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Death Where is Your Sting?: Evidence from the Makeham's Curve of Death Callisthenics

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Abstract: The curve of death rate represents an effective device in evaluating the modal age at death for the insured population evolving out of the instantaneous measure of the functional relationship $(\mu_x)^2 = \mu'_x$ in a particular instant. Results in mortality analysis for a life-aged x are commonly examined but the generation of individual curves of death and gender comparisons of their corresponding auxiliary parameters under Makeham's parsimonious parametric law are rarely investigated. In this paper, the algebraic method is adopted to measure the curve of death in both males and females using the 2020 survival data of the US actuarial surveillance area for both sexes. This paper attempts to arouse computational derivations using R software for further applications in life insurance valuations within the framework of our technique. The specific objectives were to estimate the sub-parameters $\{s, g, C, k\}$ of the Makeham's law, to compute the individual survival function based on the sub-parameters as against interval estimate and to compute the individual curve of death. From the trajectories of the computed curve of death results, the modal age at death for male is roughly 87 years while that of female is 89.5 years. This difference accounts for the risk of exposure in male than in female. Although the female's ageing rate C and the sub-parameters $\{s, g\}$ for female are higher than those of male, computational evidence from the results confirms that the male's overall curve of death is correspondingly still higher than female's curve of death.

Keywords: *Curve of death, Makeham's law, Modal age, Mortality analysis, Auxilliary parameters*

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1. Introduction

There are two fundamental demographic indicators commonly employed in the analysis of life expectancy $\bar{e}_x$: The initial life expectancy $\bar{e}_0$ and the modal age at death $\alpha_M = \max_x f_{T(x)}(\xi)$ (Diaconu, Ouellette, Camarda & Bourbeau, 2016). Computationally, the modal age at death is usually obtained through the curve of death function, and typically, the curve of death intensities at any age describes the product of the survivors l_x surviving to age x and the corresponding mortality rate μ_x. The modal age at death is obtained by comparing the number of deaths d_x at ages x so that we can obtain the age that accounts for most deaths. In Canudas-Romo (2010), Ediev (2011) and Canberra (2020), the analysis of the curve of death is important when it is required to determine the age at which most of the population dies over a defined period and to ensure computationally accurate prediction of the social security program in the general population. This may enable actuaries to predict if the population is growing at senescence so as to plan for an appropriate health care system. We infer in Diaconu, Bourbeau, Ouellette, Camarda, (2015), that when the modal age at death remains constant, then regulators seem to be approaching their maximum capacity at the current level of social welfare program for the population. Horiuchi, Ouellette, Cheung and Robine (2013) and Riffe (2015) observed in demographic analysis that the distribution of the numbers of deaths is bimodal which are usually fixed at ages 0 and at adult ages. Following Canudas-Romo (2008), the modal age at death is important in evaluating the mortality of old lives as it accounts for the level of mortality at senescence.

Suppose an insured life is aged $x+\xi$ and assume that

$$\mathbf{P}\left(T_{x+\xi} \le y\right) = \mathbf{P}\left(T_x - \xi \le y \middle| T_x > \xi\right) \tag{1}$$

for $\xi > 0$. Then $\mathbf{P}\left(0 < T_{x+\xi} \le \delta\xi\right) = \mathbf{P}\left(\xi < T_x \le \xi + \delta\xi \middle| T_x > \xi\right)$ (2)

$$\mathbf{P}\left(0 < T_{x+\xi} \le \delta\xi\right) = \frac{1}{\delta\xi}\frac{F_{T_x}\left(\xi+\delta\xi\right) - F_{T_x}\left(\xi\right)}{1 - F_{T_x}\left(\xi\right)} \tag{3}$$

$$\mu_{x+\xi} = \lim_{\delta\xi\to 0}\left(\frac{\mathbf{P}\left(\xi < T_x \le \xi+\delta\xi \middle| T_x > \xi\right)}{\delta\xi}\right) = \frac{\dfrac{F_{T_x}\left(\xi+\delta\xi\right) - F_{T_x}\left(\xi\right)}{\delta\xi}}{1 - F_{T_x}\left(\xi\right)} \tag{4}$$

where $F'_{T_x}\left(\xi^+\right) \to \dfrac{d}{d\xi}F_{T_x}\left(\xi\right)$, consequently

$$\mu_{x+\xi} = \frac{F'_{T_x}\left(\xi\right)}{S_{T_x}\left(\xi\right)} = \frac{f_{T_x}\left(\xi\right)}{S_{T_x}\left(\xi\right)} = -\frac{d}{d\xi}\log_e\left(1 - F_{T_x}\left(\xi\right)\right) \tag{5}$$

$$f_{T_x}\left(\xi\right) = \mu_{x+\xi} \times S_{T_x}\left(\xi\right) = \mu_{x+\xi} \times {}_{\xi}\,p_x \tag{6}$$

$${}_{\xi}\,p_x = \exp\left(-\int_0^{\xi}\mu_{x+\theta}d\theta\right) = \exp\left(-\int_x^{x+\xi}\mu_{\theta}d\theta\right) \tag{7}$$

Given that $\mu_x = A + BC^x$ and that the modal age at death is α defines the dispersion co-efficient, then we can re-express this as

$$\mu_x = h + \frac{1}{\beta}\exp\left(\frac{x-\alpha}{\beta}\right) \tag{8}$$

where $\alpha > 0$; $h > 0$; $\beta > 0$; $h = A$; $B = \dfrac{1}{\beta}\exp\left(\dfrac{-\alpha}{\beta}\right)$ and $C = \exp\left(\dfrac{1}{\beta}\right)$

$${}_{\xi}\,p_x = \exp\left(-\int_x^{x+\xi} h + \frac{1}{\beta}\exp\left(\frac{U-\alpha}{\beta}\right)dU\right) \tag{9}$$

$${}_{\xi}\,p_x = \exp\left(-h\xi + \mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\left(1 - e^{\frac{\xi}{\beta}}\right)\right) \tag{10}$$

$$\lim_{\xi\to 0}\left({}_{\xi}\,p_x\right) = \lim_{\xi\to 0}\exp\left(-h\xi + \mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\left(1 - e^{\frac{\xi}{\beta}}\right)\right) = 1 \tag{11}$$

$$\lim_{\xi\to\infty}\left({}_{\xi}\,p_x\right) = \lim_{\xi\to\infty}\exp-\left(h\xi - \mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\left(1 - e^{\frac{\xi}{\beta}}\right)\right) = e^{-\infty} = 0 \tag{12}$$

The smooth density function defining the distribution of deaths over all ages is obtained from cause specific smooth mortality intensity as follows

$$f_{T_x}(\xi)=\left\{h+\frac{1}{\beta}\exp\left(\frac{x-\alpha}{\beta}\right)\right\}\left\{\exp\left(-h\xi+\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\left(1-e^{\frac{\xi}{\beta}}\right)\right)\right\} \quad (13)$$

Since T_x is the complete future lifetime, then its first moment is

$$\bar{e}_x=\mathbf{E}(T_x)=\int_0^\infty \exp\left(-h\xi+\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\left(1-e^{\frac{\xi}{\beta}}\right)\right)d\xi \quad (14)$$

$$\bar{e}_x=\exp\left\{\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\right\}\int_0^\infty \exp\left(-h\xi+\mathrm{e}^{\left(\frac{x-\alpha+\xi}{\beta}\right)}\right)d\xi \quad (15)$$

$$\bar{e}_x=\beta\exp\left\{\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}+h(x-\alpha)\right\}\int_{\exp\left(\frac{x-m}{\beta}\right)}^\infty e^{-S}S^{-h\beta-1}dS \quad (16)$$

Given that

$$G(A,B)=\int_B^\infty e^{-\xi}\xi^{A-1}d\xi \quad (17)$$

$$\bar{e}_x=\beta\exp\left\{\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}+a(x-\alpha)\right\}G\left(-a\beta,\mathrm{e}^{\left(\frac{x-\alpha}{\beta}\right)}\right) \quad (18)$$

1.1 Orthogonality Conditions In Generalized Makeham's Law of Mortality

Theorem 1

If the Makeham's law of mortality satisfies the condition

$$GM(q,p)=\sum_{k=1}^{q}\beta_k C_{k-1}(\xi(x))+\exp\sum_{k=q+1}^{q+p}\beta_k C_{k-q-1}(\xi(x)) \quad (18a)$$

and suppose

$$\xi(\beta)=\frac{\beta-\left(\frac{\beta+\alpha}{2}\right)}{\left(\frac{\beta-\alpha}{2}\right)}=1 \quad \xi(\alpha)=\frac{\alpha-\left(\frac{\beta+\alpha}{2}\right)}{\left(\frac{\beta-\alpha}{2}\right)}=-1 \quad (18b)$$

then

$$\int_0^\pi \cos 2\phi\cos\phi d\phi=\frac{(\sin 2\phi+2\phi)}{4}=\frac{\pi}{2} \quad (18c)$$

