A FEW METHODS OF SOLVING OPTIMAL CONTROL PROBLEM IN HAMILTON-JACOBI-BELLMAN EQUATION FORM

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Abstract

Non-linear optimal control problem arises in many different areas, for example, engineering, medical sciences, economics, industries, etc. The solution of Hamilton-Jacobi-Bellman equation is connected with the non -linear optimal control problem. In this paper, we formulate the Hamilton-Jacobi-Bellman equation using nonlinear optimal control problem. We also discuss its solutions using Adomian decomposition method, Laplace transform-Homotopy perturbation method and variational iteration method.

Keywords: Optimal control system, Hamilton-Jacobi-Bellman equation, Adomian Decomposition, Laplace transform-Homotopy perturbation, Variational Iteration.

1. Introduction

Optimal control theory has been widely used in different sectors, for example, engineering, medical sciences, economics, industries, etc, see Li & Young (1995). However, optimal control of non-linear system is still supper challenging task for solving from decades. Solving methods of optimal control is mainly divided in two categories: direct method (optimization technique) and indirect method (calculus technique). Indirect method leads to Hamilton-Jacobi-Bellman (HJB) equation. The viscosity solution of HJB equation is called value function. A value function is connected to nonlinear optimal control problem (OCP). Some examples of optimal control problem and their corresponding HJB equations are as follows:

- Infinite Horizon Problem (Bardi & Capuzzo–Dolcetta, 1997): For any $x \in \mathbb{R}^n$ and $r \in (0, +\infty)$

$$v(x) = \inf_{u \in \mathbb{U}(t,\tau)} \left\{ \int_0^{\tau} e^{-\gamma s} \varphi\big(y_x^u(s), u(s)\big)\, ds + e^{-\gamma s} v\big(y_x^u(\tau)\big) \right\}$$

Suppose that the value functions are continuously differentiable then its HJB equation is given by
$\gamma v(x) + H\big(x. \nabla v(x)\big) = 0,\ x \in \mathbb{R}^n$ with
$H(x,p) = sup\{\langle f\,(x,u), p\rangle - \varphi(x,u) | u \in U\}$.

- Bolza Problem (Bardi & Capuzzo-Dolcetta 1997): For any $x \in \mathbb{R}^n$ and $\tau \in (t,T)$,

$$v(t,x) = \inf_{u \in \mathbb{U}(t,\tau)} \left\{ \int_t^{\tau} \varphi\big(s, y_x^u(s), u(s)\big) ds + v(\tau, y_x^u(\tau)) \right\}$$

Its HJB equation can be formulated as

$$-\partial_t v(t,x) + H\big(t,x,\nabla_x v(t,x)\big) = 0,$$

$(t,x) \in (-\infty, T) \times \mathbb{R}^n$, with

$$H(t,x,p) = \sup\{-\langle f(t,x,u), p\rangle - \varphi(t,x,u) | u \in U\}.$$

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- Minimum time problem (Bokanowski & Zidani 2011): Suppose time is not fixed to reach a given target $\mathbb{O} \subseteq \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $\tau \in (0, T^{\mathbb{O}}(x))$

$$\boldsymbol{T}^{\mathbb{O}}(\boldsymbol{x}) = \inf_{u \in \mathbb{U}(t,\tau)} \left\{\tau + T^{\mathbb{O}}\left(y_x^u(\tau)\right)\right\}.$$

