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Mathematical Derivations of Actuarial Present Value for the Fully Continuous Whole Life Assurances from the Theoretical Market Price

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Abstract: Many current mortality tables are computationally Makehamised but technically devoid of continuous key life table functions such as $\mu(x)$ and a_x because of the underlying sophisticated mortality functions and the complex methods of computation which usually do not incorporate the market price. The computationally advanced Moore's model, which does not endorse the market pricing mechanisms offers one of the most complex analytical results that seems not user friendly in actuarial literature. As a result of the Moore's complexity, we offer the Gradshteyn and Ryzhik's analytic integral as an alternative solution to examine the contingency issues involving the estimation of actuarial present values of the continuous whole life annuities. In this study, the objective are to (i) estimate the parameters of GM (1,2) through the method of equidistant points according to the natural order of human age (ii) Apply the mean value theorem to construct the survival probability function under the framework of policy modifications (iii) Develop a closed form pricing formula for the continuous whole life annuity and continuous whole life insurance. The Gradshteyn and Ryzhik's analytic integral has eliminated the Moore's complexities associated with symbolic computations such that the continuity assumption applied allows us to compute the continuous life annuity with expedience. Computational evidence further shows that the continuous life annuity reduces with age confirming that life annuities is a decreasing function.

Keywords: *Exponentiated polynomial, GM (1,2), Polynomial, Whole life insurance, Whole life annuity*

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1. Introduction

$GM(m,n)$ Mortality Class

The annuitization of the accumulated pension fund philosophy and practice is notably growing by reason of the increasing risk side of not surviving to the expected number of years or surviving beyond the estimated life span. Tables of annuities are used life insurance firms and pension funds to compute benefits and actuarial present values of premium. However, it is observed that many current mortality or annuity tables are computationally Makehamised but technically devoid of continuous key life table functions such as $\mu(x)$ and $\bar{a}_x$ because of the underlying sophisticated mortality functions and the complex methods of computation which usually do not incorporate the market price. This paper presents the Gradshteyn and Ryzhik's analytic integral as an alternative solution to examine the contingency issues involving the estimation of actuarial present values of the continuous whole life annuities in order to **(i)** estimate the parameters of $GM(1,2)$ through the method of equidistant points according to the natural order of human age **(ii)** Apply the mean value theorem to construct the survival probability function under the framework of policy modifications **(iii)** Develop a closed form pricing formula for the continuous whole life annuity and continuous whole life insurance. Mortality table consists of aggregated data where death counts and exposures at each age of the insured represent a significant source of information on the mortality intensity trajectories of insured population and such that measures of mortality like age specific death rates and fully continuous life annuity at each age are rigorously obtained. The essence is to afford life offices and pension fund to compute mortality measures derived from mortality table. Following (Mircea, Covrig and Serban, 2015), computing the curve of deaths in life contingencies, the death densities as a function of age for a given cohort represents a major problem hence parametric functions are used to solve issues relating to dimensionality, mis-estimation risk and hence smoothen statistical fluctuations. In view of (Missov, Nemeth, Danko, 2016), mathematical techniques try to model mortality intensities so as to obtain prospective life tables through advanced numerical methods of actuarial risk management of life insurance and pension funds. The $GM(m,v)$ for death probabilities in classical life contingencies is defined by the form $Polynomial_1(x)+\exp onential\left(Polynomial_2(x)\right)$. However, following (Forfar, Mccutcheon & Wilkie, 1988), the generalized Makeham function is defined as follows $GM(m,v)=\sum_{u=1}^{m}\sigma_u x^{u-1}+e^{\gamma}$ where $\gamma=\sum_{u=m+1}^{m+v}\sigma_u x^{u-m-1}$ where the polynomials are of orders m and v respectively and are positive integers. Observe that the order of the polynomials is $order=\deg ree+1$

Therefore $GM(1,2)=\exp\left(\sum_{u=2}^{3}\sigma_u x^{u-2}\right)+\sum_{u=1}^{1}\sigma_u x^{u-1}$ (1)

$$GM(1,2)=\exp\left(\sigma_2 x^0+\sigma_3 x^1\right)+\sigma_1 x^0 \quad (2)$$

$$GM(1,2)=\mathrm{e}^{\sigma_2+\sigma_3 x}+\sigma_1 \quad (3)$$

$$GM(1,2)=\mathrm{e}^{\sigma_2}\,\mathrm{e}^{\sigma_3 x}+\sigma_1=\mathrm{e}^{\sigma_2}\left(\mathrm{e}^{\sigma_3}\right)^x+\sigma_1 \quad (4)$$

This is of the form $\mu_x=\rho+GH^x$ (5)

Using the logit of $GM(1,2)$, we obtain $\frac{q_x}{p_x}=\frac{q_x}{1-q_x}=GM(1,2)$ (6)

$$q_x=GM(1,2)-q_x\times GM(1,2) \quad (7)$$

$$q_x+q_x\times GM(1,2)=GM(1,2) \quad (8)$$

$$q_x\left(1+GM(1,2)\right)=GM(1,2) \quad (9)$$

$$q_x=\frac{GM(1,2)}{\left(1+GM(1,2)\right)} \quad (10)$$

In (Forfar, Mccutcheon and Wilkie,1988; Debon, Montes and Sala, 2005), the following conditions are imposed. If $m=0$, then $GM(m,v)=\sum_{u=1}^{v}\sigma_u x^{u-1}$ and if $v=0$, $GM(m,v)=\sum_{u=1}^{m}\sigma_u x^{u-1}$

The Probability of Survival

$$\mu_x=\rho+GH^x \quad (11)$$

ρ is the accidental hump and GH^x is the hazard of ageing

Let $\zeta=e^{-\rho}$ and $G=-\log_e g\log_e H$, $\zeta>0$ and $g>0$ (11a)

Substitute the above definitions into (11), we have

$$\mu_x = -\log_e \zeta + \left(-\log_e g \log_e H\right) H^x \tag{12}$$

By definition, the force of mortality is given by

$$\mu_x = -\frac{1}{\left(\int_0^\infty l_{x+t}\mu_{x+t}dt\right)} \frac{d}{dx}\left(\int_0^\infty l_{x+t}\mu_{x+t}dt\right) = -\frac{d\log_e}{dx}\left(\int_0^\infty l_{x+t}\mu_{x+t}dt\right) \tag{13}$$

Since the expected number of survivors at age x, $l_x = \int_0^\infty l_{x+t}\mu_{x+t}dt$, equation (13) is expressed as

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{d_x} = -\frac{d\log_e l_x}{dx} \tag{14}$$

Using equation (12), we have

$$\mu_x = -\frac{d\log_e l_x}{dx} = -\log_e \zeta + \left(-\log_e g \log_e H\right) H^x \tag{14a}$$

Taking K as the constant of integration and integrating (14a)

$$-\int \frac{d\log_e l_x}{dx} dx = \int -\log_e \zeta + \left(-\log_e g \log_e H\right) H^x dx - K_1 \tag{15}$$

$$-\int \frac{d\log_e l_x}{dx} dx = -\log_e \int \zeta\, dx - \left(\log_e g \log_e H\right) \int H^x dx - K_1 \tag{16}$$

$$\log_e l_x = x\log_e \zeta + \left(\log_e g \log_e H\right)\frac{H^x}{\log_e H} + \log_e \lambda \tag{17}$$

$$-K_1 = K = \log_e \lambda \tag{18}$$

$$\log_e l_x = \log_e \zeta^x + \left(\log_e g\right)\left(\log_e H\right)\frac{H^x}{\log_e H} + \log_e \lambda \tag{19}$$

$$\log_e l_x = \log_e \zeta^x + \left(\log_e g\right) H^x + \log_e \lambda \tag{20}$$

where $\log_e \lambda$, is the constant of integration.

$$\log_e l_x = \log_e \zeta^x + \left(\log_e g^{H^x}\right) + \log_e \lambda = \log_e \lambda \zeta^x g^{H^x} \tag{21}$$

Now, equating both sides in (21), we obtain the number of insured lives l_x surviving to age x

$$l_x = \lambda\zeta^x g^{H^x} = \int_0^\infty l_{x+s}\mu_{x+s}ds \tag{22}$$

$$l_x = \lambda\zeta^x g^{H^x} \tag{23}$$

$$l_{x+t} = \lambda\zeta^{x+t} g^{H^{x+t}} \tag{24}$$

$${}_tP_x = \frac{l_{x+t}}{l_x} = \frac{\lambda\zeta^{x+t} g^{H^{x+t}}}{\lambda\zeta^x g^{H^x}} = \frac{\zeta^t g^{H^{x+t}}}{g^{H^x}} = \zeta^t g^{H^x\left(H^t-1\right)} \tag{25}$$

Mathematical Preliminaries:

The Parametric Numerical Estimation Procedure for the $GM(1,2)$ Function

Let $Z = \log_e l_x$ (26)

$$Z(x_i) = Z_i; i = 1,2,3,4 \tag{27}$$

Since ages are consecutive in mortality tables, we consider four ages x_1, x_2, x_3, x_4 with equidistant points such that

$$x_n = (n-1)y + x_1; x > 1 \tag{28}$$

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = y \tag{29}$$

$$x_4 = x_3 + y = x_2 + y + y = x_1 + y + y + y = 3y + x_1 \tag{30}$$

$$x_3 = x_2 + y = x_1 + y + y = 2y + x_1 \tag{31}$$

$$x_2 = x_1 + y \tag{32}$$

$$\log_e l_x = x\log_e\zeta + (\log_e g)H^x + \log_e\lambda \tag{33}$$

$$Z = x\log_e\zeta + (\log_e g)H^x + \log_e\lambda \tag{34}$$

$$Z_1 = x_1 \log_e \zeta + (\log_e g) H^{x_1} + \log_e \lambda \tag{35}$$

$$Z_2 = x_2 \log_e \zeta + (\log_e g) H^{x_2} + \log_e \lambda \tag{36}$$

$$Z_3 = x_3 \log_e \zeta + (\log_e g) H^{x_3} + \log_e \lambda \tag{37}$$

$$Z_4 = x_4 \log_e \zeta + (\log_e g) H^{x_4} + \log_e \lambda \tag{38}$$

$$Z_2 - Z_1 = \log_e \zeta^{x_2} - \log_e \zeta^{x_1} + (\log_e g) H^{x_2} - (\log_e g) H^{x_1} \tag{39}$$

$$Z_3 - Z_2 = \log_e \zeta^{x_3} - \log_e \zeta^{x_2} + (\log_e g) H^{x_3} - (\log_e g) H^{x_2} \tag{40}$$

$$Z_4 - Z_3 = \log_e \zeta^{x_4} - \log_e \zeta^{x_3} + (\log_e g) H^{x_4} - (\log_e g) H^{x_3} \tag{41}$$

We can then rewrite (39), (40), (41) as follows

$$Z_2 - Z_1 = \log_e \zeta^{x_2 - x_1} + (\log_e g)\left(H^{x_2} - H^{x_1}\right) \tag{42}$$

$$Z_3 - Z_2 = \log_e \zeta^{x_3 - x_2} + (\log_e g)\left(H^{x_3} - H^{x_2}\right) \tag{43}$$

$$Z_4 - Z_3 = \log_e \zeta^{x_4 - x_3} + (\log_e g)\left(H^{x_4} - H^{x_3}\right) \tag{44}$$

$$Z_2 - Z_1 = \log_e \zeta^{y} + (\log_e g)\left(H^{x_2} - H^{x_1}\right) \tag{45}$$

$$Z_3 - Z_2 = \log_e \zeta^{y} + (\log_e g)\left(H^{x_3} - H^{x_2}\right) \tag{46}$$

$$Z_4 - Z_3 = \log_e \zeta^{y} + (\log_e g)\left(H^{x_4} - H^{x_3}\right) \tag{47}$$

$$Z_3 - 2Z_2 + Z_1 = (\log_e g)\left(H^{x_3} - H^{x_2}\right) - (\log_e g)\left(H^{x_2} - H^{x_1}\right) \tag{48}$$

$$Z_3 - 2Z_2 + Z_1 = (\log_e g)\left(H^{x_3} - 2H^{x_2} + H^{x_1}\right) \tag{49}$$

$$Z_4 - 2Z_3 + Z_2 = (\log_e g)\left(H^{x_4} - 2H^{x_3} + H^{x_2}\right) \tag{50}$$

$$Z_2 - Z_1 = \log_e \zeta^{y} + \log_e g^{\left[H^{x2} - H^{x1}\right]} \tag{51}$$

$$Z_3 - Z_2 = \log_e \zeta^y + \log_e g^{\left[H^{x_3} - H^{x_2}\right]} \tag{52}$$

$$Z_4 - Z_3 = \log_e \zeta^y + \log_e g^{\left[H^{x_4} - H^{x_3}\right]} \tag{53}$$

$$Z_2 - Z_1 = \log_e \zeta^y g^{\left[H^{x_2} - H^{x_1}\right]} \tag{54}$$

$$Z_3 - Z_2 = \log_e \zeta^y g^{\left[H^{x_3} - H^{x_2}\right]} \tag{55}$$

$$Z_4 - Z_3 = \log_e \zeta^y g^{\left[H^{x_4} - H^{x_3}\right]} \Rightarrow \zeta^y g^{\left[H^{x_4} - H^{x_3}\right]} = e^{Z_4 - Z_3} \tag{56}$$

$$\zeta^y = \frac{e^{Z_4 - Z_3}}{g^{\left[H^{x_4} - H^{x_3}\right]}} \Rightarrow \zeta = \left(\frac{e^{Z_4 - Z_3}}{g^{\left[H^{x_4} - H^{x_3}\right]}} \right)^{\frac{1}{y}} \tag{57}$$

$$\begin{aligned} Z_3 - 2Z_2 + Z_1 &= x_3 \log_e \zeta + \left(\log_e g\right) H^{x_3} + \log_e \lambda \\ &-2\left[x_2 \log_e \zeta + \left(\log_e g\right) H^{x_2} + \log_e \lambda\right] + \\ &x_1 \log_e \zeta + \left(\log_e g\right) H^{x_1} + \log_e \lambda \end{aligned} \tag{58}$$

$$\begin{aligned} Z_3 - 2Z_2 + Z_1 &= x_3 \log_e \zeta + \left(\log_e g\right) H^{x_3} \\ &-2x_2 \log_e \zeta - 2\left(\log_e g\right) H^{x_2} + \\ &x_1 \log_e \zeta + \left(\log_e g\right) H^{x_1} \end{aligned} \tag{59}$$

$$\begin{aligned} Z_3 - 2Z_2 + Z_1 &= \left(2y + x_1\right) \log_e \zeta + \left(\log_e g\right) H^{2y + x_1} \\ &-2\left(y + x_1\right) \log_e \zeta - 2\left(\log_e g\right) H^{y + x_1} + \\ &x_1 \log_e \zeta + \left(\log_e g\right) H^{x_1} \end{aligned} \tag{60}$$

$$\begin{aligned} Z_3 - 2Z_2 + Z_1 &= 2y \log_e \zeta + x_1 \log_e \zeta + \left(\log_e g\right) H^{2y + x_1} \\ &-2y \log_e \zeta - 2x_1 \log_e \zeta - 2\left(\log_e g\right) H^{y + x_1} + \\ &x_1 \log_e \zeta + \left(\log_e g\right) H^{x_1} \end{aligned} \tag{61}$$

$$Z_3 - 2Z_2 + Z_1 = \left(\log_e g\right) H^{2y + x_1} - 2\left(\log_e g\right) H^{y + x_1} + \left(\log_e g\right) H^{x_1} \tag{62}$$

$$Z_3 - 2Z_2 + Z_1 = (\log_e g) H^{x_1} \left[H^{2y} - 2H^y + 1 \right] = (\log_e g) H^{x_1} \left(H^y - 1 \right)^2 \tag{63}$$

$$Z_3 - 2Z_2 + Z_1 = (\log_e g) H^{x_1} \left(H^y - 1 \right)^2 \tag{64}$$

Similarly,

$$\begin{aligned} Z_4 - 2Z_3 + Z_2 &= x_4 \log_e \zeta + (\log_e g) H^{x_4} + \log_e \lambda - 2\left[x_3 \log_e \zeta + (\log_e g) H^{x_3} + \log_e \lambda \right] \\ &+ x_2 \log_e \zeta + (\log_e g) H^{x_2} + \log_e \lambda \end{aligned} \tag{65}$$

$$\begin{aligned} Z_4 - 2Z_3 + Z_2 &= x_4 \log_e \zeta + (\log_e g) H^{x_4} + \log_e \lambda - 2x_3 \log_e \zeta - 2(\log_e g) H^{x_3} - 2\log_e \lambda \\ &+ x_2 \log_e \zeta + (\log_e g) H^{x_2} + \log_e \lambda \end{aligned} \tag{66}$$

$$\begin{aligned} Z_4 - 2Z_3 + Z_2 &= (3y + x_1)\log_e \zeta + (\log_e g) H^{3y+x_1} - 2(2y + x_1)\log_e \zeta \\ &- 2(\log_e g) H^{2y+x_1} + (y + x_1)\log_e \zeta + (\log_e g) H^{y+x_1} \end{aligned} \tag{67}$$

$$Z_4 - 2Z_3 + Z_2 = (\log_e g) H^{3y+x_1} - 2(\log_e g) H^{2y+x_1} + (\log_e g) H^{y+x_1} \tag{68}$$

$$Z_4 - 2Z_3 + Z_2 = H^{y+x_1} (\log_e g)\left[H^{2y} - 2H^y + 1 \right] \tag{69}$$

$$Z_4 - 2Z_3 + Z_2 = H^{y+x_1} (\log_e g)\left(H^y - 1 \right)^2 \tag{70}$$

$$\frac{Z_4 - 2Z_3 + Z_2}{Z_3 - 2Z_2 + Z_1} = \frac{H^{y+x_1} (\log_e g)\left(H^y - 1 \right)^2}{(\log_e g) H^{x_1} \left(H^y - 1 \right)^2} = \frac{H^{y+x_1}}{H^{x_1}} = H^y \tag{71}$$