Proof

When we investigate mortality trends of the assured between integral ages β and α with $\beta>\alpha$ under the force function the generalized Makeham's law $GM(q,p)$, we can then employ transformation

$$\xi(x)=\frac{x-\left(\frac{\beta+\alpha}{2}\right)}{\left(\frac{\beta-\alpha}{2}\right)} \quad (19)$$

so that

$$\xi\left(\beta\right)=\frac{\beta-\left(\dfrac{\beta+\alpha}{2}\right)}{\left(\dfrac{\beta-\alpha}{2}\right)}=1 \tag{19a}$$

and

$$\xi\left(\alpha\right)=\frac{\alpha-\left(\dfrac{\beta+\alpha}{2}\right)}{\left(\dfrac{\beta-\alpha}{2}\right)}=-1 \tag{19b}$$

so that, the force of mortality $GM\left(q,p\right)$ will be orthogonally compatible with Chebyshev polynomial $C_q\left(x\right)$ of the first kind in the mortality modelling with the age weighting $\sqrt{1-x^2}$. Consequently, we can express the forcing function as $GM\left(q,p\right)=\sum_{k=1}^{q}\beta_k C_{k-1}\left(\xi\left(x\right)\right)+\exp\sum_{k=q+1}^{q+p}\beta_k C_{k-q-1}\left(\xi\left(x\right)\right)$ (20)

$\xi\left(x\right)\in\left[0,1\right]$; $C_0\left(x\right)=1$; $C_1\left(x\right)=x$ and $C_{q+1}\left(x\right)=2xC_q\left(x\right)-C_{q-1}\left(x\right)$, $q\geq 1$ and the orthogonality condition with respect to age of the assured is given as follows

$$\int_{-1}^{1}\frac{C_p\left(x\right)C_q\left(x\right)}{\sqrt{1-x^2}}dx=\begin{cases}0 & if \quad q\neq p\\ \dfrac{\pi}{2} & if \quad q=p\end{cases} \tag{21}$$

We let $x=\cos\phi$, $dx=-\sin\phi d\phi$

$$\int_{-1}^{1}\frac{C_p\left(x\right)C_q\left(x\right)}{\sqrt{1-x^2}}dx=-\int_{\pi}^{0}\frac{\cos p\phi\cos q\phi}{\sqrt{1-\cos^2\phi}}\sin\phi d\phi \tag{22}$$

$$\int_{-1}^{1}\frac{C_p\left(x\right)C_q\left(x\right)}{\sqrt{1-x^2}}dx=\int_{0}^{\pi}\frac{\cos p\phi\cos q\phi}{\sqrt{\sin^2\phi}}\sin\phi d\phi=\int_{0}^{\pi}\frac{\cos p\phi\cos q\phi}{\sin\phi}\sin\phi d\phi$$

$$=\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi \tag{23}$$

For $q\neq p$, we integrate by parts two times

$$\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi=\left[\frac{1}{m}\cos p\phi\sin q\phi\right]_0^{\pi}+\int_{0}^{\pi}\frac{p}{m}\sin p\phi\sin q\phi d\phi \tag{24}$$

$$\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi=0+\left[-\frac{p}{q^2}\sin p\phi\cos q\phi\right]_0^{\pi}+\int_{0}^{\pi}\frac{p^2}{q^2}\cos p\phi\cos q\phi d\phi \tag{25}$$

$$\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi-\int_{0}^{\pi}\frac{p^2}{q^2}\cos p\phi\cos q\phi d\phi=0 \tag{26}$$

$$\left(1-\frac{p^2}{q^2}\right)\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi=0 \tag{27}$$

Since

$$q\neq p,\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi=0 \tag{28}$$

For $q=p$, $\int_{0}^{\pi}\cos p\phi\cos q\phi d\phi=\int_{0}^{\pi}\cos^2 q\phi d\phi$ (29)

Observe that

$$\cos 2\phi = 2\cos^2\phi - 1 \qquad (30)$$

$$\int_0^{\pi} \cos p\phi \cos q\phi d\phi = \int_0^{\pi} \cos^2 q\phi d\phi = \int_0^{\pi} \left(\frac{\cos 2q\phi + 1}{2} \right) d\phi \qquad (31)$$

$$\int_0^{\pi} \cos p\phi \cos q\phi d\phi = \left[\frac{\sin 2q\phi}{4q} + \frac{\phi}{2} \right] = 0 + \frac{\pi}{2} - 0 = \frac{\pi}{2} \qquad (32)$$

hence for Makeham, $p = 1; q = 2$ and

$$\int_0^{\pi} \cos 2\phi \cos \phi d\phi = \left[\frac{\sin 2\phi}{4} + \frac{\phi}{2} \right] = \frac{\pi}{2} \qquad (33)$$

$Q.E.D$

1.2 Theorem 2

If $\frac{dl_x}{dx} = -l_x\mu_x$ and $\frac{d^2 l_x}{dx^2} = 0$ when $\frac{d}{dx}(l_x\mu_x) = 0$ and $e^{\beta} = b$ then $2a^2 + \beta^2 = 4a\beta$

Proof

$$\frac{d}{dx}(l_x\mu_x) = l'_x\mu_x + l_x\mu'_x \qquad (34a)$$

$$-l_x(\mu_x)^2 + l_x\mu'_x = 0 \Rightarrow (\mu_x)^2 = \mu'_x$$

Observe that

$$b\beta e^{\beta x} = \left(a + be^{\beta x}\right)^2 \qquad (34b)$$

$$b\beta e^{\beta x} = a^2 + b^2 e^{2\beta x} + 2abe^{\beta x} \qquad (35)$$

$$b^2 e^{2\beta x} + \left(2ab - b\beta\right)e^{\beta x} + a^2 = 0 \qquad (36)$$

$$e^{\beta x} = \frac{-\left(2ab - b\beta\right) \pm \sqrt{\left(2ab - b\beta\right)^2 - 4b^2a^2}}{2b^2} \qquad (37)$$

$$e^{\beta x} = \frac{-\left(2ab - b\beta\right) \pm \sqrt{\left(2ab - b\beta\right)^2 - 4b^2a^2}}{2b^2} \qquad (38)$$

$$x_M = \frac{1}{\beta}\log_e\left[\frac{-\left(2ab - b\beta\right) \pm \sqrt{\left(2ab - b\beta\right)^2 - 4b^2a^2}}{2b^2} \right] \qquad (39)$$

$$\mu_x = h + \frac{1}{\beta}\exp\left(\frac{x - \frac{1}{\beta}\log_e\left[\frac{-\left(2ab - b\beta\right) \pm \sqrt{\left(2ab - b\beta\right)^2 - 4b^2a^2}}{2b^2} \right]}{\beta} \right) \qquad (40)$$

For the Makeham's law, observe that

$$\frac{1}{\beta}\log_e\left[\frac{-(2ab-b\beta)+\sqrt{(2ab-b\beta)^2-4b^2a^2}}{2b^2}\right]$$
$$=\frac{1}{\beta}\log_e\left[\frac{-(2ab-b\beta)-\sqrt{(2ab-b\beta)^2-4b^2a^2}}{2b^2}\right] \quad (41)$$

$$\sqrt{(2ab-b\beta)^2-4b^2a^2}+\sqrt{(2ab-b\beta)^2-4b^2a^2}=0 \quad (42)$$

$$2\sqrt{2(2ab-b\beta)^2-4b^2a^2}=0 \quad (43)$$

$$2(2ab-b\beta)^2-4b^2a^2=0 \quad (44)$$

$$(2ab-b\beta)^2-2b^2a^2=0 \quad (45)$$

$$4a^2b^2+b^2\beta^2-4ab^2\beta-2b^2a^2=0 \quad (46)$$

$$2a^2+\beta^2-4a\beta=0 \quad (47)$$

$$2a^2+\beta^2=4a\beta \quad (48)$$

$Q.E.D$

1.3 Modification Theorem

When policy holders change from one policy to another, the term and conditions are modified. The probability of survival or otherwise in such conditions are determined through modification theorem

Theorem 3

$$e^{-\left(a\Delta+\frac{e^{b+cx+c\frac{\Delta}{2}+c\Delta}}{c}-\frac{e^{b+cx+c\frac{\Delta}{2}}}{c}\right)}\simeq\frac{\left(1-\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)\right)}{\left[\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)+1\right]} \quad (49)$$