Its HJB equation is

$$-1 + H\left(x, \nabla T^{\mathbb{O}}(x)\right) = 0, x \in int(dom\, T^{\mathbb{O}}), \text{ with}$$

$$H(x,p) = \sup\{-\langle f(x,u), p\rangle | u \in U\}.$$

Different methods have been proposed to solve the Hamilton-Jacobi-Bellman equation such as modification of variational iteration method using He's polynomials for a class of nonlinear optimal control problems. Matinfar & Saeidy (2014) discussed that this method is very efficient and powerful technique in finding the solution to the nonlinear optimal control problems. Crandall et al. (1984) described two approximations of solutions of Hamilton-Jacobi equations. Atangana et al. (2014) described the homotopy perturbation method (HPM) for the HJB equation. For solving the problem of optimal portfolio construction, Kiliannova & Sevcovic (2013) proposed and analyzed a method for the Riccati transformation, which is known as transformation method for HJB equation. The problem of optimal reinsurance design has been studied by using risk measures such as the Valueat-Risk (VaR), Conditional Value-at-Risk (CVaR) and conditional tail expectation (CTE). Wen & Yin (2019) presented the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle and obtained the closed form expression of the optimal reinsurance strategy and corresponding survival probability under proportional reinsurance. Yousefia et al. (2010) described the variational iteration method to find the optimal control of linear system and tested the example for the validity of solution with the exact solution. Jafari et al. (2013) applied the standard Adomian decomposition method and the homotopy perturbation method to obtain the solution of nonlinear functional equations and shown that the standard HPM provides exactly the same solutions as the standard Adomian decomposition method for solving functional equations.

In the present work, we are going to use Adomian decomposition method (ADM), Laplace homotopy perturbation method (LHPM) and variational iteration method (VIM) to solve the HJB equations. These methods give approximate solution as an infinite series which are converging to the exact solution. Then we will analyze the error with its solution with the exact solutions.

2. Notation and Assumption

A control system is the output of the calculus of variation. When we use the controller in the differential equation then it will affect the evolution of the system.

Consider U as a family of admissible control functions defined by

$$U \coloneqq \{u: \mathbb{R} \longrightarrow \mathbb{R}^m, \quad u \text{ is measurable}, \; u(t) \in U(t) \text{ for a.e. } t\}.$$

Now, we consider non-linear control system in the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), a.e\ t \in \left(0, T_f\right], \\ x(0) = x_0. \end{cases} \tag{2.1}$$

Here we can see that the rate of change $\dot{x}(t)$ depends not only on the state variables x itself, but depends on some extra parameters, $u = \{u_1, u_2, \dots, u_m\}$ also. Researchers often describe such problems in the form of a cost function that can represent time, energy and money costs or their respective combinations, see Evans (2010). The task of finding the controls relates to the minimization of the cost functional.

The cost function in its general form is given by

$$J(x(t), u, t) = \int_0^{T_f} L\big(t, x(t), u(t)\big)dt + h\left(x\left(T_f\right)\right), \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent state and control variables, respectively at time $t, L: [0, T_f] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ and $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ represents real valued functions defining the Lagrangian or running cost per unit time and terminal cost of the system, respectively.

Bardi & Capuzzo-Dolcetta (1997) defined the OCP as

$$\inf_{u \in \mathbb{U}} J(x(t), u, t) = \inf_{u \in \mathbb{U}} \left\{\int_0^{T_f} L\big(t, x(t), u(t)\big)dt + h(x(T_f))\right\},$$

Subject to $\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), a.e\ t \in (0, T_f], \\ x(0) = x_0. \end{cases}$

(2.3)

That means,

Under which optimal control value of u that minimize the cost function given by (2.2) that also satisfies the condition (2.1).

Now, let us consider the value function $v: [0, T_f] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$v := \inf_{u \in \mathbb{U}} J(x, u, t).$$

We have,

$$v(x(t), t) = \inf_{u \in \mathbb{U}} \Big(h\big(x(T_f), T_f\big) + \int_t^{T_f} L(t, x(w),\ u(w),\ w) dw \Big). \quad (2.4)$$