$$H^{y}=\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\Rightarrow H=\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}} \quad (72)$$

$$Z_3-2Z_2+Z_1=\left(\log_e g\right)H^{x_1}\left(H^{y}-1\right)^2\Rightarrow\left(\log_e g\right)=\frac{\left(Z_3-2Z_2+Z_1\right)}{H^{x_1}\left(H^{y}-1\right)^2} \quad (73)$$

$$g=\exp\left(\frac{\left(Z_3-2Z_2+Z_1\right)}{H^{x_1}\left(H^{y}-1\right)^2}\right) \quad (74)$$

$$Z_1=x_1\log_e\zeta+\left(\log_e g\right)H^{x_1}+\log_e\lambda \quad (75)$$

$$\log_e\lambda=Z_1-x_1\log_e\zeta-\left(\log_e g\right)H^{x_1} \quad (76)$$

$$\zeta^{y}=\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\Rightarrow\zeta=\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}} \quad (77)$$

$$\log_e\lambda=Z_1-x_1\log_e\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}-\left(\frac{\left(Z_3-2Z_2+Z_1\right)}{H^{x_1}\left(H^{y}-1\right)^2}\right)H^{x_1} \quad (78)$$

$$\lambda=\exp\left[Z_1-x_1\log_e\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}-\left(\frac{\left(Z_3-2Z_2+Z_1\right)}{H^{x_1}\left(H^{y}-1\right)^2}\right)H^{x_1}\right] \quad (79)$$

$$G=-\log_e\exp\left(\frac{\left(Z_3-2Z_2+Z_1\right)}{H^{x_1}\left(H^{y}-1\right)^2}\right)\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}} \quad (80)$$

The Interval of Convergence for the Series $\sum_{t=1}^{\Omega-x-1}\left(\frac{1}{1+i}\right)^{t}\left({}_{t}P_{x}\right)$

Let $T(x) = X - x$ (81)

the complete future life time of a scheme holder aged x that is independent of the evolution of an account value governed by a unit linked mechanism,

then $T(o) = X = T$ (82)

represents the age at death of a new born life, $\Pr\left(T(x) > t\right) = \left({}_t p_x\right)$

By definition $a_{\overline{t|}} = \sum_{k=1}^{t}\left(\frac{1}{1+i}\right)$ (83)

The expected value of $E\left(a_{\overline{T(0)|}}\right) = (1+\alpha)\left\{\sum_{t=1}^{\Omega-x-1} a_{\overline{t|}}\frac{l_{x+t}}{l_x} - \sum_{t=1}^{\Omega-x-1} a_{\overline{t|}}\frac{l_{x+t+1}}{l_x}\right\}$ (84)

$$E\left(a_{\overline{T(0)|}}\right) = (1+\alpha)\sum_{t=1}^{\Omega-x-1}\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)^k\frac{d_{x+t}}{l_x} = (1+\alpha)\sum_{t=1}^{\Omega-x-1}\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)^k\left({}_{t|1}q_x\right) = (1+\alpha)\sum_{t=1}^{\Omega-x-1}\sum_{k=1}^{t}v^k\left({}_{t|1}q_x\right) \tag{85}$$

$$\begin{aligned}E\left(a_{\overline{T(0)|}}\right) &= (1+\alpha)\sum_{t=1}^{\Omega-x-1}\sum_{k=1}^{t}\left(\frac{1}{1+i}\right)\frac{d_{x+t}}{l_x} = (1+\alpha)\left(\frac{1}{1+i}\right)\left({}_{1|1}q_x\right) + (1+\alpha)\left[\left(\frac{1}{1+i}\right)+\left(\frac{1}{1+i}\right)^2\right]\left({}_{2|1}q_x\right)\\
&+(1+\alpha)\left[\left(\frac{1}{1+i}\right)+\left(\frac{1}{1+i}\right)^2+\left(\frac{1}{1+i}\right)^3\right]\left({}_{3|1}q_x\right)+(1+\alpha)\left[\left(\frac{1}{1+i}\right)+\left(\frac{1}{1+i}\right)^2+\left(\frac{1}{1+i}\right)^3+\left(\frac{1}{1+i}\right)^4\right]\left({}_{4|1}q_x\right)\\
&+\ldots+(1+\alpha)\left[\left(\frac{1}{1+i}\right)+\left(\frac{1}{1+i}\right)^2+\left(\frac{1}{1+i}\right)^3+\left(\frac{1}{1+i}\right)^4+\ldots+\left(\frac{1}{1+i}\right)^{\Omega-x-1}\right]\left[\frac{\left({}_{1|}q_{\Omega-1}\right)}{l_x}\right]\end{aligned} \tag{86}$$

$$\begin{aligned}E\left(a_{\overline{T(0)|}}\right) &= (1+\alpha)\left(\frac{1}{1+i}\right)^1\left[\frac{\left({}_{1|1}q_x\right)}{l_x}+\frac{\left({}_{2|1}q_x\right)}{l_x}+\frac{\left({}_{3|1}q_x\right)}{l_x}+\ldots+\frac{\left({}_{1|}q_{\Omega-1}\right)}{l_x}\right]\\
&+(1+\alpha)\left(\frac{1}{1+i}\right)^2\left[\frac{\left({}_{2|1}q_x\right)}{l_x}+\frac{\left({}_{3|1}q_x\right)}{l_x}+\ldots+\frac{\left({}_{1|}q_{\Omega-1}\right)}{l_x}\right]\\
&+(1+\alpha)\left(\frac{1}{1+i}\right)^3\left[\frac{\left({}_{3|1}q_x\right)}{l_x}+\frac{\left({}_{4|1}q_x\right)}{l_x}+\ldots+\frac{\left({}_{1|}q_{\Omega-1}\right)}{l_x}\right]+\ldots+(1+\alpha)\left(\frac{1}{1+i}\right)^{\Omega-x-1}\left[\frac{\left({}_{1|}q_{\Omega-1}\right)}{l_x}\right]\end{aligned} \tag{87}$$

$$E\left(a_{\overline{T(0)|}}\right)=(1+\alpha)\left(\frac{1}{1+i}\right)\left[\frac{(l_{x+1})}{l_x}\right]+(1+\alpha)\left(\frac{1}{1+i}\right)^2\left[\frac{(l_{x+2})}{l_x}\right]$$
$$+(1+\alpha)\left(\frac{1}{1+i}\right)^3\left[\frac{(l_{x+3})}{l_x}\right]+\ldots+(1+\alpha)\left(\frac{1}{1+i}\right)^{\Omega-x-1}\left[\frac{(l_{\Omega-1})}{l_x}\right] \quad (88)$$

$$E\left(a_{\overline{T(0)|}}\right)=(1+\alpha)\sum_{t=1}^{\Omega-x-1}\left(\frac{1}{1+i}\right)^t\left({}_tP_x\right)=(1+\alpha)\sum_{t=1}^{\Omega-x-1}\left(\frac{1}{1+i}\right)^t\left\{e^{-\int_0^t \mu(x+t)dt}\right\} \quad (89)$$

Since $\left(\frac{1}{1+i}\right)^t\left({}_tP_x\right)<\left(\frac{1}{1+i}\right)^t$ for $0<\frac{1}{1+i}<1$, the series $\sum_{t=1}^{\Omega-x-1}\left(\frac{1}{1+i}\right)^t\left({}_tP_x\right)$ is convergent.

As a consequence of (89), suppose that the regular annuity payments to the annuitant from annuity fund commence at age $z>x$, the expected remaining life time with regular payments for a life annuity scheme purchased at age x is obtained as

$$\mathbf{P}\left(z-x\le T(x)\le\Omega-x\right)=\frac{l_z}{l_x}\sum_{u=0}^{\Omega-z}\left({}_u p_z\right)=\left({}_{z-x}p_x\right)\sum_{u=0}^{\Omega-z}\mathbf{P}\left(T(z)>u\right) \quad (89a)$$

$$\mathbf{P}\left(z-x\le T(x)\le\Omega-x\right)=\sum_{u=0}^{\Omega-z}\left({}_{z-x}p_x\right)\left({}_u p_z\right)=\sum_{u=0}^{\Omega-z}\frac{l_z}{l_x}\times\frac{l_{z+u}}{l_z}=\sum_{u=0}^{\Omega-z}\frac{l_{z+u}}{l_x} \quad (89b)$$

$$\mathbf{P}\left(y-x\le T(x)\le\Omega-x\right)=\sum_{u=0}^{\Omega-z}\left({}_{z-x}p_x\right)\left({}_u p_z\right)=\sum_{u=0}^{\Omega-z}\left({}_{z-x+u}p_x\right)=\sum_{u=0}^{\Omega-z}\frac{l_{z+u}}{l_x} \quad (89c)$$

$$\mathbf{P}\left(z-x\le T(x)\le\Omega-x\right)=\sum_{u=0}^{\Omega-z}\left\{\prod_{s=0}^{z-x+u-1}\left({}_1p_{x+s}\right)\right\} \quad (89d)$$

Because regular payments of annuity will be made at the commencement of age R in future, the cash-flows could be discounted such that the present value of the whole life annuity scheme is then computed as the discounted expected value of the regular payments to the insured annuitant. Suppose the age of receiving the initial annuity income is R such that the regular annuity payments commences at age R until Ω at the beginning of each year, then the net single premium of whole life annuity with entry age $\mathbf{e}$ for $\mathbf{e}\le\Omega$ is expressed by ${}_{R-\mathbf{e}|}\ddot{a}_{\mathbf{e}}=\theta\times\sum_{i=0}^{\Omega-R}\left({}_{R-\mathbf{e}+i}P_{\mathbf{e}}\right)\times e^{-\delta(R-\mathbf{e}+i)}$ where θ is the unit annuity claim.

