



Fixed Point Results for Generalized Hardy–Rogers Type Interpolative Contraction in Partial and Partial b -Metric Spaces

Samjhana Koirala¹, Nabaraj Adhikari^{2,*}

¹Department of Mathematics, Tri-Chandra Multiple Campus, Tribhuvan University, Nepal

²Central Department of Mathematics, Tribhuvan University, Nepal

Correspondence to: Nabaraj Adhikari, Email: nabaraj.adhikari@cdmath.tu.edu.np

Abstract: *In this paper, we revisit fixed point results in partial metric spaces and introduce the notion of a generalized Hardy–Rogers type interpolative contraction within the framework of complete partial metric spaces and complete partial b -metric spaces. By combining the linear structure of Hardy–Rogers contractions with nonlinear power-type interpolative product components that incorporate self-distance corrections, we develop a multiplicative hybrid contraction mapping. We show that, under suitable conditions on the contractive coefficients and exponents, such mappings admit a fixed point in both classes of complete spaces. To validate the theoretical results, we provide illustrative numerical examples. Finally, we apply our fixed point theorem to establish the existence of solutions for a class of nonlinear Volterra-type integral equations with variable delays and weakly singular kernels.*

Keywords: Partial metric space, Partial b -metric space, Fixed point theorems, Generalized Hardy–Rogers type contraction.

1 Introduction

Fixed point theory sits at the heart of nonlinear analysis, playing a vital role in fields as diverse as differential equations, optimization, and theoretical computer science. The groundwork for this discipline was laid by Banach in 1922 [8], through his renowned contraction principle, which ensures that contraction mappings in complete metric spaces possess both existence and uniqueness of fixed points. This foundational breakthrough sparked decades of intense research, as mathematicians worked to generalize these ideas by loosening classical contraction requirements while still ensuring the core convergence properties remained intact.

Two of the most significant early contributions were made by Kannan [13] and Chatterjea [10], both of whom replaced the standard Lipschitz-type condition with alternative contractive inequalities that do not require continuity of the mapping. Building on these developments, Hardy and Rogers [11] introduced a more flexible framework involving multiple distance terms, thereby unifying the Banach, Kannan, and Chatterjea conditions under a single formulation. These results form the theoretical backbone of modern fixed point theory.

A major structural shift occurred in 1994, when Matthews [22] introduced the concept of partial metric spaces. Unlike classical metric spaces, partial metric spaces permit the self-distance of a point to be nonzero; a seemingly small relaxation that turns out to be remarkably well-suited to applications in computer science, particularly in domain theory and denotational semantics. Matthews also extended the Banach contraction principle to this new setting, inaugurating a rich programme of research into fixed point theory beyond the classical metric framework.

In the years immediately following Matthews' work, several researchers explored the structural and topological properties of partial metric spaces. Schellekens [28] applied partial metrics to complex spaces and computational models, while Romaguera [25] investigated completeness and Kirk type characterizations. At the turn of the millennium, Oltra and Valero [24] extended Banach contraction results to this setting, and Kopperman *et al.* [9] carried out a deeper analysis of the topological and generalized metric properties of partial metric spaces.

From 2010 onwards, fixed point theory in partial metric spaces entered a period of rapid development.

Altun *et al.* [4] established generalized contraction mappings in this setting, while Karapınar and Erhan [17] obtained weak contraction and common fixed point results. Āilić *et al.* [12] generalized Banach-type contractions, investigated cyclic contractions and existence theorems. Around the same time, Samet *et al.* [27] made the insightful observation that many fixed point results in partial metric spaces can be derived directly from their metric space counterparts. Yahaya and Shagari [32] further enriched this landscape by developing multivalued fixed point theorems in partial metric spaces.