Therefore, we have

$$v(x(t), t) = \inf_{u \in \mathbb{U}} \Big(h\big(x(T_f), T_f\big) + \int_{t+\delta t}^{T_f} L(t, x(w), u(w),\ w) dw + \int_t^{t+\delta t} L(t, x(w),\ u(w),\ w) dw \Big),$$

then by using Bellman's Principle of Optimality (Bellman et al., 1965) given by the equation (2.4),

$$v(x(t), t) = \inf_{u \in \mathbb{U}} \left(\int_t^{t+\delta t} L(t, x(w), u(w), w) dw + v\big(x\,(t + \delta t), (t + \delta t)\big) \right).$$

Using Taylor series,

$$v(x(t), t) = \inf_{u \in \mathbb{U}} \left(\int_t^{t+\delta t} L(t, x(w), u(w), w) dw + v(x\,(t),\ t) + v_t \delta t + v_x [x(t + \delta t) - x(t)] + O(\delta t) \right)$$

Suppose $\delta t \rightarrow 0$ then $w \rightarrow t$

$$v(x(t), t) = \inf_{u \in \mathbb{U}} \big(L \delta t + v(x, t) + v_t \delta t + v_x f(x, u, t) \delta t + O(\delta t) \big).$$

By dividing both sides by δt, we have

$$-\frac{\partial v}{\partial t} = \inf_{u \in \mathbb{U}} \left(L(x(t), u(t), t) + \frac{\partial v}{\partial x} f(x, u, t) \right)$$

then

$$\frac{\partial v}{\partial t} + H(x(t), u(t), v_x, t) = 0,$$

where

$$H(x(t), u(t), v_x, t) = \inf_{u \in \mathbb{U}} \left(L(x(t), u(t), t) + \frac{\partial v}{\partial x} f(x, u, t) \right).$$

This is called the HJB equation with the boundary condition $v\big(x(T_f), T_f\big) := \inf_{u \in \mathbb{U}} J\big(x(T_f), T_f\big) = h\big(x(T_f), T_f\big).$

Our aim is to solve the HJB equation by Adomian decomposition method, Laplace transform-Homotopy perturbation method and variational iteration method.

3. Basic Concept of Adomian Decomposition Method

Consider a nonlinear differential equation (Adomian, 1988) in the form of

$$Dv + Lv + Nv = f(x), \quad (3.1)$$

where D is the highest order derivatives which is considered as invertible, L is the linear differential operator of lesser than D, N represents the nonlinear terms and f is the input terms or source terms. Then taking inverse operator D^{-1} to the both side of (3.1), we get,

$$v = g(x) - D^{-1}(Lv) - D^{-1}(Nv),$$

which is calculated after integration from $f(x)$ and using given initial conditions. Here integral operator D^{-1} is taken as definite integrals from t_0 to t. In the Adomian decomposition method (Adomian, 1988), nonlinear term $N(v)$ is defined by

$$N(v) = \sum_{j=0}^{\infty} A_j.$$

The components $v_0, v_1, v_2, \dots$ are determined by the recursive relation:

$$v_0 = g(x), \qquad v_{n+1} = -D^{-1}(Lv) - D^{-1}(A_n).$$

Here A_n is called Adomian Polynomial, and defined by

$$A_n = \left(\frac{1}{n!}\right) \frac{d^n}{dp^n} \left[N\left(\sum_{i=0}^{\infty} v_i p^i \right) \right]_{p=0}, n = 0, 1, 2, \dots \quad (3.2)$$

And finally, we consider the solution $v(x)$ by the series $v(x) = \sum_{n=0}^{\infty} v_n$.

4. Adomian Method for HJB Equation

Consider HJB equation of the form:

$$\frac{\partial v}{\partial t} = H(x(t), u(t), v_x, t),$$

with initial condition

$$v(x(0), 0) = h(x(0), 0).$$

Consider $D_t = \frac{\partial}{\partial t}$, then we have,

$$D_t v = H(x(t), u(t), v_x, t). \quad (4.1)$$

Then separating right hand side of (4.1) as linear

and nonlinear term,
i.e.

$$D_t v = L(v) + N(v). \tag{4.2}$$

Now, taking inverse of operator $D_t^{-1} = \int_0^t (.)dt$
On both sides of (4.2), we get

$$D^{-1}D_t v = D_t^{-1}L(v) + D_t^{-1}N(v).$$

Consider

$v(x,t) = \sum_{n=0}^{\infty} v_n$.