The Time Rate of Change of the Fully Continuous Whole Life Insurance

Suppose $\mu_x=GM(1,2)$ and ${}_tP_x=\zeta^t g^{H^x\left(H^t-1\right)}$ (90)

Define $\eta(i)=(\log_e g)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt$ (91)

with the following conditions $\rho=-\log_e\zeta$ and

$G=-\log_e g\log_e H\Rightarrow\frac{-G}{\log_e H}=\log_e g$ (92) $\sigma=\frac{GH^x}{\log_e H}\Rightarrow-\sigma=H^x\log_e g$ (93)

$$\theta=\frac{\delta+\rho}{\log_e H} \quad (94)$$

$\zeta>0$ and $g>0$. Then

$$\frac{d}{dx}\overline{A}_x=\delta\left(\log_e\frac{1}{H}\right)\left(\log_e g\right)H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)g^{H^x\left(H^t-1\right)}dt+\frac{\delta}{12}\left(G\left(\log_e H\right)H^x\right) \quad (95)$$

Proof

$$\int_0^\infty e^{-\delta t}\left({}_tP_x\right)\mu\left(x+t\right)dt=\overline{A}_x=1-\delta\overline{a}_x \quad (96)$$

$$\frac{d}{dx}\overline{A}_x=-\delta\frac{d}{dx}\overline{a}_x \quad (97)$$

In (Bowers, Hickman, Gerber, Jones & Nesbit, 1997, pp. 99; Gauger, Lewis, Willder & Lewry, 2011, pp. 53; Kara, 2021), the continuous whole life annuity are defined as

$\overline{a}_x=\int_0^\infty e^{-\delta\zeta}\left({}_\zeta P_x\right)d\zeta$ while the discrete whole life annuity is defined as $a_x=\sum_{k=0}^{\infty}v^{k+1}\left({}_kP_x\right)$

Neil (1979, pp. 78) estimated the continuous whole life annuity purchased by the insured as

$$\overline{a}_x=a_x+\frac{1}{2}-\frac{1}{12}\left(\mu_x+\delta\right)=a_x+\frac{1}{2}-\frac{1}{12}\left(\rho+GH^x+\delta\right) \quad (98)$$

Therefore by (96) $\frac{d}{dx}\overline{A}_x=-\delta\frac{d}{dx}\left(a_x+\frac{1}{2}-\frac{1}{12}\left(\mu_x+\delta\right)\right)=\left(-\delta\frac{d}{dx}a_x+\frac{\delta}{12}\frac{d}{dx}\mu_x\right)$ (99)

We need to differentiate a_x in (98). Now putting (25) in (97), we have

$$\overline{a}_x=\int_0^\infty e^{-\delta t}\zeta^t g^{H^x\left(H^t-1\right)}dt \quad (100)$$

$$\frac{d}{dx}\overline{a}_x=\int_0^\infty e^{-\delta t}\zeta^t g^{H^x\left(H^t-1\right)}\left(\log_e g\right)H^x\left(H^t-1\right)\left(\log_e H\right)dt \quad (101)$$

Following (Moore, 1979), $\eta\left(i\right)=\left(\log_e g\right)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt$ (102)

$$i\eta\left(i\right)=i\left(\log_e g\right)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt \quad (103)$$

$$\eta\left(i+1\right)=\left(\log_e g\right)^{(i+1)}H^{(i+1)x}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^{(i+1)}g^{H^x\left(H^t-1\right)}dt \quad (104)$$

$$\begin{aligned}i\eta\left(i\right)+\eta\left(i+1\right)&=i\left(\log_e g\right)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt\\&+\left(\log_e g\right)^{(i+1)}H^{(i+1)x}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^{(i+1)}g^{H^x\left(H^t-1\right)}dt\end{aligned} \quad (105)$$

Substituting $i=1$ in (102), we obtain

$$\eta\left(1\right)=\left(\log_e g\right)H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)g^{H^x\left(H^t-1\right)}dt \quad (106)$$

Differentiating (102), we obtain

$$\begin{aligned}\frac{d}{dx}\eta\left(i\right)&=\left(\log_e g\right)^i\left(\log_e H\right)H^{ix}i\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt\\&+\left(\log_e g\right)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i\left(\log_e g\right)g^{H^x\left(H^t-1\right)}\left(\log_e H\right)H^x\left(H^t-1\right)dt\end{aligned} \quad (107)$$

$$\frac{d}{dx}\eta(i)=(\log_e H)\left\{\begin{array}{l} i(\log_e g)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt \\ +(\log_e g)^i H^{ix}H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i(\log_e g)g^{H^x\left(H^t-1\right)}\left(H^t-1\right)dt \end{array}\right\} \quad (108)$$

$$\frac{d}{dx}\eta(i)=(\log_e H)\left\{\begin{array}{l} i(\log_e g)^i H^{ix}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^i g^{H^x\left(H^t-1\right)}dt \\ +(\log_e g)^{i+1} H^{ix+x}\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)^{i+1} g^{H^x\left(H^t-1\right)}dt \end{array}\right\} \quad (110)$$

$$\frac{d}{dx}\eta(i)=(\log_e H)\{i\eta(i)+\eta(i+1)\} \quad (111)$$

$$\frac{d}{dx}a_x=(\log_e H)\eta(1)=(\log_e H)(\log_e g)H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)g^{H^x\left(H^t-1\right)}dt \quad (112)$$

Therefore $\frac{d}{dx}\bar{A}_x=-\delta(\log_e H)(\log_e g)H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)g^{H^x\left(H^t-1\right)}dt+\frac{\delta}{12}\mu'_x$ (113)

Hence using (11) $\frac{d}{dx}\bar{A}_x=\delta\times\left(\log_e\frac{1}{H}\right)\times\eta(1)+\frac{\delta}{12}\left(G(\log_e H)H^x\right)$ (114)

Q.E.D

where $\eta(1)=(\log_e g)H^x\int_0^\infty e^{-\delta t}\zeta^t\left(H^t-1\right)g^{H^x\left(H^t-1\right)}dt$ which seems to offer *no analytic solution* and consequently, we apply the Gradshteyn and Ryzhik's analytic integral

Modification Theorem for $GM(1,2)$ Mortality Intensities

Theorem

If $\mu_x=-\frac{1}{l_x}\frac{dl_x}{dx}=-\frac{d\ln l_x}{dx}$ holds under $GM(1,2)$

Then $e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)}\cong\frac{2-\Delta\times\left(a+e^{b+cx}\right)}{2+\Delta\times\left(a+e^{b+c(x+\Delta)}\right)}$ (115)

Proof

$$GM(1,2)=\sum_{j=1}^{1}\beta_j x^{j-1}+\exp\left(\sum_{j=2}^{3}\beta_j x^{j-2}\right) \quad (116)$$

$$\mu_x=GM(1,2)=a+e^b e^{Cx} \quad (117)$$

$\mu_x=\beta+GH^x$ where $G=e^b$, $H=e^c$ and $\beta=a$

$$l_x\mu_x=-\frac{dl_x}{dx};l_0=1 \quad (118)$$

Following (Walters, & Wilkie, 1987; Castro-Perez, Aguilar-Sanchez & Gonzalez-Nucamendi, 2020) and the mean value theorem for integral, we obtain

$$\frac{l_{x+\Delta}-l_x}{\Delta}=\frac{dl_x}{dx}\Big|_{x=U}=-l_U\,\mu_U \tag{119}$$

where $U\in[x,x+\Delta]$ (120)

Consequently, (119) defines the initial value problem $\begin{cases}-dl_x=(\mu_x l_x)\,dx\\ l_0=10^n\end{cases}$

where $l_x=\alpha e^{-\int_0^x \mu_\xi d\xi}$ and $n=0,1,2,3,4,6$

Rearranging (119), we obtain $l_{x+\Delta}-l_x=-\Delta\times l_u\mu_u$ (121)

$$l_x-l_{x+\Delta}=\Delta\times l_u\mu_u \tag{122}$$

Since Δ is very small, we can then approximate $\Delta\times l_u\mu_u$ using the following schemes

For any mortality table, the following relationship holds

$$l_U\,\mu_U=\frac{l_x\mu_x+l_{x+\Delta}\mu_{x+\Delta}}{2} \tag{123}$$

Multiplying both sides of (123) by Δ

$$\Delta\times l_U\,\mu_U=\frac{\Delta\times l_x\mu_x+\Delta\times l_{x+\Delta}\mu_{x+\Delta}}{2} \tag{124}$$