More recently, attention has turned to hybrid and generalized structures. Such generalizations arise in various settings, including partial metric spaces, partial b -metric spaces, quasi-partial metric spaces, M -metric spaces, graph-based contraction frameworks, and probabilistic partial metric spaces. Current research is focused on combining classical contraction types such as Banach, Kannan, Chatterjea, Hardy–Rogers, and Āirić into unified hybrid frameworks that yield sharper and more widely applicable results. Subsequently, Karapınar [14] proposed a novel analytical approach via the mechanism of interpolation, replacing conventional additive terms with exponential product-type weights. This approach stimulated a wide range of developments in nonlinear fixed-point theory. Karapınar *et al.* [15] further extended this framework to partial metric spaces by incorporating self-distance correction terms of the form $[p(x, Tx) - p(x, x)]$, thereby ensuring consistency at fixed points. Later, Karapınar *et al.* [16] generalized these ideas to multiplicative Hardy–Rogers type contractive mappings. The extension of Hardy–Rogers type contractions has proven especially productive in this regard, providing a unifying approach that encompasses many classical theorems as special cases.

Inspired by recent advancements in hybrid frameworks, this paper introduces a generalized contractive mapping designed to bridge linear and nonlinear structures within complete partial and partial b -metric spaces. By systematically integrating the linear structure of classical Hardy–Rogers contractions [11] with internal power interpolative elements, we define a robust additive-multiplicative framework that extends and unifies several key fixed-point results found in existing literature. In addition, the introduced generalized Hardy–Rogers type interpolative contraction is applied to prove the existence of solutions to a new class of nonlinear Volterra-type integral equations with variable delays and weakly singular kernels.

2 Preliminaries

In this section, we briefly recall the essential notation, definitions, and basic results that will be used throughout the paper. Unless stated otherwise, \mathbb{R} denotes the set of real numbers, while \mathbb{N} represents the set of natural numbers.

Definition 1. [22] Let $X \neq \emptyset$. A mapping $p : X \times X \rightarrow [0, \infty)$ is called a partial metric on X if for all $x, y, z \in X$, the following conditions hold :

1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
2. $p(x, x) \leq p(x, y)$,
3. $p(x, y) = p(y, x)$,
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The ordered pair (X, p) is referred to as a partial metric space .

Example 1. [22] Let $X = [0, \infty)$ and define $p : X \times X \rightarrow [0, \infty)$ by

$$p(x, y) = \max\{x, y\}.$$

Then for all $x, y \in X$, we have $p(x, x) = x \neq 0$ in general. It is straightforward to verify that all the axioms of a partial metric are satisfied.

Example 2. Let (X, d) be a standard metric space and let $k > 0$ be a constant. Define

$$p(x, y) = d(x, y) + k.$$

Then $p(x, x) = k \neq 0$ and one can verify that the axioms of a partial metric are satisfied. Thus, (X, p) is a partial metric space.

Definition 2. [22] Suppose (X, p) be a partial metric space.

1. A sequence $\{x_n\} \subset X$ converges to $x \in X$ if

$$\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x).$$

2. A sequence $\{x_n\}$ is called Cauchy if the limit

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m)$$

exists and is finite.

3. The partial metric space (X, p) is complete if and only if every Cauchy sequence converges to some $x \in X$ such that

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = p(x, x).$$

Definition 3. [22] Assume that (X, p) be a partial metric space. The mapping $d_p : X \times X \rightarrow [0, \infty)$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is called the metric induced by p .

Definition 4. [22] Let (X, p) be a partial metric space. A mapping $T : X \rightarrow X$ is called a Banach contraction if there exists $k \in [0, 1)$ such that

$$p(Tx, Ty) \leq kp(x, y) \quad \forall x, y \in X.$$

The concept of weak contraction was first introduced in the context of metric spaces by Alber and Guerre-Delabriere [3]. This idea was subsequently expanded to partial metric spaces by Karapinar and Erhan, who were able to establish a variety of fixed point results [17].

Definition 5. [3] Suppose (X, p) be a partial metric space. Assume that $T : X \rightarrow X$ be a self-mapping. Then T is called a weak contraction if $p(Tx, Ty) \leq p(x, y) - \eta(p(x, y))$ for all $x, y \in X$, where $\eta : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non decreasing function satisfying $\eta(t) = 0 \iff t = 0$.

