there exists $x_1 \in (a,b)$ such that $g(x_1) > g(b)$. Since g is continuous on $[a, x_1)$ and as we have $0=g(a) < g(b) <g(x_1)$ then we conclude from Darboux's intermediate value theorem there exists $x_2 \in (a,x_1)$ such that $g(x_2)=g(b)$. Since $g \in C[x_2 ,b] \cap D(x_2, b)$, from Rolle's theorem we have $g'(c)=0$. For some $c \in (x_2 ,b) \subseteq (a, b)$.

Geometrical Meaning of Flett's Theorem:

If a curve y= f(x) has a tangent at each point of [a, b] and tangents at the end points [a, f(a)] and [b, f(b)] are parallel, then Flett's theorem guarantees the existence of such a point $c \in (a,b)$ such that the tangent drawn to the graph of f at that point passes through the point [a, f(a)] which has in the form of $y=f(a)+ f'(c)(x-a)$.

Which is shown in the following figure.

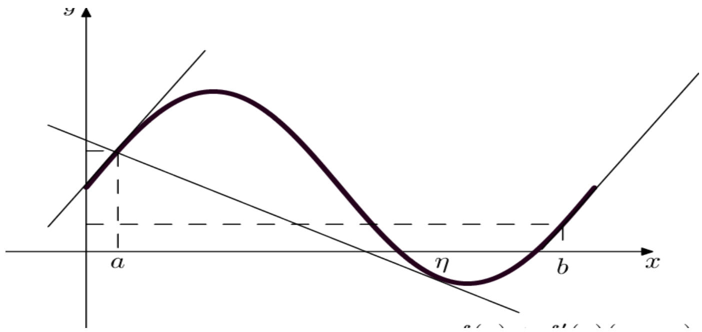

Figure (ii)

It goes without saying that Flett's Theorem's conditions might be true even if its premise is false. There exists an infinite number of locations c in (a, b) for which the tangent formed in the point c passes through the point [a, -a]. If we look at the non-differentiable function f(x)=|x| on the interval [a, b], where a < b.

Conjecture : If f has a valid or improper derivative at each point of the interval [a, b] and the tangents at the end points are parallel, then there exists a point c in (a,b).

$$f'(c)= {}_a^bT(f).$$

The Flett's mean value theorem can be expressed as the following equivalent forms:

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow f(a)= T_1 (f, b)(a)$$

$$\Leftrightarrow \begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0$$

In the second expression $T_1(f, x_0)(x)$ is the first Taylor's polynomial of function f at the point x_0 as a function of x.

The last expression resembles an equivalent formulation of the last assertion of Lagrange's theorem in the form of determinant. i.e. there exists a point $c \in (a, b)$ such that

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} =0$$

Theorem 4: (reidel-Sahoo, 1998) If $f \in D[a, b]$ then there exists $c \in (a, b)$ such that

$${}_a^cT(f) = f'(c) - {}_a^bT(f').\frac{c-a}{2}.$$

Proof: Let us define a function F by

$$F(x) = \begin{vmatrix} f(x) & x^2 & x & 1 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \\ f'(b) & 2b & 1 & 0 \end{vmatrix}, \quad x \in [a, b].$$

Clearly, $F \in D[a, b]$ and

$$F'(x) = \begin{vmatrix} f'(x) & 2x & 1 & 0 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \\ f'(b) & 2b & 1 & 0 \end{vmatrix}, \quad x \in [a, b].$$

Thus F'(a)= F'(b)= 0, and by Flett's theorem there exists $c \in (a, b)$ such that

$F'(c)= {}_a^bT(F)$, which is equivalent to the assertion of Riedel- Sahoo's theorem (2001).

References

Abel, U. & M. Riedel. (2004). The mean value theorem of Flett and divided differences.*J. Math Anal. Appl.295*, 1 − 9 .

Davitt, et. al, (1999). First mean value theorem for holomorphic functions. *Math . Mag.*72 ,304-309.

Das, M., Riedel, T. & Sahoo, P.K. (2001). Flett's mean value theorem for approximately differential functions. *Red. Mat.10*, 157-164.

Gologan, R. & Lupu,C. (2009). An olyampiad problem: Zeroes of functions on the image of a Volterra operator. *Gazeta Mathematica 27*, 227-238.

Lupu, J.C. & Lupu, G. (2009). Men value theorems for some linear integral operators. Election. *J. Differential equations 117*, 1-15.

Powers, R.C., Riedel T.& Sahoo,P.K. (2001). Flett's mean value theorem on to R. p ological vector space. Internal. J. Math, Set.*27*, 689-894.