Substituting (125) into the L.H.S of (124)

$$l_x-l_{x+\Delta}=\frac{\Delta l_x\mu_x}{2}+\frac{\Delta l_{x+\Delta}\mu_{x+\Delta}}{2} \tag{125}$$

$$l_x-\frac{\Delta l_x\mu_x}{2}=l_{x+\Delta}+\frac{\Delta l_{x+\Delta}\mu_{x+\Delta}}{2} \tag{126}$$

Factorising the $l_{(.)}$ in (126)

$$l_x\left(1-\frac{\Delta\mu_x}{2}\right)=l_{x+\Delta}\left(1+\frac{\Delta\mu_{x+\Delta}}{2}\right) \tag{127}$$

Dividing through by $l_{x+\Delta}$, we have

$$\frac{l_{x+\Delta}}{l_x}=\frac{\left(1-\frac{\Delta\mu_x}{2}\right)}{\left(1+\frac{\Delta\mu_{x+\Delta}}{2}\right)} \tag{128}$$

Therefore, the survival probability is $_{\Delta}P_x = \frac{\left(1-\frac{\Delta\mu_x}{2}\right)}{\left(1+\frac{\Delta\mu_{x+\Delta}}{2}\right)}$ (129)

Substituting (117) in (129)

$$_{\Delta}P_x = \frac{\left(1-\frac{\Delta\left(a+e^{b+cx}\right)}{2}\right)}{\left(1+\frac{\Delta\left(a+e^{b+c(x+\Delta)}\right)}{2}\right)} = \frac{\left(\frac{2-\Delta\left(a+e^{b+cx}\right)}{2}\right)}{\left(\frac{2+\Delta\left(a+e^{b+c(x+\Delta)}\right)}{2}\right)} = \frac{2-\Delta\times\left(a+e^{b+cx}\right)}{2+\Delta\times\left(a+e^{b+c(x+\Delta)}\right)} \quad (130)$$

$$_{\Delta}P_x = e^{-\int_0^{\Delta}\mu_{x+s}ds} \quad (131)$$

$$\mu_{x+s} = a+e^{b+c(x+s)} \quad (132)$$

$$\int_0^{\Delta}\mu_{x+s}ds = \int_0^{\Delta}ads+\int_0^{\Delta}e^{b+cx+cs}ds = a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c} \quad (133)$$

Using (131) $_{\Delta}P_x = e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)}$ (134)

Death probability is $_{\Delta}q_x = 1-e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}-\frac{e^{b+cx}}{c}\right)}$ (135)

Equating (130) and (134) therefore, $e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}\frac{e^{b+cx}}{c}\right)} \cong \frac{2-\Delta\times\left(a+e^{b+cx}\right)}{2+\Delta\times\left(a+e^{b+c(x+\Delta)}\right)}$ (136)

$$Error_1 = \left[e^{-\left(a\Delta+\frac{e^{b+cx+c\Delta}}{c}\frac{e^{b+cx}}{c}\right)}\right]-\left[\frac{2-\Delta\times\left(a+e^{b+cx}\right)}{2+\Delta\times\left(a+e^{b+c(x+\Delta)}\right)}\right] \quad (137)$$

Q.E.D

Theorem

Let $\mu:\mathbf{R}^{+}\to\mathbf{R}^{+}$ and $g:\mathbf{R}^{2}\to\mathbf{R}^{+}$ then at ages x,y,z the mortality rate intensity $GM(1,2)$ satisfies the following algebraic equation $\mu(x+y)+\mu(x+z)=\theta+g(y,z)\times GM(0,2)$

Proof

By definition $\mu(x)=\rho+GH^{x}$ (138)

Let $g(y,z)=H^{y}+H^{z}$ and let $\theta=2\rho$

$$\mu(x+y)=\rho+GH^{x+y}=\rho+GH^{x}H^{y} \tag{139}$$

$$\mu(x+z)=\rho+GH^{x+z}=\rho+GH^{x}H^{z} \tag{140}$$

$$\mu(x+y)+\mu(x+z)=2\rho+GH^{x}H^{y}+GH^{x}H^{z} \tag{141}$$

$$\mu(x+y)+\mu(x+z)=2\rho+GH^{x}\left(H^{y}+H^{z}\right) \tag{142}$$

$$\mu(x+y)+\mu(x+z)=2\rho+\left(H^{y}+H^{z}\right)\times GM(0,2) \tag{143}$$

$$\mu(x+y)+\mu(x+z)=\theta+g(y,z)\times GM(0,2) \tag{144}$$

$$H=\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}} \tag{145}$$

Q.E.D

Theorem

If $i+1=\left(1-\frac{d^{(K)}}{K}\right)^{-K}$ and δ is the force of interest, then $\lim\limits_{K\to\infty}\left(\delta-\frac{\delta^2}{2!K}+\frac{\delta^3}{3!K^2}-\frac{\delta^4}{4!K^3}\right)=\log_e(1+i)$

Proof

Observe that $i+1=\left(1-\frac{d^{(K)}}{K}\right)^{-K}=\left(1+\frac{i^{(K)}}{K}\right)^{K}$ (145a)

$$\left(1-\frac{d^{(K)}}{K}\right)^{-K}=1-K\times\left(-\frac{d^{(K)}}{K}\right)^{1}+\frac{-K(-K-1)}{2!}\times\left(-\frac{d^{(K)}}{K}\right)^{2}$$
$$+\frac{-K(-K-1)(-K-2)}{3!}\times\left(-\frac{d^{(K)}}{K}\right)^{3}+\ldots \tag{145b}$$

$$\left(1-\frac{d^{(K)}}{K}\right)^{-K}=1+K\times\left(\frac{d^{(K)}}{K}\right)^{1}+\frac{K(K+1)}{K^2}\times\left(\frac{d^{(K)}}{2!}\right)^{2}+\frac{K(K+1)(K+2)}{K^3}\times\left(\frac{d^{(K)}}{3!}\right)^{3}+\ldots \tag{145c}$$

Taking the limit above and $K\to\infty$

$$\lim_{K\to\infty}\left(1-\frac{d^{(K)}}{K}\right)^{-K}=\lim_{K\to\infty}\left\{1+d^{(K)}+\frac{K(K+1)}{K^2}\times\left(\frac{d^{(K)}}{2!}\right)^{2}+\frac{K(K+1)(K+2)}{K^3}\times\left(\frac{d^{(K)}}{3!}\right)^{3}+\ldots\right\} \tag{145d}$$

$$\lim_{K\to\infty}\left(1-\frac{d^{(K)}}{K}\right)^{-K}=\lim_{K\to\infty}\left\{1+d^{(K)}+\frac{K^2\left(1+\frac{1}{K}\right)}{K^2}\times\left(\frac{d^{(K)}}{2!}\right)^2+\frac{K^3\left(1+\frac{1}{K}\right)\left(1+\frac{2}{K}\right)}{K^3}\times\left(\frac{d^{(K)}}{3!}\right)^3+\ldots\right\} \quad (145e)$$

$$\lim_{K\to\infty}\left(1-\frac{d^{(K)}}{K}\right)^{-K}=1+d^{(\infty)}+\left(\frac{d^{(\infty)}}{2!}\right)^2+\left(\frac{d^{(\infty)}}{3!}\right)^3+\left(\frac{d^{(\infty)}}{4!}\right)^4\ldots=e^{\delta} \quad (145f)$$

$$1-\frac{d^{(K)}}{K}=e^{-\frac{\delta}{K}}=1-\frac{\delta}{K}+\frac{\delta^2}{2!K^2}-\frac{\delta^3}{3!K^3}+\frac{\delta^4}{4!K^4} \quad (145g)$$

$$\frac{d^{(K)}}{1}=\frac{\delta}{1}-\frac{\delta^2}{2!K}+\frac{\delta^3}{3!K^2}-\frac{\delta^4}{4!K^3} \quad (145h)$$

The force of mortality becomes $\lim_{K\to\infty}d^{(K)}=\lim_{K\to\infty}\left(\delta-\frac{\delta^2}{2!K}+\frac{\delta^3}{3!K^2}-\frac{\delta^4}{4!K^3}\right)=\log_e(1+i)$ **(145i)**

where $d^{(K)}=i^{(K)}\times v^{\frac{1}{K}}$

Q.E.D

3. Materials and methods

The Annuity Estimation Procedure

Under the usual actuarial notation, we define and continuous life annuity function $\bar{a}_x$ assuming that Ω is the omega age in a mortality table and $l_x=\int_0^\infty l_{x+t}\mu_{x+t}dt$ be the survival function representing the population surviving to age x. Suppose that Let $\alpha(x)$ is the age dependent security loading, then the market price of the fully continuous whole life annuity is defined by

$$\bar{a}_x=(1+\alpha)\int_0^\infty\left(\frac{1}{1+i}\right)^t\left(\frac{\int_0^\infty l_{x+t+s}\mu_{x+t}(s)ds}{\int_0^\infty l_{x+t}\mu_{x+t}dt}\right)dt \quad (146)$$

$$\bar{a}_x=\int_0^\infty\left(\frac{1}{1+i}\right)^t(1+\alpha)\frac{l_{x+t}}{l_x}dt \quad (147)$$

where $l_{x+t}=\int_0^\infty l_{x+t+s}\mu_{x+t}(s)ds$ is the number of lives surviving to age $x+t$