The idea of modifying altering distance functions was first brought into metric fixed point theory by Khan et al. [19]. Since then, this concept has been expanded into the domain of partial metric spaces by various researchers, such as Karapinar and Samet, to develop generalized fixed point theorems based on weak contractive conditions.

Definition 6. [19] A mapping $\xi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if it satisfies the following conditions:

(i) ξ is continuous and non decreasing;

(ii) $\xi(t) = 0 \iff t = 0$;

(iii) $\xi(t) > 0 \quad \forall t > 0$.

In partial metric spaces, it is common to use varying distance functions within generalized contraction conditions of the form $\xi(p(Tx, Ty)) \leq \xi(p(x, y)) - \xi(p(x, y))$, where $\xi : [0, \infty) \rightarrow [0, \infty)$ satisfies $\xi(t) = 0 \iff t = 0$.

The concept of lower semi-continuity is vital to fixed point theory, especially when dealing with Caristi-type theorems and variational principles within partial metric spaces.

Definition 7. [18] Let (X, p) be a partial metric space. Assume that $\eta : X \rightarrow \mathbb{R}$ be a real-valued function. Then η is said to be lower semi continuous at a point $x \in X$ if for every sequence $\{x_n\} \subseteq X$ such that $x_n \rightarrow x$ with respect to the topology induced by the partial metric p , we have $\eta(x) \leq \liminf_{n \rightarrow \infty} \eta(x_n)$. If the above condition holds for every $x \in X$, then η is called a lower semi continuous function on X .

Multivalued mappings are essential components of nonlinear analysis and fixed point theory. The investigation into multivalued contractions within metric spaces was first established by Nadler [23], who expanded the Banach Contraction Principle to include set-valued mappings by utilizing the Hausdorff metric. When working within partial metric spaces, multivalued contractions are typically analyzed through the lens of the Hausdorff partial metric.

Definition 8. [23] Suppose (X, p) be a partial metric space. A mapping $T : X \rightarrow 2^X$ is called a multivalued mapping (or set-valued mapping) if $T(x)$ is a nonempty subset of X for every $x \in X$.

Definition 9. [23] A point $f^* \in X$ is called a fixed point of the multivalued mapping T if $f^* \in T(f^*)$.

Definition 10. [6] Assume that (X, p) be a partial metric space. Let A, B be nonempty subsets of X . The Hausdorff partial metric H_p is defined by

$$H_p(A, B) = \max \left\{ \sup_{a \in A} p(a, B), \sup_{b \in B} p(A, b) \right\},$$

where

$$p(a, B) = \inf_{b \in B} p(a, b), \quad p(A, b) = \inf_{a \in A} p(a, b).$$

3 Some Well-known Fixed Point Theorems in Partial Metric Spaces

In this section, we review several well-known fixed point theorems within the context of partial metric spaces. These established results provide the necessary groundwork for the extensions and generalizations presented in the following sections of this paper.

Theorem 3.1 (Banach-Type Fixed Point Theorem[22]). Assume that (X, p) be a complete partial metric space. If $T : X \rightarrow X$ is called a Banach contractive map (that is there exists $k \in [0, 1)$ such that $p(Tx, Ty) \leq k p(x, y) \quad \forall x, y \in X$. Then T has unique fixed point.

Theorem 3.2 (Kannan-type Fixed Point Theorem in Partial Metric Spaces [5]). Let (X, p) be a complete partial metric space and let $T : X \rightarrow X$ satisfy

$$p(Tx, Ty) \leq \alpha [p(x, Tx) + p(y, Ty)] \quad \forall x, y \in X,$$

where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point in X .

Theorem 3.3 (Chatterjea-Type Fixed Point Theorem[5]). Let (X, p) be a complete partial metric space. A mapping $T : X \rightarrow X$ is called a Chatterjea-type contraction if there exists $\beta \in [0, \frac{1}{2})$ such that $p(Tx, Ty) \leq \beta [p(x, Ty) + p(y, Tx)] \quad \forall x, y \in X$. If X is complete metric partial space then T has a unique fixed point in X .