Then we have,

$$\sum_{n=0}^{\infty} v_n = v_0 + D_t^{-1}L\left(\sum_{n=0}^{\infty} v_n\right) + D_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{4.4}$$

where

$v_0 = h(x(0),0)$.

Now, we can compute $v_n'\, s$ as follows:

$$v_1 = D_t^{-1}L(v_0) + D_t^{-1}A_0,$$

$$v_2 = D_t^{-1}L(v_1) + D_t^{-1}A_1,$$

$$v_3 = D_t^{-1}L(v_2) + D_t^{-1}A_2,$$

$$\vdots$$

$$v_{n+1} = D_t^{-1}L(v_n) + D_t^{-1}A_n,$$

and Adomian Polynomial A_n is obtained by (3.2).

5. Experimental Evaluation of ADM

Consider the control problem

$$\dot{x} = -\frac{1}{2}u$$

$$J = \int_0^1 \left(-x^2 - \frac{1}{4}u^2\right) dt.$$

Its Hamiltonian is

$$H = -x^2 - \frac{1}{4}u^2 - \frac{1}{2}\frac{\partial v}{\partial x}u.$$

Its Hamiltonian-Jacobi-Bellman equation (Adomian, 1988) is

$$\frac{\partial v}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2, \qquad v(x(0),0) = 0. \tag{5.1}$$

This can be written as

$$D_t v = x^2 - \frac{1}{4}(v_x)^2,$$

where $D_t = \frac{\partial}{\partial t}$.

Now, taking the inverse $D_t^{-1} = \int_0^t [.]dt$ on both sides

$$D_t^{-1}D_t v = D_t^{-1}(x^2) - \frac{1}{4}D^{-1}(v_x)^2.$$

Here, $D_t^{-1}D_t v = v - v(x(0),0) = v$ and put $v = \sum_{n=0}^{\infty} v_n$ we have

$$\sum_{n=0}^{\infty} v_n = v_0 - \tfrac{1}{4}\sum_{n=0}^{\infty} A_n,$$

where v_0 is represented by $v_0 = D_t^{-1}(x^2) = x^2 t$ and $(v_x)^2$ is the nonlinearity part which is replaced by A_n polynomials. The polynomials $A_n's$ for v^2 are defined by (3.2)

$$A_0 = v_0^2;$$
$$A_1 = 2v_0 v_1;$$
$$A_2 = v_1^2 + 2v_0 v_2;$$
$$\vdots$$

And so on.
Consequently,

$$v_1 = -\frac{1}{4}D_t^{-1}(v_{0x})^2 = -\frac{1}{4}D_t^{-1}(2xt) = -\frac{1}{3}x^2t^3;$$

$$v_2 = -\frac{1}{4}D_t^{-1}(2v_{0x}v_{1x}) = -\frac{1}{4}D_t^{-1}\left(-\frac{8}{3}x^2t^4\right) = \frac{2}{15}x^2t^5;$$

$$v_3 = -\frac{1}{4}D_t^{-1}(2v_{0x}v_{2x} + (v_{1x})^2) = -\frac{1}{4}D_t^{-1}\left(\frac{4}{9}xt^6 + \frac{16}{15}x^2t^6\right) = -\frac{17}{315}x^2t^7;$$

$$\vdots$$

and so on.
Now, substituting these value in $v = \sum_{n=0}^{\infty} v_n$, we get $v = x^2t - \frac{1}{3}x^2t^3 + \frac{2}{15}x^2t^5 - \frac{17}{315}x^2t^7 + \cdots$ is the required solution. It is obvious that a higher number of iterations make $v(x,t)$ converge to the exact solution.