Proof

Following (Walters, & Wilkie, 1987; Castro-Perez., Aguilar-Sanchez G., & Gonzalez-Nucamendi 2020), for a mortality table, the following relationship holds

$$l_U\mu_U=\left(\mu_{x+\frac{\Delta}{2}}\right)\times\frac{[l_x+l_{x+\Delta}]}{2} \quad (50)$$

Multiplying both sides of (50) by Δ

$$\Delta l_U\mu_U=\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)\times\frac{[l_x+l_{x+\Delta}]}{2} \quad (51)$$

$$l_x-l_{x+\Delta}=\Delta\times l_U\mu_U \quad (52)$$

$$l_x-l_{x+\Delta}=\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)\times\frac{[l_x+l_{x+\Delta}]}{2} \quad (53)$$

$$l_x-l_{x+\Delta}=\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)\frac{l_x}{2}+\frac{l_{x+\Delta}}{2}\Delta\left(\mu_{x+\frac{\Delta}{2}}\right) \quad (54)$$

$$l_x-\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)\frac{l_x}{2}=\frac{l_{x+\Delta}}{2}\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)+l_{x+\Delta} \quad (55)$$

Factorising the $l_{(.)}$ in (55)

$$l_x\left(1-\frac{\Delta}{2}\left(\mu_{x+\frac{\Delta}{2}}\right)\right)=\left[\frac{1}{2}\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)+1\right]l_{x+\Delta} \tag{56}$$

$$\left[\frac{1}{2}\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)+1\right]\frac{l_{x+\Delta}}{l_x}=\left(1-\frac{\Delta}{2}\left(\mu_{x+\frac{\Delta}{2}}\right)\right) \tag{57}$$

$$\left[\frac{1}{2}\Delta\left(\mu_{x+\frac{\Delta}{2}}\right)+1\right]\left({}_{\Delta}P_x\right)=\left(1-\frac{\Delta}{2}\left(\mu_{x+\frac{\Delta}{2}}\right)\right) \tag{58}$$

$${}_{\Delta}p_x=\frac{\left(1-\frac{\Delta}{2}\left(\mu_{x+\frac{\Delta}{2}}\right)\right)}{\left[\frac{\Delta}{2}\left(\mu_{x+\frac{\Delta}{2}}\right)+1\right]} \tag{59}$$

Substituting $\mu_x=GM\left(1,2\right)=a+e^be^{Cx}$ in (59), we have
the survival probability is

$${}_{\Delta}p_x=\frac{\left(1-\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)\right)}{\left[\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)+1\right]} \tag{60}$$

Death probability is

$${}_{\Delta}q_x=1-\frac{\left(1-\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)\right)}{\left[\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)+1\right]}=\frac{\Delta a+\Delta e^{b+cx+c\frac{\Delta}{2}}}{\left[\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)+1\right]} \tag{61}$$

$$\mu_{x+s}=a+e^{b+c(x+s)}\Rightarrow\mu_{x+\frac{\Delta}{2}+s}=a+e^{b+c\left(cx+c\frac{\Delta}{2}+cs\right)} \tag{62}$$

$$\int_0^{\Delta}\mu_{x+\frac{\Delta}{2}+s}ds=\int_0^{\Delta}ads+\int_0^{\Delta}e^{b+\left(cx+c\frac{\Delta}{2}+cs\right)}ds=a\Delta+\frac{e^{b+cx+c\frac{\Delta}{2}+c\Delta}}{c}-\frac{e^{b+cx+c\frac{\Delta}{2}}}{c} \tag{63}$$

$${}_{\Delta}p_x=e^{-\left(a\Delta+\frac{e^{b+cx+c\frac{\Delta}{2}+c\Delta}}{c}-\frac{e^{b+cx+c\frac{\Delta}{2}}}{c}\right)} \tag{64}$$

Equating (64) and (60)
Hence

$$e^{-\left(a\Delta+\frac{e^{b+cx+c\frac{\Delta}{2}+c\Delta}}{c}-\frac{e^{b+cx+c\frac{\Delta}{2}}}{c}\right)}\cong\frac{\left(1-\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)\right)}{\left[\frac{\Delta}{2}\left(a+e^{b+cx+c\frac{\Delta}{2}}\right)+1\right]} \tag{65}$$

$$Error = \left(e^{-\left(a\Delta + \frac{e^{b+cx+c\frac{\Delta}{2}+c\Delta}}{c} - \frac{e^{b+cx+c\frac{\Delta}{2}}}{c} \right)} \right) - \frac{\left(1 - \frac{\Delta}{2}\left(a + e^{b+cx+c\frac{\Delta}{2}} \right) \right)}{\left[\frac{\Delta}{2}\left(a + e^{b+cx+c\frac{\Delta}{2}} \right) + 1 \right]} \tag{66}$$

Q.E.D.

2. Materials and Methods

The Makeham's law for the force of mortality is given as $\mu_x = a + bc^x$ where the second term of the equation represents the Gompertz's term and a is the autonomous risk of mortality parameter in the young adulthood. The integrated hazard is defined as follows.

$$\int_0^x \mu_\xi d\xi = \int_0^x \left(a + bc^\xi \right) d\xi \tag{67}$$

$$\int_0^x \mu_\xi d\xi = ax + \frac{bc^x}{\log_e c} - \frac{b}{\log_e c} \tag{68}$$

$$\int_0^x \mu_\xi d\xi = -\log_e s^x - \log_e g^{c^x - 1} \tag{69}$$

where $(\log_e g) \times (\log_e c) = -b$ and $-a = \log_e S$ then,

$$l_x = l_0 \times e^{-\int_0^x \mu_\xi d\xi} \tag{70}$$

$$l_x = l_0 \times e^{\log_e s^x + \log_e g^{c^x - 1}} \tag{71}$$

$$l_x = l_0 s^x g^{c^x - 1} \tag{72}$$

Define $kg = l_0$, then

$$l_x = ks^x g^{c^x} \tag{73}$$

$$l_{x+\xi} = ks^{x+\xi} g^{c^{x+\xi}} \tag{74}$$

$${}_\xi p_x = \frac{ks^{x+\xi} g^{c^{x+\xi}}}{ks^x g^{c^x}} \tag{75}$$

$${}_\xi p_x = s^x g^{c^x \left(c^\xi - 1 \right)} \tag{76}$$

$$\log_e l_x = \log_e \kappa + x(\log_e s) + (\log_e g) c^x \tag{77}$$

where $\log_e \kappa$ defines the constant of integration

$$\log_e l_x = \log_e \kappa s^x g^{c^x} \tag{78}$$

$$\log_e l_x = \log_e \kappa s^x g^{c^x} \tag{79}$$

$$l_x = \kappa s^x g^{c^x} \tag{80}$$

Since human age has chronological order, we define four equidistant values of $\log_e l_x$

$$\log_e l_x = \log_e k + x \log_e s + c^x \log_e g \tag{81}$$

$$\log_e l_{x+\xi} = \log_e k + (x+\xi)\log_e s + c^{x+\xi}\log_e g \quad (82)$$

$$\log_e l_{x+2\xi} = \log_e k + (x+2\xi)\log_e s + c^{x+2\xi}\log_e g \quad (83)$$

$$\log_e l_{x+3\xi} = \log_e k + (x+3\xi)\log_e s + c^{x+3\xi}\log_e g \quad (84)$$

Taking now the differences pairwise as follows, we have

$$\log_e l_{x+\xi} - \log_e l_x = \xi\log_e s + c^{x}\left(c^{\xi}-1\right)\times\log_e g \quad (85)$$

$$\log_e l_{x+2\xi} - \log_e l_x = \xi\log_e s + c^{x+\xi}\left(c^{\xi}-1\right)\times\log_e g \quad (86)$$

$$\log_e l_{x+3\xi} - \log_e l_x = \xi\log_e s + c^{x+2\xi}\left(c^{\xi}-1\right)\times\log_e g \quad (87)$$

Taking differences again, we have

$$\log_e l_{x+2\xi} - 2\log_e l_{x+\xi} + \log_e l_x = c^{x}\left(c^{\xi}-1\right)^2\log_e g \quad (88)$$

$$\log_e l_{x+3\xi} - 2\log_e l_{x+2\xi} + \log_e l_x = c^{x+\xi}\left(c^{\xi}-1\right)^2\log_e g \quad (89)$$

Dividing the second by the first, we have

$$c^{\xi} = \left(\frac{\log_e l_{x+3\xi} - 2\log_e l_{x+2\xi} + \log_e l_{x+\xi}}{\log_e l_{x+2\xi} - 2\log_e l_{x+2\xi} + \log_e l_x}\right) \quad (90)$$