In (Bowers, Hickman, Gerber, Jones & Nesbit, 1997, pp. 99; Gauger, Lewis, Willder & Lewry, 2011, pp. 53; Kara, 2021), the continuous whole life annuity is defined as the Laplace transform of the probability of survival function ${}_tP_x$ given by $\bar{a}_x=\int_0^{\Omega-x}e^{-\delta t}e^{-\int_0^t\mu_{x+\xi}d\xi}dt$ where δ is the interest rate intensity defined in (145i).

$$\bar{a}_x=(1+\alpha)\int_0^\infty\int_0^t e^{-\delta t}ds\,d\left({}_tq_x\right)=\int_0^{\Omega-x}\left(\frac{1}{1+i}\right)^t(1+\alpha)\left({}_tP_x\right)dt \quad (148)$$

$$\bar{a}_x = \int_0^{\Omega-x} \zeta^t g^{H^x\left(H^t-1\right)}\left(1+\alpha\right)\left(\frac{1}{1+i}\right)^t dt \tag{149}$$

$${}_tP_x\left(\frac{1}{1+i}\right)^t = \left(\frac{1}{1+i}\right)^t \zeta^t g^{H^x\left(H^t-1\right)} = \left(\frac{1}{1+i}\right)^t \zeta^t g^{\left(H^{x+t}-H^x\right)} = \frac{\left(\frac{1}{1+i}\right)^t \zeta^t g^{\left(H^{x+t}\right)}}{g^{H^x}} \tag{150}$$

$${}_tP_x\left(\frac{1}{1+i}\right)^t = \left(\frac{1}{1+i}\right)^t \zeta^t g^{H^x\left(H^t-1\right)} = \frac{\left(\zeta\times\frac{1}{1+i}\right)^t g^{\left(H^{x+t}\right)}}{g^{H^x}} = \frac{\exp\left(H^t H^x \log_e g + t\log_e\left(\frac{\zeta}{1+i}\right)\right)}{g^{H^x}} \tag{151}$$

Observe that $H^t = e^{\log_e H^t} = e^{t\log_e H}$

$${}_tP_x\left(\frac{1}{1+i}\right)^t = \frac{\exp\left\{\left(e^{t\log_e H}\right)\left(H^x\times\log_e g\right)+t\log_e\left(\frac{1}{1+i}\right)\zeta\right\}}{g^{H^x}} \tag{152}$$

$$\bar{a}_x = \int_0^{\Omega-x}\left(1+\alpha\right)\frac{\exp\left\{\left(e^{t\log_e H}\right)\left(H^x\times\log_e g\right)+t\log_e\left(\frac{1}{1+i}\right)\zeta\right\}}{g^{H^x}}dt \tag{153}$$

Let $\eta = t\log_e H \Rightarrow \frac{\eta}{\log_e H} = t$ (154)

when $t=0$, $\eta=0$ and when $t=\Omega-x$,

$$\eta = \left(\Omega-x\right)\log_e H = \log_e H^{(\Omega-x)} \tag{155}$$

where Ω is the highest age in the mortality table

$$\frac{d\eta}{dt} = \log_e H \Rightarrow d\eta = \log_e H dt \Rightarrow \frac{d\eta}{\log_e H} = dt \tag{156}$$

$$\bar{a}_x = \int_0^{\log_e H^{(\Omega-x)}} (1+\alpha) \frac{\exp\left(e^{\eta}\left(H^x \times \log_e g\right) + \frac{\eta}{\log_e H} \times \log_e\left(\frac{1}{1+i}\right)\zeta\right)}{g^{H^x}} \frac{d\eta}{\log_e H} \tag{157}$$

$$\bar{a}_x = \frac{(1+\alpha)}{(\log_e H)\left(g^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left\{\left(e^{\eta} \times H^x \times \log_e g\right) + \frac{\eta}{\log_e H} \times \log_e\left(\frac{1}{1+i}\right)\zeta\right\} d\eta \tag{158}$$

$$\bar{a}_x = \frac{(1+\alpha)}{(\log_e H)\left(g^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left\{\left(H^x \times \log_e g\right)e^{\eta} + \left(\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right) \times \eta\right\} d\eta \tag{159}$$

Following (Gradshteyn and Ryzhik, n.d, pp. 356, formula, $ETI147(37)$)

$$\alpha^{-\delta} \int_0^{\infty} \exp\left(-\alpha e^{Y} - \delta Y\right) dY = \Gamma(-\delta, \alpha) \tag{160}$$

$\alpha = a + ib$, $i = \sqrt{-1}$ and $a > 0$

$$\bar{a}_x = \frac{(1+\alpha)}{(\log_e H)\left(g^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \exp\left\{-\left(-H^x \times \log_e g\right)e^{\eta} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right\} d\eta \tag{161}$$

$$\int_0^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta =$$

$$\left\{\begin{array}{l}\displaystyle\int_0^{\log_e H^{(\Omega-x)}} \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta \\ \displaystyle+\int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta\end{array}\right\} \tag{162}$$

Consequently,

$$\bar{a}_x = \frac{(1+\alpha)}{(\log_e H)\left(g^{H^x}\right)}\left\{\begin{array}{l}\displaystyle\int_0^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta - \\ \displaystyle\int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta\end{array}\right\} \tag{163}$$

In order to evaluate equation (163) within $(0,\infty)$, we break the integral into two as follows

$$I_1 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta =$$

$$\left[\left(-H^x \times \log_e g\right)\right]^{\left(-\frac{\log_e \zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right),\left(-H^x \times \log_e g\right)\right) \tag{164}$$

$$I_2 = \int_{\log_e H^{(\Omega-x)}}^\infty \exp\left(-\left(-H^x \times \log_e g\right)e^\eta - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)\eta\right)d\eta \tag{165}$$

Using the change of variable below

Let $\xi = \eta - \log_e H^{\Omega - x}$ (166)

Let $\xi + \log_e H^{\Omega - x} = \eta$ (167)

$d\xi = d\eta$ (168)

when $\eta = \infty$, $\xi = \infty$ (169)

when $\eta = \log_e H^{\Omega - x}$, $\xi = 0$

Therefore, $I_2 = \int_0^\infty \exp\left(-\left(-H^x \times \log_e g\right) e^{\xi + \log_e H^{\Omega-x}} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\left(\xi + \log_e H^{\Omega-x}\right) \right) d\xi$ (170)

$$I_2 = \int_0^\infty \exp\left\{ -\left(-H^x \times \log_e g\right) e^{\log_e H^{\Omega-x}} e^{\xi} - \left(\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi + \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right) \log_e H^{\Omega-x} \right) \right\} d\xi$$

(171)

After removing the second right bracket $(.)$, we have

$$I_2 = \int_0^\infty \exp\left\{ -\left(-H^x \times \log_e g\right) H^{\Omega-x} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\left(\Omega - x\right)\left(\log_e H\right) \right\} d\xi$$

(172)

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi - \left(-\log_e\left(\frac{\zeta}{1+i}\right)\right)\left(\Omega - x\right) \right\} d\xi \quad (173)$$

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi + \log_e\left(\frac{\zeta}{1+i}\right)^{(\Omega - x)} \right\} d\xi \tag{174}$$

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi + \log_e \frac{e^{-\rho(\Omega - x)}}{\left(1+i\right)^{(\Omega - x)}} \right\} d\xi \tag{175}$$

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi + \log_e e^{-\rho(\Omega - x)} - \log_e \left(1+i\right)^{(\Omega - x)} \right\} d\xi \tag{176}$$

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi - \rho\left(\Omega - x\right) - \left(\Omega - x\right)\log_e\left(1+i\right) \right\} d\xi \tag{177}$$

$$I_2 = \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} e^{\xi} - \left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H} \right)\xi - \rho\left(\Omega - x\right) - \delta \times \left(\Omega - x\right) \right\} d\xi \tag{178}$$

$$I_2 = e^{-\rho(\Omega - x) - \delta \times (\Omega - x)} \int_0^\infty \exp\left\{ -\left(-\log_e g\right) H^{\Omega} \times e^{\xi} - \left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H} \right)\xi \right\} d\xi \tag{179}$$