6. Basic Concept of Homotopy Perturbation Method.

The combination of the perturbation method and the homotopy method is called homotopy perturbation method. This method is constructed in the following way:
Consider the nonlinear differential equation in the form (Fakharian et al., 2010):

$$Lv + Nv - g(x) = 0, \quad x \in \Omega,$$

where L is linear part and N is nonlinear part, $g(x)$ is analytic function and Ω is bounded domain.
In the homotopy technique, we construct the

homotopy $u(x,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$H(u,p) = L(u) - L(v_0) + pL(v_0) + p[N(u) - g(x)] = 0, \quad (6.1)$$

where v_0 is initial approximation, $p \in [0,1]$ is embedding parameter. So, we have

$$H(u,0) = L(u) - L(v_0) = 0,$$
$$H(u,1) = L(u) + N(u) - g(x) = 0.$$

While $L(u) - L(v_0)$ and $L(u) + N(u) - g(x)$ are homotopy then the changing process of p from zero to unity is $u(x,p)$ from v_0 to $v(x)$ is deformation. Now, we assume the solution of (6.1) is in the power series form in p:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \infty. \quad (6.2)$$

In (6.2), assume p tends to 1, we get

$$v = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \cdots$$

This is the solution of the given nonlinear differential equation.

7. Basic Concept of Laplace Transform-Homotopy Perturbation Method (LT-HPM)

Consider a nonlinear non-homogenous partial differential equation with initial condition (Khan & Wu, 2011):

$$\frac{\partial v}{\partial t} + Rv(x,t) + Nv(x,t) = g(x,t),$$
$$v(x,0) = \alpha(x), \quad (7.1)$$

where $R = \frac{\partial}{\partial x}$ is the linear differential operator, N is the nonlinear differential operator and $g(x,t)$ is the source term.

Now, taking the Laplace transform on both sides (7.1) and then using initial condition, we get

$$L[v(x,t)] = \frac{1}{s}(\alpha(x) + L[g(x)]) + \frac{1}{s}(L[Rv(x,t)] + L[Nv(x,t)]). \quad (7.2)$$

Taking inverse Laplace transform in (7.2), we have

$$v(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s}(L[Rv(x,t)] + L[Nv(x,t)])\right], \quad (7.3)$$

where $G(x,t)$ is prescribed from the source term and initial condition. Now, we apply homotopy perturbation given by

$$v(x,t) = \sum_{n=0}^{\infty} p^n H_n(x).$$

Using these coupling of the Laplace transform and the homotopy perturbation, we get

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s}L\left[R\sum_{n=0}^{\infty} p^n v_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right) \quad (7.4)$$

where H_n is He's polynomials and given by

$$H_n = \left(\frac{1}{n!}\right)\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty} v_i p^i\right)\right]_{p=0}, \quad n = 0,1,2,\cdots \quad (7.5)$$

Comparing the coefficients of like power of p in (7.4), we have

$$p^0: v_0(x,t) = G(x,t),$$
$$p^1: v_1(x,t) = -L^{-1}\left(\frac{1}{s}L[Rv_0(x,t) + H_0(v)]\right),$$
$$p^2: v_2(x,t) = -L^{-1}\left(\frac{1}{s}L[Rv_1(x,t) + H_1(v)]\right),$$
$$p^3: v_3(x,t) = -L^{-1}\left(\frac{1}{s}L[Rv_2(x,t) + H_2(v)]\right),$$
$$\vdots$$

and so on.