To estimate the parameters, we use the following transformation

$$_{\xi}p_x = s^x g^{c^x\left(c^{\xi}-1\right)} \quad (91)$$

Let $U = e^{-A}$ and $V = e^{\frac{b}{\log_e c}}$

Substitute this in equation (76), we obtain

$$_{\xi}p_x = U^{\xi}V^{x\left(1-c^{\xi}\right)} \quad (92)$$

Omitting the too lengthy callisthenics of estimation details, we have

MEN

$$l_x = ks^x g^{c^x} \quad (93)$$

$$\log_e l_x = \log_e k + x\log_e s + c^x\log_e g \quad (94)$$

$$\ln l_{20} = \ln k + 20\ln s + c^{20}\ln g \quad (95)$$

$$\ln l_{40} = \ln k + 40\ln s + c^{40}\ln g \quad (96)$$

$$\ln l_{60} = \ln k + 60\ln s + c^{60}\ln g \quad (97)$$

$$\ln l_{80} = \ln k + 80\ln s + c^{80}\ln g \quad (98)$$

$$c = 1.09854562 \quad (99)$$

$$g = 0.9996478151 \quad (100)$$

$$s = 0.999381513 \quad (101)$$

$$k = 100369.8071 \quad (101)$$

Recall

$$\mu_x = a + bc^x \quad (103)$$

The value of a, b and c are found as follows

$$a = -\ln s \quad (104)$$

$$a = -\left(-0.000618678316\right) \quad (105)$$

$$a = 0.000618678316 \quad (106)$$

$$b = (-\ln g)(\ln c) \quad (107)$$

$$b = -(-0.000352246929)(\ln 1.098554562) \quad (108)$$

$$b = 0.00331066819 \quad (109)$$

WOMEN

$$l_x = ks^x g^{c^x} \quad (110)$$

$$\ln l_x = \ln k + x \ln s + c^x \ln g \quad (111)$$

$$\ln l_{20} = \ln k + 20 \ln s + c^{20} \ln g \quad (112)$$

$$\ln l_{40} = \ln k + 40 \ln s + c^{40} \ln g \quad (113)$$

$$\ln l_{60} = \ln k + 60 \ln s + c^{60} \ln g \quad (114)$$

$$\ln l_{80} = \ln k + 80 \ln s + c^{80} \ln g \quad (115)$$

$$c = 1.1036915 \quad (116)$$

$$g = 0.9998230123 \quad (117)$$

$$S = 0.999732372 \quad (118)$$

$$k = 99895.35074 \quad (119)$$

Recall

$$\mu_x = a + bc^x \quad (120)$$

The value of a , b and c are found as follows

$$a = -\ln s \quad (121)$$

$$a = -(-0.00026766376) \quad (122)$$

$$a = 0.00026766376 \quad (123)$$

$$b = (\ln g)(\ln c) \quad (124)$$

$$b = -(-0.0001680018067)(\ln 1.1036915) \quad (125)$$

$$b = (0.0001680018067)(0.0986604704) \quad (126)$$

$$b = 0.00001657513728 \quad (127)$$

3. Results

Table 1: Male's Survivor Function

x	k	s^x	c^x	g^{C^x}	l_x
0	100369.807100	1.000000	1.000000	0.999648	100334.000000
1	100369.807100	0.999382	1.098546	0.999613	100269.000000
2	100369.807100	0.998763	1.206803	0.999575	100203.000000
3	100369.807100	0.998146	1.325728	0.999533	100137.000000
4	100369.807100	0.997528	1.456372	0.999487	100070.000000
5	100369.807100	0.996911	1.599891	0.999437	100003.000000
6	100369.807100	0.996295	1.757554	0.999381	99936.000000
7	100369.807100	0.995679	1.930753	0.999320	99868.000000
8	100369.807100	0.995063	2.121020	0.999253	99800.000000

9	100369.807100	0.994447	2.330037	0.999180	99731.000000
10	100369.807100	0.993832	2.559652	0.999099	99661.000000
11	100369.807100	0.993218	2.811895	0.999010	99590.000000
12	100369.807100	0.992603	3.088995	0.998913	99519.000000
13	100369.807100	0.991989	3.393402	0.998805	99447.000000
14	100369.807100	0.991376	3.727807	0.998688	99374.000000
15	100369.807100	0.990763	4.095166	0.998559	99299.000000
16	100369.807100	0.990150	4.498726	0.998417	99224.000000
17	100369.807100	0.989538	4.942056	0.998261	99147.000000
18	100369.807100	0.988926	5.429074	0.998090	99069.000000
19	100369.807100	0.988314	5.964085	0.997901	98989.000000
20	100369.807100	0.987703	6.551820	0.997695	98907.000000
21	100369.807100	0.987092	7.197473	0.997468	98823.000000
22	100369.807100	0.986481	7.906752	0.997219	98738.000000
23	100369.807100	0.985871	8.685928	0.996945	98649.000000
24	100369.807100	0.985261	9.541888	0.996645	98559.000000
25	100369.807100	0.984652	10.482200	0.996315	98465.000000
26	100369.807100	0.984043	11.515174	0.995952	98368.000000
27	100369.807100	0.983434	12.649944	0.995554	98268.000000
28	100369.807100	0.982826	13.896541	0.995117	98164.000000
29	100369.807100	0.982218	15.265984	0.994637	98056.000000
30	100369.807100	0.981611	16.770380	0.994110	97944.000000
31	100369.807100	0.981004	18.423028	0.993532	97826.000000
32	100369.807100	0.980397	20.238536	0.992896	97703.000000
33	100369.807100	0.979791	22.232955	0.992199	97574.000000
34	100369.807100	0.979185	24.423916	0.991434	97439.000000
35	100369.807100	0.978579	26.830786	0.990594	97296.000000
36	100369.807100	0.977974	29.474842	0.989671	97145.000000
37	100369.807100	0.977369	32.379459	0.988659	96986.000000
38	100369.807100	0.976764	35.570313	0.987549	96817.000000
39	100369.807100	0.976160	39.075611	0.986330	96638.000000
40	100369.807100	0.975557	42.926341	0.984993	96447.000000
41	100369.807100	0.974953	47.156544	0.983526	96244.000000
42	100369.807100	0.974350	51.803615	0.981918	96027.000000
43	100369.807100	0.973748	56.908635	0.980154	95795.000000
44	100369.807100	0.973145	62.516731	0.978219	95547.000000
45	100369.807100	0.972544	68.677481	0.976099	95281.000000
46	100369.807100	0.971942	75.445346	0.973775	94995.000000
47	100369.807100	0.971341	82.880155	0.971228	94688.000000
48	100369.807100	0.970740	91.047631	0.968438	94358.000000
49	100369.807100	0.970140	100.019976	0.965382	94002.000000
50	100369.807100	0.969540	109.876507	0.962036	93618.000000
51	100369.807100	0.968940	120.704355	0.958374	93204.000000