Since, the Moore's technique is too complex,
we apply (Gradshteyn and Ryzhik, n.d, pp. 356, formula, $ETI147\left(37\right)$)

$$I_2 = e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\int_0^{\infty}\exp\left\{-\left(-\log_e g\right)H^{\Omega}\times e^{\xi}-\left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right)\xi\right\}d\xi =$$

$$e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\left[\left(-\log_e g\right)H^{\Omega}\right]^{\left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right),\left(-\log_e g\right)H^{\Omega}\right) \tag{180}$$

$$\bar{a}_x = \frac{(1+\alpha)}{\left(\log_e H\right)\left(g^{H^x}\right)}\left(I_1 - I_2\right) \tag{181}$$

$$\bar{a}_x = \frac{(1+\alpha)}{\left(\log_e H\right)\left(g^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x\times\log_e g\right)\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right),\left(-H^x\times\log_e g\right)\right)- \\ e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\left[\left(-\log_e g\right)H^{\Omega}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)}\Gamma\left(-\left(-\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right),\left(-\log_e g\right)H^{\Omega}\right)\end{array}\right\}$$

(182)

$$\bar{a}_x = \frac{(1+\alpha)}{\left(\log_e H\right)\left(g^{H^x}\right)}\left\{\begin{array}{l}\left[\left(-H^x\times\log_e g\right)\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right),\left(-H^x\times\log_e g\right)\right)- \\ e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\left[\left(-\log_e g\right)H^{\Omega}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e\left(\frac{\zeta}{1+i}\right)}{\log_e H}\right),\left(-\log_e g\right)H^{\Omega}\right)\end{array}\right\}$$

(183)

Following de Souza (2018), the continuous annuity in equation (183) is bounded as follows

$$\left(\frac{l_{x+m}-l_{x+m}e^{-\delta m}+l_{x+m}\delta e^{-\delta m}\bar{a}_{x+m}}{\delta l_x}\right)\le \bar{a}_x \le \left(\frac{l_x - l_x e^{-\delta m}+\delta l_{x+m}e^{-\delta m}\bar{a}_{x+m}}{l_x\delta}\right) \tag{183a}$$

$$g = \exp\left(\frac{(Z_3 - 2Z_2 + Z_1)}{H^{x_1}\left(H^y - 1\right)^2}\right) \tag{184}$$

$$\zeta = \left(\frac{e^{Z_4 - Z_3}}{g^{\left[H^{x_4} - H^{x_3}\right]}}\right)^{\frac{1}{y}} \tag{185}$$

$$H = \left(\frac{Z_4 - 2Z_3 + Z_2}{Z_3 - 2Z_2 + Z_1}\right)^{\frac{1}{y}} \tag{186}$$

For ease of computation, equation (181) was broken down into segments as shown in columns $1-10$ and in column 11 the continuous life annuity is computed.

4. Results and discussion

$\bar{a}_x$ **Annuity**

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
x										$\bar{a}_x = 8 \times 9 \times 10$
30	0.098010	0.000246	0.000019	2.940302	18.921560	0.003626	10.240100	0.007047	1.075795	0.077627
31	0.098010	0.000246	0.000019	3.038312	20.869980	0.004000	10.243930	0.007683	1.075077	0.084609
32	0.098010	0.000246	0.000019	3.136322	23.019050	0.004412	10.248150	0.008376	1.074294	0.092217
33	0.098010	0.000246	0.000019	3.234332	25.389410	0.004866	10.252800	0.009132	1.073440	0.100507
34	0.098010	0.000246	0.000019	3.332342	28.003850	0.005367	10.257940	0.009957	1.072510	0.109540
35	0.098010	0.000246	0.000019	3.430352	30.887520	0.005920	10.263610	0.010855	1.071496	0.119381
36	0.098010	0.000246	0.000019	3.528362	34.068130	0.006529	10.269870	0.011835	1.070392	0.130102
37	0.098010	0.000246	0.000019	3.626372	37.576250	0.007202	10.276780	0.012904	1.069189	0.141782
38	0.098010	0.000246	0.000019	3.724382	41.445630	0.007943	10.284400	0.014068	1.067878	0.154506
39	0.098010	0.000246	0.000019	3.822392	45.713440	0.008761	10.292820	0.015338	1.066450	0.168365
40	0.098010	0.000246	0.000019	3.920403	50.420740	0.009663	10.302110	0.016723	1.064894	0.183461
41	0.098010	0.000246	0.000019	4.018413	55.612750	0.010658	10.312360	0.018232	1.063199	0.199902
42	0.098010	0.000246	0.000019	4.116423	61.339420	0.011756	10.323690	0.019878	1.061354	0.217808

43	0.098010	0.000246	0.000019	4.214433	67.655770	0.012966	10.336190	0.021673	1.059344	0.237306
44	0.098010	0.000246	0.000019	4.312443	74.622550	0.014302	10.350000	0.023629	1.057155	0.258538
45	0.098010	0.000246	0.000019	4.410453	82.306730	0.015774	10.365260	0.025762	1.054772	0.281655
46	0.098010	0.000246	0.000019	4.508463	90.782170	0.017399	10.382110	0.028087	1.052178	0.306822
47	0.098010	0.000246	0.000019	4.606473	100.130400	0.019190	10.400720	0.030623	1.049355	0.334219
48	0.098010	0.000246	0.000019	4.704483	110.441200	0.021166	10.421300	0.033387	1.046283	0.364041
49	0.098010	0.000246	0.000019	4.802493	121.813700	0.023346	10.444040	0.036401	1.042940	0.396497
50	0.098010	0.000246	0.000019	4.900503	134.357400	0.025750	10.469170	0.039687	1.039304	0.431818
51	0.098010	0.000246	0.000019	4.998513	148.192700	0.028402	10.496970	0.043269	1.035350	0.470252
52	0.098010	0.000246	0.000019	5.096523	163.452600	0.031326	10.527720	0.047175	1.031051	0.512067
53	0.098010	0.000246	0.000019	5.194533	180.284000	0.034552	10.561730	0.051433	1.026378	0.557556
54	0.098010	0.000246	0.000019	5.292543	198.848500	0.038110	10.599380	0.056076	1.021301	0.607035
55	0.098010	0.000246	0.000019	5.390553	219.324700	0.042034	10.641050	0.061138	1.015786	0.660845
56	0.098010	0.000246	0.000019	5.488564	241.909500	0.046363	10.687210	0.066657	1.009797	0.719358
57	0.098010	0.000246	0.000019	5.586574	266.819800	0.051137	10.738360	0.072674	1.003298	0.782975
58	0.098010	0.000246	0.000019	5.684584	294.295300	0.056403	10.795050	0.079234	0.996248	0.852129
59	0.098010	0.000246	0.000019	5.782594	324.600000	0.062211	10.857930	0.086387	0.988604	0.927291
60	0.098010	0.000246	0.000019	5.880604	358.025300	0.068617	10.927710	0.094185	0.980321	1.008968
61	0.098010	0.000246	0.000019	5.978614	394.892600	0.075682	11.005200	0.102687	0.971350	1.097708
62	0.098010	0.000246	0.000019	6.076624	435.556200	0.083476	11.091300	0.111956	0.961642	1.194106
63	0.098010	0.000246	0.000019	6.174634	480.407100	0.092072	11.187050	0.122062	0.951144	1.298800
64	0.098010	0.000246	0.000019	6.272644	529.876500	0.101553	11.293620	0.133080	0.939801	1.412482
65	0.098010	0.000246	0.000019	6.370654	584.440000	0.112010	11.412340	0.145093	0.927557	1.535899
66	0.098010	0.000246	0.000019	6.468664	644.622000	0.123544	11.544730	0.158191	0.914354	1.669856
67	0.098010	0.000246	0.000019	6.566674	711.001300	0.136266	11.692540	0.172470	0.900133	1.815221
68	0.098010	0.000246	0.000019	6.664684	784.215800	0.150298	11.857760	0.188039	0.884834	1.972931

69	0.098010	0.000246	0.000019	6.762694	864.969600	0.165774	12.042710	0.205013	0.868398	2.143995
70	0.098010	0.000246	0.000019	6.860704	954.038800	0.182845	12.250050	0.223519	0.850767	2.329501
71	0.098010	0.000246	0.000019	6.958714	1052.280000	0.201673	12.482880	0.243696	0.831887	2.530620
72	0.098010	0.000246	0.000019	7.056725	1160.637000	0.222440	12.744820	0.265694	0.811706	2.748614
73	0.098010	0.000246	0.000019	7.154735	1280.153000	0.245345	13.040120	0.289677	0.790179	2.984842
74	0.098010	0.000246	0.000019	7.252745	1411.975000	0.270610	13.373760	0.315826	0.767266	3.240766
75	0.098010	0.000246	0.000019	7.350755	1557.371000	0.298475	13.751670	0.344335	0.742940	3.517959
76	0.098010	0.000246	0.000019	7.448765	1717.740000	0.329210	14.180890	0.375418	0.717184	3.818115
77	0.098010	0.000246	0.000019	7.546775	1894.622000	0.363111	14.669870	0.409306	0.689995	4.143055
78	0.098010	0.000246	0.000019	7.644785	2089.719000	0.400501	15.228770	0.446254	0.661390	4.494737
79	0.098010	0.000246	0.000019	7.742795	2304.906000	0.441743	15.869960	0.486536	0.631404	4.875269
80	0.098010	0.000246	0.000019	7.840805	2542.251000	0.487231	16.608520	0.530455	0.600099	5.286916
81	0.098010	0.000246	0.000019	7.938815	2804.036000	0.537403	17.463060	0.578339	0.567561	5.732115
82	0.098010	0.000246	0.000019	8.036825	3092.778000	0.592741	18.456680	0.630544	0.533908	6.213485
83	0.098010	0.000246	0.000019	8.134835	3411.254000	0.653778	19.618310	0.687462	0.499290	6.733843
84	0.098010	0.000246	0.000019	8.232845	3762.524000	0.721100	20.984520	0.749519	0.463891	7.296219
85	0.098010	0.000246	0.000019	8.330855	4149.966000	0.795354	22.602030	0.817176	0.427934	7.903870
86	0.098010	0.000246	0.000019	8.428865	4577.304000	0.877255	24.531070	0.890942	0.391673	8.560299
87	0.098010	0.000246	0.000019	8.526875	5048.646000	0.967590	26.850240	0.971366	0.355398	9.269273
88	0.098010	0.000246	0.000019	8.624886	5568.525000	1.067226	29.663320	1.059049	0.319429	10.034850
89	0.098010	0.000246	0.000019	8.722896	6141.938000	1.177122	33.109080	1.154648	0.284111	10.861380
90	0.098010	0.000246	0.000019	8.820906	6774.397000	1.298335	37.375690	1.258876	0.249803	11.753560