8. Experimental Evaluation for LT-HPM

Consider the following example (5.1):

$$\frac{\partial v}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2, v(x(0),0) = 0.$$

Taking Laplace transform on both sides,

$$L[v_t(x,t)] = L[x^2] - \frac{1}{4}L[(v_x)^2]. \quad (8.1)$$

Using initial condition in (8.1), we get

$$V = \frac{1}{s^2}x^2 - \frac{1}{4s}L[(v_x)^2]. \quad (8.2)$$

Taking inverse Laplace transform on (8.2), we get,

$$v(x,t) = x^2t - L^{-1}\left[\frac{1}{4s}L[(v_x)^2]\right]. \quad (8.3)$$

Using homotopy perturbation method in (8.3), we get

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = x^2 t - p\left[L^{-1}\left[\frac{1}{4s}L\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right], \quad (8.4)$$

where $H_n(v)$ are He's polynomials (7.5) and calculated by

$$H_0(v):\ [v_0(x)]^2 = 4x^2t^2,$$
$$H_1(v): 2v_{0x}v_{1x} = -\frac{8}{3}x^2t^4,$$
$$H_2(v): 2v_{0x}v_{2x} + (v_{1x})^2 = \frac{68}{45}x^2t^6,$$
$$\vdots$$

and so on.

And finally comparing the coefficients of like power of p in (8.4)

$$p^0:\ v_0(x,t) = x^2t,$$
$$p^1:\ v_1(x,t) = -L^{-1}\left[\frac{1}{4s}L[H_0(v)]\right] = \frac{1}{3}x^2t^3,$$
$$p^2:\ v_2(x,t) = -L^{-1}\left[\frac{1}{4s}L[H_1(v)]\right] = \frac{2}{15}x^2t^5,$$
$$p^3:\ v_3(x,t) = -L^{-1}\left[\frac{1}{4s}L[H_2(v)]\right] = -\frac{17}{315}x^2t^7,$$
$$\vdots$$

and so on.

Thus, the solution of (5.1) is given by

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + \cdots = x^2t - \frac{1}{3}x^2t^3 + \frac{2}{15}x^2t^5 - \frac{17}{315}x^2t^7 + \cdots$$

Which is exactly same as the solution given by Adomian decomposition method. In the paper (Mishra & Nagar, 2012), it is shown that the standard HPM provides exactly same solutions as the standard ADM for solving functional equations. It has been proved that He's polynomials are only Adomians polynomials with different name.

9. Variational Iteration Method (VIM)

The variational iteration method is the approximation method for solving linear and nonlinear problems. It has been the centre of attention of many researchers. This method was introduced by the Chinese mathematician He (1999). The main idea in the variational iteration method is to construct an iterative sequence of functions converging to an exact solution.

Consider the differential equation (He, 1999):

$$Lv + Nv = g(x),$$

where $L = \frac{d^m}{dx^m}, m \in \mathbb{N}$, is linear operator, N is nonlinear operator and $g(x)$ is sourcing term. Using VIM, we can construct a correct functional as follows:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda\left(Lv_n(x,w) + N\overline{v_n}(x,w) - g(x)\right)dw,$$

where λ is a Lagrange multiplier. It is calculated by

$\lambda = -1$, for $m = 1$

$\lambda = w - t$, for $m = 2$,

and in general

$\lambda = \frac{(-1)^m}{(m-1)!}(w-t)^{m-1}$, for $m \geq 1$.

Here $\overline{v_n}$ is supposed to be restricted variation that means $\delta\overline{v_n} = 0$. In this method, first to determine the Lagrange multiplier λ is determined optimally via variational theory. The successive approximation v_{n+1}, $n \geq 0$ of the solution u is obtained by using determined Lagrange multiplier and selective initial approximation v_0. Then the solution is given by

$$v = \lim_{n\to\infty} v_n.$$

10. Experimental Evaluation For VIM

Consider the example (5.1):

$$\frac{\partial v}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial v}{\partial x}\right)^2, v(x(0),0) = 0.$$

Its correction functional can be written as

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda\left[v_{nw}(x,w) + \frac{1}{4}\left(v_{nx}(x,w)\right)^2 - x^2\right]dw.$$