52	100369.807100	0.968341	132.599240	0.954366	92757.000000
53	100369.807100	0.967742	145.666315	0.949984	92274.000000
54	100369.807100	0.967143	160.021092	0.945192	91752.000000
55	100369.807100	0.966545	175.790470	0.939957	91187.000000
56	100369.807100	0.965947	193.113851	0.934238	90576.000000
57	100369.807100	0.965350	212.144375	0.927997	89915.000000
58	100369.807100	0.964753	233.050274	0.921188	89201.000000
59	100369.807100	0.964156	256.016358	0.913766	88427.000000
60	100369.807100	0.963560	281.245648	0.905681	87591.000000
61	100369.807100	0.962964	308.961175	0.896882	86686.000000
62	100369.807100	0.962368	339.407946	0.887315	85708.000000
63	100369.807100	0.961773	372.855112	0.876922	84652.000000
64	100369.807100	0.961178	409.598350	0.865646	83512.000000
65	100369.807100	0.960584	449.962474	0.853425	82282.000000
66	100369.807100	0.959990	494.304304	0.840198	80956.000000
67	100369.807100	0.959396	543.015829	0.825905	79530.000000
68	100369.807100	0.958803	596.527660	0.810483	77997.000000
69	100369.807100	0.958210	655.312848	0.793873	76351.000000
70	100369.807100	0.957617	719.891059	0.776018	74588.000000
71	100369.807100	0.957025	790.833170	0.756866	72702.000000
72	100369.807100	0.956433	868.766315	0.736372	70689.000000
73	100369.807100	0.955841	954.379430	0.714497	68547.000000
74	100369.807100	0.955250	1048.429343	0.691214	66272.000000
75	100369.807100	0.954659	1151.747462	0.666510	63864.000000
76	100369.807100	0.954069	1265.247130	0.640389	61323.000000
77	100369.807100	0.953479	1389.931693	0.612872	58652.000000
78	100369.807100	0.952889	1526.903373	0.584004	55855.000000
79	100369.807100	0.952300	1677.373013	0.553857	52939.000000
80	100369.807100	0.951711	1842.670776	0.522529	49914.000000
81	100369.807100	0.951122	2024.257911	0.490153	46792.000000
82	100369.807100	0.950534	2223.739661	0.456893	43590.000000
83	100369.807100	0.949946	2442.879465	0.422952	40327.000000
84	100369.807100	0.949358	2683.614536	0.388565	37025.000000
85	100369.807100	0.948771	2948.072995	0.354003	33711.000000
86	100369.807100	0.948184	3238.592676	0.319568	30413.000000
87	100369.807100	0.947598	3557.741799	0.285588	27162.000000
88	100369.807100	0.947012	3908.341670	0.252410	23992.000000
89	100369.807100	0.946426	4293.491623	0.220387	20935.000000
90	100369.807100	0.945841	4716.596417	0.189872	18025.000000
91	100369.807100	0.945256	5181.396336	0.161197	15294.000000
92	100369.807100	0.944671	5692.000250	0.134662	12768.000000
93	100369.807100	0.944087	6252.921944	0.110519	10472.000000
94	100369.807100	0.943503	6869.120014	0.088955	8424.000000

95	100369.807100	0.942919	7546.041704	0.070083	6633.000000
96	100369.807100	0.942336	8289.671062	0.053933	5101.000000
97	100369.807100	0.941753	9106.581837	0.040447	3823.000000
98	100369.807100	0.941171	10003.995590	0.029485	2785.000000
99	100369.807100	0.940589	10989.845538	0.020835	1967.000000
100	100369.807100	0.940007	12072.846680	0.014227	1342.000000
101	100369.807100	0.939426	13262.572841	0.009356	882.000000
102	100369.807100	0.938845	14569.541305	0.005904	556.000000
103	100369.807100	0.938264	16005.305786	0.003561	335.000000
104	100369.807100	0.937684	17582.558568	0.002043	192.000000
105	100369.807100	0.937104	19315.242703	0.001110	104.000000
106	100369.807100	0.936524	21218.675271	0.000568	53.000000
107	100369.807100	0.935945	23309.682781	0.000272	26.000000
108	100369.807100	0.935366	25606.749922	0.000121	11.000000
109	100369.807100	0.934788	28130.182970	0.000050	5.000000
110	100369.807100	0.934209	30902.289291	0.000019	2.000000
111	100369.807100	0.933632	33947.574549	0.000006	1.000000
112	100369.807100	0.933054	37292.959330	0.000002	0.000000
113	100369.807100	0.932477	40968.017129	0.000001	0.000000
114	100369.807100	0.931900	45005.235777	0.000000	0.000000
115	100369.807100	0.931324	49440.304640	0.000000	0.000000
116	100369.807100	0.930748	54312.430113	0.000000	0.000000
117	100369.807100	0.930172	59664.682213	0.000000	0.000000
118	100369.807100	0.929597	65544.375313	0.000000	0.000000
119	100369.807100	0.929000	72003.486400	0.000000	0.000000

Source: Authors' Computation

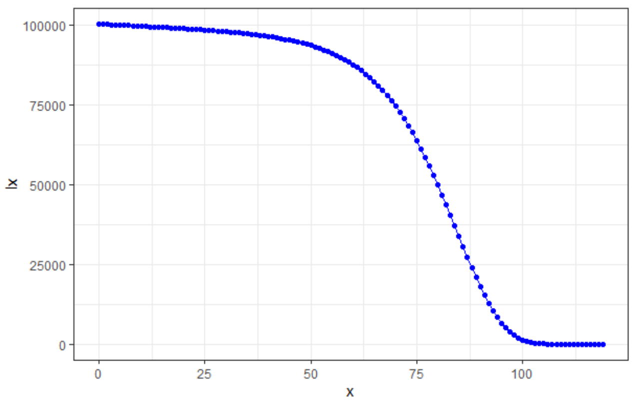

Figure 1: Male's Survivor's Trajectory

Table 2: Male's Curve of Death

x	l_x	μ_x	$l_x\mu_x$

0	100334.000000	0.000652	65.396495
1	100269.000000	0.000655	65.680909
2	100203.000000	0.000659	65.996913
3	100137.000000	0.000663	66.347593
4	100070.000000	0.000667	66.736332
5	100003.000000	0.000672	67.166845
6	99936.000000	0.000677	67.643209
7	99868.000000	0.000683	68.169897
8	99800.000000	0.000689	68.751819
9	99731.000000	0.000696	69.394361
10	99661.000000	0.000703	70.103433
11	99590.000000	0.000712	70.885521
12	99519.000000	0.000721	71.747739
13	99447.000000	0.000731	72.697892
14	99374.000000	0.000742	73.744540
15	99299.000000	0.000754	74.897076
16	99224.000000	0.000768	76.165799
17	99147.000000	0.000782	77.562003
18	99069.000000	0.000798	79.098073
19	98989.000000	0.000816	80.787589
20	98907.000000	0.000836	82.645436
21	98823.000000	0.000857	84.687931
22	98738.000000	0.000880	86.932954
23	98649.000000	0.000906	89.400098
24	98559.000000	0.000935	92.110827
25	98465.000000	0.000966	95.088649
26	98368.000000	0.001000	98.359305
27	98268.000000	0.001037	101.950976
28	98164.000000	0.001079	105.894501
29	98056.000000	0.001124	110.223621
30	97944.000000	0.001174	114.975237
31	97826.000000	0.001229	120.189689
32	97703.000000	0.001289	125.911063
33	97574.000000	0.001355	132.187509
34	97439.000000	0.001427	139.071594
35	97296.000000	0.001507	146.620670
36	97145.000000	0.001594	154.897273
37	96986.000000	0.001691	163.969540
38	96817.000000	0.001796	173.911654
39	96638.000000	0.001912	184.804311
40	96447.000000	0.002040	196.735203
41	96244.000000	0.002180	209.799529
42	96027.000000	0.002334	224.100506
43	95795.000000	0.002503	239.749898

44	95547.000000	0.002688	256.868545
45	95281.000000	0.002892	275.586877
46	94995.000000	0.003116	296.045414
47	94688.000000	0.003363	318.395223
48	94358.000000	0.003633	342.798325
49	94002.000000	0.003930	369.428022
50	93618.000000	0.004256	398.469106
51	93204.000000	0.004615	430.117940
52	92757.000000	0.005009	464.582333
53	92274.000000	0.005441	502.081198
54	91752.000000	0.005916	542.843890
55	91187.000000	0.006439	587.109191
56	90576.000000	0.007012	635.123830
57	89915.000000	0.007642	687.140456
58	89201.000000	0.008334	743.414950
59	88427.000000	0.009095	804.202944
60	87591.000000	0.009930	869.755413
61	86686.000000	0.010847	940.313175
62	85708.000000	0.011855	1016.100138
63	84652.000000	0.012963	1097.315101
64	83512.000000	0.014179	1184.121940
65	82282.000000	0.015515	1276.637973
66	80956.000000	0.016983	1374.920357
67	79530.000000	0.018596	1478.950352
68	77997.000000	0.020368	1588.615379
69	76351.000000	0.022314	1703.688838
70	74588.000000	0.024452	1823.807787
71	72702.000000	0.026801	1948.448727
72	70689.000000	0.029381	2076.901925
73	68547.000000	0.032215	2208.245004
74	66272.000000	0.035329	2341.316796
75	63864.000000	0.038749	2474.692913
76	61323.000000	0.042507	2606.664883
77	58652.000000	0.046635	2735.225249
78	55855.000000	0.051169	2858.061546
79	52939.000000	0.056151	2972.562599
80	49914.000000	0.061623	3075.841018
81	46792.000000	0.067635	3164.776062
82	43590.000000	0.074239	3236.080992
83	40327.000000	0.081494	3286.398618
84	37025.000000	0.089464	3312.427638
85	33711.000000	0.098220	3311.080559
86	30413.000000	0.107838	3279.671213
87	27162.000000	0.118404	3216.126087