From the results computed in table 1, we observe that the annuity values progressively reduces downwards from ages $30-95$. This pattern confirms the actuarial behavior that $\bar{a}_x$ is a decreasing function.

The incomplete lower Gamma integral $\gamma\left(Z,x\right)=\int_0^x t^{Z-1}e^{-t}dt$ (187)

The incomplete upper Gamma integral $\Gamma\left(Z,x\right)=\int_x^\infty t^{Z-1}e^{-t}dt$ (188)

The Gamma integral $\Gamma(Z)=\int_0^\infty t^{Z-1}e^{-t}dt=\lim_{m\to\infty}\left\{\frac{m!m^Z}{Z(Z+1)(Z+2)(Z+3)\dots(Z+m)}\right\}$ (189)

$$\Gamma(Z,x)=\frac{1}{Z+Z^2}\exp\left(Z(1-\gamma)+\sum_{m=2}^{\infty}(-1)^m\left[\varsigma(m)-1\right]\frac{Z^m}{m}\right)-\sum_{n=0}^{\infty}(-1)^n\frac{x^{Z+n}}{n!(Z+n)} \quad (190)$$

$$\Gamma(Z)=\frac{1}{Z}\exp\left(\log_e(1+Z)^{-1}+Z(1-\gamma)\right)+\sum_{m=2}^{\infty}(-1)^m\left[\varsigma(m)-1\right]\frac{Z^m}{m} \quad (191)$$

$\varsigma(m)=\sum_{K=1}^{\infty}K^{-m}$ defines the Rieman Zeta function

If $\gamma(Z,x)=\int_0^x t^{Z-1}e^{-t}dt$ then (192)

$$\gamma(Z+1,x)=\int_0^x t^Z e^{-t}dt \quad (193)$$

$$\gamma(Z+1,x)=\int_0^x t^Z e^{-t}dt \quad (194)$$

$$\gamma(Z+1,x)=\int_0^{x_1} t^Z e^{-t}dt-\int_0^{x_0} t^Z e^{-t}dt \quad (195)$$

$$\gamma(Z+1,x)=\gamma(Z+1,x_1)-\gamma(Z+1,x_0) \quad (196)$$

$$\gamma(Z+1,x)=\Delta\gamma(Z+1,x_1,x_0) \quad (197)$$

Substituting equations (184), (185) and (186) in equation (183), we obtain

$$\bar{a}_x = \frac{(1+\alpha)}{\left(\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}\right)\left(\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)^{\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}}\right)}\times$$

$$\left\{\begin{array}{l}\left[\left(-\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\log_e\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)\right)\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left(\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right),\left(-\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\log_e\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)\right)\right)- \\ e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\times \\ \left[\left(-\log_e\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)\right)\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left(\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right),\left(-\log_e\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)\right)\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right)\end{array}\right\} \quad (198)$$

$$\bar{a}_x = \frac{(1+\alpha)}{\left(\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}\right)\left(\exp\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)^{\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}}\right)}\times$$

$$\left\{\begin{array}{l}\left[\left(-\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\left\{\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right\}\right)\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left\{\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right\},\left(-\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\left\{\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right\}\right)\right)- \\ e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\times \\ \left[\left(-\left\{\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right\}\right)\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left\{\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right\},\left(-\left(\frac{(Z_3-2Z_2+Z_1)}{H^{x_1}(H^y-1)^2}\right)\right)\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right)\end{array}\right\} \quad (199)$$

The fully continuous whole life insurance is given by

$$\bar{A}_x = \int_0^\infty \left(\frac{1}{1+i}\right)^t \left(\frac{\int_0^\infty l_{x+t+s}\mu_{x+t}(s)ds}{\int_0^\infty l_{x+t}\mu_{x+t}dt}\right)\mu_{x+t}dt \quad (200)$$

This implies that $\bar{A}_x = \int_0^\infty e^{-\delta t}\left({}_tP_x\right)\mu(x+t)dt$ (201)

Following (Koshkin, Gubina, 2016; Dmitriev, Kohkin, 2018; Castellares, Patricio and Lemonte, 2022), the continuous whole life insurance at age x is given by $\overline{A}_x = 1 - \delta \overline{a}_x$ (202)

$$\overline{A}_x =$$

$$1-\delta\left\{\begin{array}{l}\dfrac{(1+\alpha)}{\left(\log_e\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}\right)\left(\exp\left(\dfrac{(Z_3-2Z_2+Z_1)}{H^{x_1}\left(H^y-1\right)^2}\right)^{\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}}\right)}\times \\ \left\{\begin{array}{l}\left[\left\{-\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\left(\dfrac{(Z_3-2Z_2+Z_1)}{H^{x_1}\left(H^y-1\right)^2}\right)\right\}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left\{\dfrac{\log_e\left(\dfrac{1}{1+i}\right)\left(\dfrac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right\},\left(-\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{x}{y}}\times\left(\dfrac{(Z_3-2Z_2+Z_1)}{H^{x_1}\left(H^y-1\right)^2}\right)\right)\right)- \\ e^{-\rho(\Omega-x)-\delta\times(\Omega-x)}\times \\ \left[\left\{-\left(\dfrac{(Z_3-2Z_2+Z_1)}{H^{x_1}\left(H^y-1\right)^2}\right)\right\}\times\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right]^{\left(-\frac{\log_e\left(\frac{1}{1+i}\right)\left(\frac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\frac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right)}\times \\ \Gamma\left(\left\{\dfrac{\log_e\left(\dfrac{1}{1+i}\right)\left(\dfrac{e^{Z_4-Z_3}}{g^{\left[H^{x_4}-H^{x_3}\right]}}\right)^{\frac{1}{y}}}{\log_e\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{1}{y}}}\right\},\left(-\left(\dfrac{(Z_3-2Z_2+Z_1)}{H^{x_1}\left(H^y-1\right)^2}\right)\right)\left(\dfrac{Z_4-2Z_3+Z_2}{Z_3-2Z_2+Z_1}\right)^{\frac{\Omega}{y}}\right)\end{array}\right\}\end{array}\right\} \quad (203)$$

5. Conclusion

The numerical technique used here offers the benchmarks against which the estimates of $\overline{a}_x$ and $\overline{A}_x$ could be formulated as this is rarely considered in actuarial life underwriting practice. The essence is to obtain cost effective approximations of life table functions. Consequently, this paper discussed the application of the Gradshteyn and Ryzhik's analytic integral as an alternative to the Moore's scheme to enable us compute the market price of the fully continuous life assurances associated with $GM(1,2)$ which have been expressed in terms of Gamma function. In life underwriting practice, the net single premium $\int_0^{\Omega-x} e^{-\delta t}\left({}_tP_x\right)dt$ and $\int_0^{\Omega-x} e^{-\delta t}\left({}_tP_x\right)\mu_{x+t}dt$ only take care of the annuity pay-out phase. It is therefore pertinent for the life office to obtain a fair market price $\left(1+\alpha\right)\int_0^{\Omega-x} e^{-\delta t}\left({}_tP_x\right)dt$ for annuity scheme that can compensate the life office for the commissions to brokers and overriding commissions to insurance agents, stamp duties and other management expenses. Under the framework of Generalised Gompertz-Makeham's mortality $GM\left(m,v\right)$ defines the parametric force of mortality where $m\in\mathbf{Z}^+$ and $v\in\mathbf{Z}^+$ are the orders of the associated polynomial and the exponentiated polynomial.

In parametric applications, a common problem is that with continuous non-linear mortality functions, the life assurances and in particular annuity being measured seem very complex to compute as a result of the evolving integral representations. However, irrespective of the difficulty level, these variables have been computed numerically and the market price of life annuity under $GM\left(1,2\right)$ obtained in closed forms in terms of the Gradshteyn and Ryzhik's integral function which eliminates the requirements of going through the Moore's intractable techniques. The consulting actuaries can then apply the variables being computed to the valuation of pensions and life insurances.

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