Choosing $\lambda = -1$ and starting with the initial approximation $v_0(x,t) = 0$,

$$v_1(x,t) = v_0(x,t) - \int_0^t \left[v_{0x}(x,w) + \frac{1}{4}\left(v_{0x}(x,w)\right)^2 - x^2\right]dw = 0 - \int_0^t (-x^2)dw, = x^2t;$$

$$v_2(x,t)$$
$$= v_1(x,t)$$
$$- \int_0^t \left[v_{1x}(x,w) + \frac{1}{4}\left(v_{1x}(x,w)\right)^2 - x^2 \right] dw$$
$$= x^2 t - \int_0^t (x^2 w^2) dw,$$
$$= x^2 t - \frac{1}{3} x^2 t^3;$$
$$v_3(x,t)$$
$$= v_2(x,t)$$
$$- \int_0^t \left[v_{2x}(x,w) + \frac{1}{4}\left(v_{2x}(x,w)\right)^2 - x^2 \right] dw$$
$$= v_2(x,t) - \int_0^t \left(-\frac{2}{3} x^2 w^4 + \frac{1}{9} x^2 w^6 \right) dw,$$
$$= x^2 t - \frac{1}{3} x^2 t^3 + \frac{2}{15} x^2 t^5 - \frac{1}{63} x^2 t^7;$$
$$\vdots$$

and so on.
The exact solution of (5.1) is $v(x,t) = x^2 \tanh t$.

11. Comparison of Solution of ADM, VIM with Exact

We compare the solutions obtained using ADM and VIM to the exact solution given by $v(x,t) = x^2 \tanh t$.

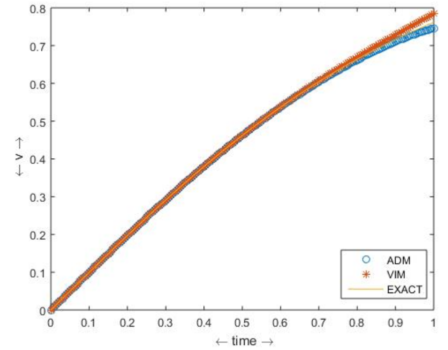

Fig. 1 Solutions given by ADM, VIM and Exact

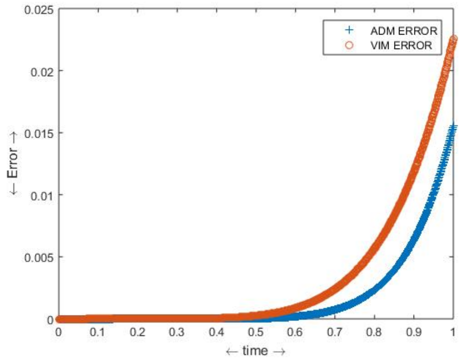

Fig. 2 Absolute Error of ADM, VIM with Exact

Table 1: Comparison Error

Time	Absolute ADM Error	Absolute VIM Error
0.1	2.18E-11	3.79E-09
0.2	1.10E-08	4.77E-07
0.3	4.15E-07	7.92E-06
0.4	5.38E-06	5.70E-05
0.5	3.88E-05	0.000259
0.6	0.000192	0.000874
0.7	0.000736	0.002401
0.8	0.002331	0.005658
0.9	0.006379	0.011842
1	0.015562	0.022533

Here, Fig. 1 represents the solutions of ADM, VIM and Exact, and Fig. 2 shows the absolute error of ADM, VIM with Exact.

From the error analysis shown in the Table 1, we can conclude that the result obtained from the ADM is better than the result obtained from the VIM. That is, ADM gives the better approximation for the HJB equation.

12. Conclusion

In this paper, nonlinear optimal control system is deduced in the HJB equation. Adomian decomposition method and the Laplace transform - homotopy perturbation transform method (LT-HPM) are successfully applied to study the Hamilton-Jacobi-Bellman equation with initial condition. Both the methods give the exactly same solution. The results show that the ADM and LT-HPM are both powerful and efficient techniques in finding exact and approximate solutions for HJB and nonlinear partial differential equations as well.

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