88	23992.000000	0.130011	3119.209035
89	20935.000000	0.142762	2988.743483
90	18025.000000	0.156770	2825.810794
91	15294.000000	0.172158	2632.898603
92	12768.000000	0.189062	2413.970123
93	10472.000000	0.207632	2174.425892
94	8424.000000	0.228032	1920.934767
95	6633.000000	0.250443	1661.122185
96	5101.000000	0.275062	1403.121034
97	3823.000000	0.302107	1155.012799
98	2785.000000	0.331818	924.210818
99	1967.000000	0.364456	716.858719
100	1342.000000	0.400311	537.329296
101	882.000000	0.439698	387.906195
102	556.000000	0.482968	268.709306
103	335.000000	0.530501	177.885495
104	192.000000	0.582719	112.036134
105	104.000000	0.640082	66.803936
106	53.000000	0.703099	37.508421
107	26.000000	0.772325	19.713507
108	11.000000	0.848373	9.635611
109	5.000000	0.931916	4.348788
110	2.000000	1.023691	1.798117
111	1.000000	1.124510	0.675273
112	0.000000	1.235265	0.000000
113	0.000000	1.356934	0.000000
114	0.000000	1.490593	0.000000
115	0.000000	1.637423	0.000000
116	0.000000	1.798723	0.000000
117	0.000000	1.975918	0.000000
118	0.000000	2.170576	0.000000
119	0.000000	2.384415	0.000000

Authors' Computation

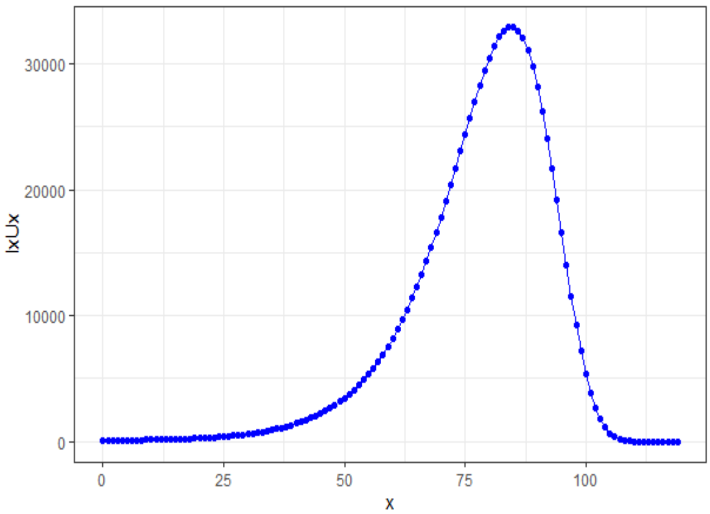

Figure 2: Male's Curve of Death

Table 3: Female's Survivors Function

x	k	s^x	c^x	g^{C^x}	l_x
0	99895.350700	1.000000	1.000000	0.999832	99879
1	99895.350700	0.999732	1.098546	0.999815	99850
2	99895.350700	0.999465	1.206803	0.999795	99821
3	99895.350700	0.999197	1.325728	0.999774	99793
4	99895.350700	0.998930	1.456372	0.999751	99764
5	99895.350700	0.998663	1.599891	0.999725	99734
6	99895.350700	0.998395	1.757554	0.999696	99705
7	99895.350700	0.998128	1.930753	0.999665	99675
8	99895.350700	0.997861	2.121020	0.999630	99645
9	99895.350700	0.997594	2.330037	0.999592	99614
10	99895.350700	0.997327	2.559652	0.999550	99583
11	99895.350700	0.997060	2.811895	0.999503	99552
12	99895.350700	0.996793	3.088995	0.999451	99520
13	99895.350700	0.996526	3.393402	0.999394	99488
14	99895.350700	0.996260	3.727807	0.999332	99455
15	99895.350700	0.995993	4.095166	0.999262	99422
16	99895.350700	0.995727	4.498726	0.999186	99387
17	99895.350700	0.995460	4.942056	0.999102	99352
18	99895.350700	0.995194	5.429074	0.999008	99317
19	99895.350700	0.994927	5.964085	0.998906	99280
20	99895.350700	0.994661	6.551820	0.998792	99242
21	99895.350700	0.994395	7.197473	0.998667	99203
22	99895.350700	0.994129	7.906752	0.998529	99163
23	99895.350700	0.993863	8.685928	0.998377	99121
24	99895.350700	0.993597	9.541888	0.998208	99078
25	99895.350700	0.993331	10.482200	0.998023	99033
26	99895.350700	0.993065	11.515174	0.997818	98986

27	99895.350700	0.992799	12.649944	0.997592	98937
28	99895.350700	0.992533	13.896541	0.997343	98886
29	99895.350700	0.992268	15.265984	0.997067	98832
30	99895.350700	0.992002	16.770380	0.996764	98776
31	99895.350700	0.991737	18.423028	0.996429	98716
32	99895.350700	0.991471	20.238536	0.996059	98653
33	99895.350700	0.991206	22.232955	0.995652	98586
34	99895.350700	0.990941	24.423916	0.995202	98515
35	99895.350700	0.990676	26.830786	0.994705	98440
36	99895.350700	0.990410	29.474842	0.994158	98359
37	99895.350700	0.990145	32.379459	0.993554	98273
38	99895.350700	0.989880	35.570313	0.992888	98181
39	99895.350700	0.989615	39.075611	0.992154	98082
40	99895.350700	0.989351	42.926341	0.991344	97976
41	99895.350700	0.989086	47.156544	0.990450	97862
42	99895.350700	0.988821	51.803615	0.989465	97738
43	99895.350700	0.988556	56.908635	0.988379	97605
44	99895.350700	0.988292	62.516731	0.987182	97460
45	99895.350700	0.988027	68.677481	0.985863	97304
46	99895.350700	0.987763	75.445346	0.984408	97134
47	99895.350700	0.987499	82.880155	0.982805	96950
48	99895.350700	0.987234	91.047631	0.981039	96750
49	99895.350700	0.986970	100.019976	0.979094	96533
50	99895.350700	0.986706	109.876507	0.976951	96295
51	99895.350700	0.986442	120.704355	0.974592	96037
52	99895.350700	0.986178	132.599240	0.971995	95756
53	99895.350700	0.985914	145.666315	0.969136	95448
54	99895.350700	0.985650	160.021092	0.965991	95113
55	99895.350700	0.985386	175.790470	0.962531	94747
56	99895.350700	0.985123	193.113851	0.958727	94348
57	99895.350700	0.984859	212.144375	0.954546	93911
58	99895.350700	0.984595	233.050274	0.949953	93434
59	99895.350700	0.984332	256.016358	0.944909	92913
60	99895.350700	0.984068	281.245648	0.939373	92344
61	99895.350700	0.983805	308.961175	0.933301	91723
62	99895.350700	0.983542	339.407946	0.926645	91044
63	99895.350700	0.983279	372.855112	0.919353	90303
64	99895.350700	0.983015	409.598350	0.911372	89496
65	99895.350700	0.982752	449.962474	0.902644	88615
66	99895.350700	0.982489	494.304304	0.893108	87655
67	99895.350700	0.982226	543.015829	0.882700	86610
68	99895.350700	0.981964	596.527660	0.871354	85474
69	99895.350700	0.981701	655.312848	0.859000	84240
70	99895.350700	0.981438	719.891059	0.845568	82900
71	99895.350700	0.981175	790.833170	0.830988	81449

72	99895.350700	0.980913	868.766315	0.815187	79879
73	99895.350700	0.980650	954.379430	0.798097	78183
74	99895.350700	0.980388	1048.429343	0.779650	76356
75	99895.350700	0.980125	1151.747462	0.759784	74390
76	99895.350700	0.979863	1265.247130	0.738446	72282
77	99895.350700	0.979601	1389.931693	0.715591	70026
78	99895.350700	0.979339	1526.903373	0.691186	67620
79	99895.350700	0.979077	1677.373013	0.665215	65061
80	99895.350700	0.978815	1842.670776	0.637683	62352
81	99895.350700	0.978553	2024.257911	0.608617	59494
82	99895.350700	0.978291	2223.739661	0.578072	56493
83	99895.350700	0.978029	2442.879465	0.546137	53358
84	99895.350700	0.977767	2683.614536	0.512935	50101
85	99895.350700	0.977505	2948.072995	0.478628	46737
86	99895.350700	0.977244	3238.592676	0.443421	43288
87	99895.350700	0.976982	3557.741799	0.407563	39776
88	99895.350700	0.976721	3908.341670	0.371343	36232
89	99895.350700	0.976459	4293.491623	0.335092	32686
90	99895.350700	0.976198	4716.596417	0.299177	29175
91	99895.350700	0.975937	5181.396336	0.263989	25737
92	99895.350700	0.975676	5692.000250	0.229937	22411
93	99895.350700	0.975415	6252.921944	0.197430	19237
94	99895.350700	0.975154	6869.120014	0.166861	16254
95	99895.350700	0.974893	7546.041704	0.138586	13496
96	99895.350700	0.974632	8289.671062	0.112907	10993
97	99895.350700	0.974371	9106.581837	0.090053	8765
98	99895.350700	0.974110	10003.995590	0.070160	6827
99	99895.350700	0.973849	10989.845538	0.053264	5182
100	99895.350700	0.973589	12072.846680	0.039299	3822
101	99895.350700	0.973328	13262.572841	0.028095	2732
102	99895.350700	0.973068	14569.541305	0.019398	1886
103	99895.350700	0.972807	16005.305786	0.012889	1253
104	99895.350700	0.972547	17582.558568	0.008209	797
105	99895.350700	0.972287	19315.242703	0.004989	485
106	99895.350700	0.972026	21218.675271	0.002879	280
107	99895.350700	0.971766	23309.682781	0.001570	152
108	99895.350700	0.971506	25606.749922	0.000804	78
109	99895.350700	0.971246	28130.182970	0.000384	37
110	99895.350700	0.970986	30902.289291	0.000170	16
111	99895.350700	0.970726	33947.574549	0.000069	7
112	99895.350700	0.970467	37292.959330	0.000026	2
113	99895.350700	0.970207	40968.017129	0.000009	1
114	99895.350700	0.969947	45005.235777	0.000003	0
115	99895.350700	0.969688	49440.304640	0.000001	0
116	99895.350700	0.969428	54312.430113	0.000000	0

117	99895.350700	0.969169	59664.682213	0.000000	0
118	99895.350700	0.968909	65544.375313	0.000000	0
119	99895.350700	0.968650	72003.486416	0.000000	0

Source: Authors' Computation

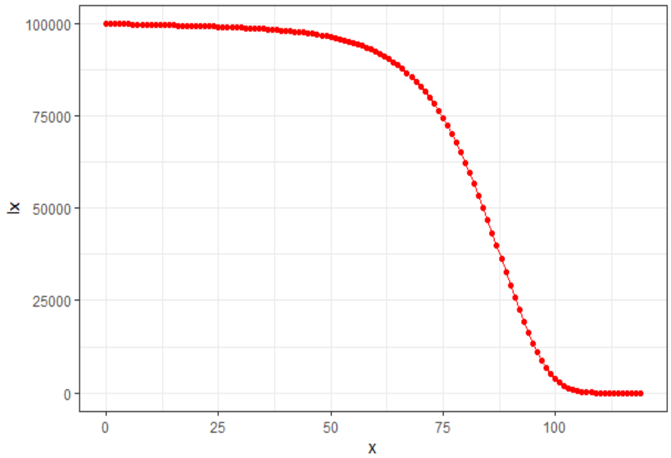

Fig 3. Graph of Female's Survivors:

Table 4: Female's Curve of Death

x	l_x	μ_x	$l_x\mu_x$
0	99879.000000	0.000284	28.389375
1	99850.000000	0.000286	28.552895
2	99821.000000	0.000288	28.734057
3	99793.000000	0.000290	28.934685
4	99764.000000	0.000292	29.156787
5	99734.000000	0.000295	29.402583
6	99705.000000	0.000298	29.674518
7	99675.000000	0.000301	29.975293
8	99645.000000	0.000304	30.307886
9	99614.000000	0.000308	30.675580
10	99583.000000	0.000312	31.081998
11	99552.000000	0.000317	31.531138
12	99520.000000	0.000322	32.027408
13	99488.000000	0.000327	32.575669
14	99455.000000	0.000334	33.181284
15	99422.000000	0.000340	33.850167
16	99387.000000	0.000348	34.588840
17	99352.000000	0.000356	35.404495
18	99317.000000	0.000366	36.305062
19	99280.000000	0.000376	37.299286
20	99242.000000	0.000387	38.396807
21	99203.000000	0.000399	39.608253
22	99163.000000	0.000413	40.945340
23	99121.000000	0.000428	42.420982
24	99078.000000	0.000445	44.049410
25	99033.000000	0.000463	45.846306
26	98986.000000	0.000483	47.828950
27	98937.000000	0.000506	50.016377
28	98886.000000	0.000530	52.429554
29	98832.000000	0.000557	55.091571

30	98776.000000	0.000587	58.027850
31	98716.000000	0.000621	61.266376
32	98653.000000	0.000657	64.837947
33	98586.000000	0.000698	68.776444
34	98515.000000	0.000742	73.119136
35	98440.000000	0.000791	77.906996
36	98359.000000	0.000846	83.185060
37	98273.000000	0.000906	89.002802
38	98181.000000	0.000972	95.414552
39	98082.000000	0.001045	102.479941
40	97976.000000	0.001125	110.264375
41	97862.000000	0.001214	118.839559
42	97738.000000	0.001313	128.284039
43	97605.000000	0.001421	138.683792
44	97460.000000	0.001540	150.132846
45	97304.000000	0.001672	162.733933
46	97134.000000	0.001818	176.599171
47	96950.000000	0.001979	191.850775
48	96750.000000	0.002156	208.621783
49	96533.000000	0.002352	227.056795
50	96295.000000	0.002568	247.312700
51	96037.000000	0.002807	269.559398
52	95756.000000	0.003070	293.980471
53	95448.000000	0.003361	320.773792
54	95113.000000	0.003681	350.152033
55	94747.000000	0.004035	382.343036
56	94348.000000	0.004426	417.589990
57	93911.000000	0.004857	456.151353
58	93434.000000	0.005333	498.300449
59	92913.000000	0.005858	544.324650
60	92344.000000	0.006438	594.524024
61	91723.000000	0.007078	649.209339
62	91044.000000	0.007784	708.699265
63	90303.000000	0.008564	773.316604
64	89496.000000	0.009424	843.383358
65	88615.000000	0.010373	919.214408
66	87655.000000	0.011421	1001.109563
67	86610.000000	0.012578	1089.343716
68	85474.000000	0.013854	1184.154827
69	84240.000000	0.015263	1285.729439
70	82900.000000	0.016818	1394.185461
71	81449.000000	0.018534	1509.551968
72	79879.000000	0.020428	1631.745830
73	78183.000000	0.022518	1760.545073
74	76356.000000	0.024825	1895.559056
75	74390.000000	0.027372	2036.195718
76	72282.000000	0.030182	2181.626483
77	70026.000000	0.033284	2330.749775
78	67620.000000	0.036708	2482.154600
79	65061.000000	0.040486	2634.086271
80	62352.000000	0.044656	2784.417026
81	59494.000000	0.049259	2930.625158
82	56493.000000	0.054339	3069.787080
83	53358.000000	0.059946	3198.587622
84	50101.000000	0.066134	3313.354518
85	46737.000000	0.072964	3410.123453
86	43288.000000	0.080502	3484.739912
87	39776.000000	0.088821	3533.003114
88	36232.000000	0.098004	3550.855339
89	32686.000000	0.108138	3534.616509

90	29175.000000	0.119323	3481.258888
91	25737.000000	0.131668	3388.710052
92	22411.000000	0.145293	3256.164207
93	19237.000000	0.160331	3084.373050
94	16254.000000	0.176929	2875.879093
95	13496.000000	0.195247	2635.148400
96	10993.000000	0.215465	2368.558591
97	8765.000000	0.237779	2084.204165
98	6827.000000	0.262407	1791.496995
99	5182.000000	0.289588	1500.565840
100	3822.000000	0.319588	1221.493491
101	2732.000000	0.352699	963.468553
102	1886.000000	0.389243	733.962736
103	1253.000000	0.429577	538.063031
104	797.000000	0.474092	378.081216
105	485.000000	0.523224	253.524925
106	280.000000	0.577450	161.448023
107	152.000000	0.637299	97.117111
108	78.000000	0.703354	54.858392
109	37.000000	0.776258	28.909591
110	16.000000	0.856721	14.111263
111	7.000000	0.945528	6.329426
112	2.000000	1.043544	2.586002
113	1.000000	1.151723	0.953139
114	0.000000	1.271119	0.000000
115	0.000000	1.402895	0.000000
116	0.000000	1.548336	0.000000
117	0.000000	1.708857	0.000000
118	0.000000	1.886024	0.000000
119	0.000000	2.081560	0.000000

Source: Authors' Computation

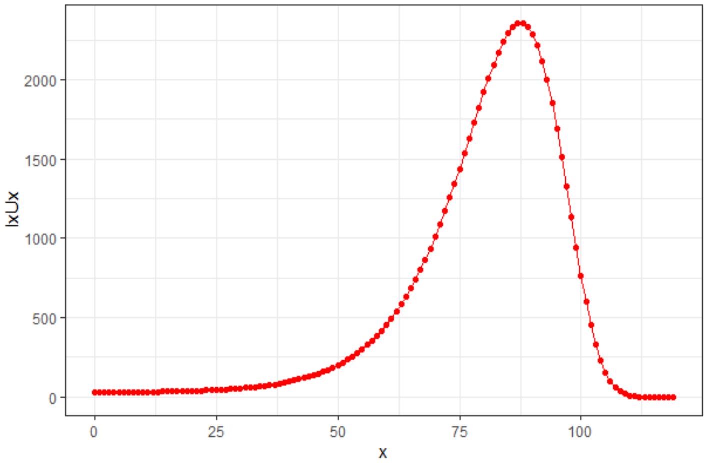

Figure 4: Female's Curve of Death

3.1 Application in Computing the Employer's Liability

From the mortality functions developed in column 3 of tables 2 and 4, the employer's liability to the employee could be derived as follows. Let $r<\infty$ define the retirement age and suppose $\Phi(r)$ represent the accrued benefits at age r, then the commutation function

$$ {}_{\zeta}E_x = \frac{D_{x+\zeta}}{D_x} \tag{128} $$

where $D_x = v^x l_x$; $v = e^{-\sigma}$ is the discount factor and l_x is the number of lives surviving to age x

Let us denote the liability of the pension plan to the employee at r by $\beta(r)$

$$\beta(r) = \frac{\Phi(r)}{D_r}\int_0^\infty D_{r+\zeta}d\zeta = \Phi(r)\int_0^\infty \frac{D_{r+\zeta}}{D_r}d\zeta \quad (129)$$

$$\beta(r) = \Phi(r)\int_0^\infty {}_\zeta E_r d\zeta \quad (130)$$

$$\left({}_\zeta E_r\right) = e^{-\int_0^\zeta (\sigma+\mu_{r+\theta})d\theta} \quad (131)$$

The force of instantaneous mortality rate is given by

$$\mu_{r+\theta} = -\frac{1}{l_{r+\theta}}\frac{dl_{r+\theta}}{d\theta} \quad (132)$$

and the force of interest $\sigma = \log_e(i+1)$ while i is the interest rate

Consequently, putting (131) in (130), the liability function becomes

$$\beta(r) = \Phi(r)\int_0^\infty \left\{ e^{-\int_0^\zeta (\mu_{r+\theta}+\sigma)d\theta} \right\} d\zeta \quad (133)$$

Inserting the $\Phi(r)$ into the integral, we have

$$\beta(r) = \int_0^\infty \Phi(r)\left\{ e^{-\int_0^\zeta \mu_{r+\theta}d\theta} \times e^{-\int_0^\zeta \sigma d\theta} \right\} d\zeta \quad (134)$$

$$\beta(r) = \int_0^\infty \Phi(r)\left\{ e^{-\int_0^\zeta \mu_{r+\theta}d\theta} \right\} e^{-\sigma\zeta} d\zeta \quad (135)$$

Observe that

$$\left({}_\zeta P_r\right) = e^{-\int_0^\zeta \mu_{r+\theta}d\theta} \quad (136)$$

$$\beta(r) = \int_0^\infty \Phi(r)\left({}_\zeta P_r\right)e^{-\sigma\zeta}d\zeta \quad (137)$$

Let x be any age such that $x > r$ and e is the entry age at which the scheme member was incepted. Let $\zeta = x - r$ then $d\zeta = dx$. When $\zeta = 0$ then $x = r$ and if $\zeta = \infty$; $x = \infty$

Assume the employee has put in $r-e$ years of service with $r-e > L$ where L is the total length of service, then the actuarial present value of the future benefits promised by the pension scheme sponsor associated with the employee at the retirement age r is given by

$$\beta(r) = \int_r^\infty \Phi(r)\left({}_{x-r}P_r\right)e^{-\sigma(x-r)}dx = \Phi(r)\int_r^\infty \left({}_{x-r}P_r\right)e^{-\sigma(x-r)}dx \quad (138)$$

$$\beta(r) = \int_r^\infty \Phi(r)\left({}_{x-r}P_r\right)e^{-\sigma(x-r)}dx = \Phi(r)\Lambda(r) \quad (139)$$

The annuity factor defining the present value at retirement age r of a life time annuity of 1 unit of money will then be

$$\Lambda(r)=\int_r^{\infty}\left({}_{x-r}P_r\right)e^{-\sigma(x-r)}dx \quad (140)$$

4. Discussion

In figures 1 and 3, the shape of the survival curves l_x progressively decreases such that it falls steeply forming asymptotes with age axis. The total area under these survival curves represents life expectancy which are required in defining trends in longevity analytics. As observed, the curves move down which imply that many deaths are likely to be observed in the young and middle ages. Specifically in both genders, the survival curves form asymptotes within the age interval $100 \le x \le 119$ on the age axis where 119 is the maximum age in our computation. In the figures 2 and 4, the maximum and minimum points of the curve would translate to the points where the survival trajectory l_x inflexes because $\frac{1}{l_x}\frac{dl_x}{dx}=-\mu_x$ so that $\frac{d^2l_x}{dx^2}=0$ as $\frac{d}{dx}\left(l_x\mu_x\right)=l_x'\mu_x+l_x\mu_x'=0$. This explains why the distribution of the curve of death has a roughly j shaped configuration during childhood mortality but progressively transiting from the childhood mortality to death in the modal age range. Consequently, if the modal age at death increases, then the distribution of deaths is shifted towards the senescent ages. It was observed that the probability that (x) will not survive age $x+1$ is

$$q_x=\int_0^1 {}_{\xi}p_x\mu_{x+\xi}d\xi=\frac{1}{l_x}\int_0^1 l_{x+\xi}\mu_{x+\xi}d\xi \quad (141)$$

$$l_xq_x=\int_0^1 l_{x+\xi}\mu_{x+\xi}d\xi \quad (142)$$

Therefore if $l_{x+\xi}q_{x+\xi}$ is increasing, then $l_x\mu_x<l_xq_x$ hence $\mu_x<q_x$ and l_x becomes concave to the x age axis. However, if $l_x\mu_x=-\frac{dl_x}{dx}$ is increasing, then the shape of the curve is tangent to the curve of death which will be decreasing instantaneously. As a result l_x will be concave to the x-axis and $\mu_x<q_x$. However, if l_x is convex to the x axis, then $q_x<\mu_x$. The sting of death is represented by the instantaneous intensities computed in column 3 of tables 2 and 4. In these tables suppose that there is a decline in mortality at ages y where $y<\alpha_M$, then the number of deaths d_x obtained at any other age x preceeding α_M compared with those at age α_M will still be smaller. The reason in this regard why Canudas-Romo (2010) argues behind this phenomenon is that as more lives survive to the modal age α_M, more deaths will correspondingly occur at this age maintaining α_M as the modal age and consequently, changes in mortality occurring at ages higher than α_M will shift the modal age at death to senescent ages. Nevertheless more lives would be required to survive at the senescent ages so as to obtain a higher number of deaths than at α_M

Table 5: Combined Summary Statistics of Research Results

Sex	a	b	c	k	g	s
Male	0.000618678316	0.000331066819	1.098554562	100369.8071	0.999648	0.999382
Female	0.00026766376	0.000016575137	1.1036915	99895.35074	0.999820	0.999730

Source: Authors' Computation

Tables 1 and 3 show the computed values of the survival function l_x using the sub-parameters. In table 5, the comparative results of the sub-parameters obtained for both *male* and *female* are shown with the following relationship.

$$k_{men}>k_{women} \quad (143)$$

$$g_{men}<g_{women} \quad (144)$$

$$s_{men}<s_{women} \quad (145)$$

However, from the trajectories of the mortality results in figures 2 and 4, the age at which the maximum of the curve of death occurs for male is roughly $\alpha = 87$ years while that of female is $\alpha = 89.5$ years. The difference accounts for the total risk of exposure in male than in female. In general, $l_x \mu_x$ is proportional to the mortality density function of a new born with random life time T_0.

5. Conclusion

This study has provided an extensive approach to analyzing the curve of death and modal age at death analytics through parsimonious parametric Makeham's mortality intensities. In the analytics, the numerical algebraic technique of mortality measurement was employed. Actuaries and life offices usually need good estimates of mortality to price life annuities, and consequently, Makeham's law has been used to fit mortality statistics.

6. References

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