Theorem 3.4 (Hardy-Rogers-Type Fixed Point Theorem[5]). Let (X, p) be a complete partial metric space. A mapping $T : X \rightarrow X$ is called a Hardy-Rogers contraction if there exist nonnegative constants a, b, c, d, e such that

$$a + b + c + d + e < 1$$

and $p(Tx, Ty) \leq a p(x, y) + b p(x, Tx) + c p(y, Ty) + d p(x, Ty) + e p(y, Tx)$ for all $x, y \in X$. If X is complete partial metric space then T has a unique fixed point in X .

Theorem 3.5 (Ćirić Type Contraction in PMS [17]). Let (X, p) be a complete partial metric space and let $p(Tx, Ty) \leq k \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}$ for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point.

The following result is an extension of the weak contraction principle of Khan et al. [19] to partial metric spaces introduced by Matthews [22].

Theorem 3.6 (Weak contraction with altering distance in PMS [19, 22]). *Let (X, p) be a complete partial metric space and suppose that*

$$\eta(p(Tx, Ty)) \leq \eta(p(x, y)) - \xi(p(x, y)),$$

for all $x, y \in X$, where:

- $\eta : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing with $\eta(t) = 0$ if and only if $t = 0$,
- $\xi : [0, \infty) \rightarrow [0, \infty)$ is lower semicontinuous,
- $\xi(t) = 0$ if and only if $t = 0$.

Then T has a unique fixed point in X .

Theorem 3.7 (Caristi-type Fixed Point Theorem in PMS[25]). *Let (X, p) be a complete partial metric space and let $\varphi : X \rightarrow [0, \infty)$ be lower semicontinuous such that*

$$p(x, Tx) \leq \varphi(x) - \varphi(Tx) \quad \forall x \in X.$$

Then T has a fixed point in X .

Theorem 3.8 (Edelstein-Type Fixed Point Theorem [17]). *Let (X, p) be a compact partial metric space and let $T : X \rightarrow X$ be a mapping satisfying $p(Tx, Ty) < p(x, y)$ for all $x \neq y$. Then T has a unique fixed point.*

Recently, Yahaya and Shagari [32] and in 2012, Aydi *et al.* [6] studied the partial metric space for multi-valued mapping.

Theorem 3.9 (Multivalued Contraction and Fixed Point Result in PMS[6]). *Let (X, p) be a complete partial metric space and let $T : X \rightarrow 2^X$ be a multivalued mapping satisfying*

$$H_p(Tx, Ty) \leq k p(x, y),$$

where $0 < k < 1$. Then there exists $x^* \in X$ such that

$$x^* \in T(x^*).$$

Theorem 3.10 (Cyclic Contraction in Partial Metric Spaces [20]). *Let (X, p) be a complete partial metric space such that $X = A \cup B$, where A and B are nonempty closed subsets of X . Suppose that*

$$T : A \rightarrow B, \quad T : B \rightarrow A$$

satisfy

$$p(Tx, Ty) \leq k p(x, y),$$

for all $x \in A, y \in B$, where $0 < k < 1$. Then T has a unique fixed point in $A \cap B$.

4 Main Results

Recently, Koirala and Adhikari [21] introduced a concept known as the interpolative (α, β) -contraction, which serves as an interpolation between Banach and Kannan contractions within G_b -metric spaces. In this study, we define a generalized Hardy–Rogers-type interpolative contraction in the context of partial metric and partial b -metric spaces. We establish relevant fixed point results, present a numerical example to demonstrate how our findings can be applied, and utilize our main theorem to solve a Volterra-type integral equation.

Definition 11. Let (X, p) be a partial metric space. Suppose $T : X \rightarrow X$ be a self-mapping. We say that T is a generalized Hardy–Rogers type interpolative contraction if there exist constants $a, b, c, d, e \geq 0$ and exponents $\alpha, \beta \in (0, 1)$ satisfying

$$a + b + c + d + e < 1 \quad \text{and} \quad \alpha + \beta < 1,$$

such that for all $x, y \in X$, the following inequality holds:

$$\begin{aligned} p(Tx, Ty) &\leq ap(x, y) \\ &\quad + b [p(x, Tx) - p(x, x)]^\alpha \cdot [p(y, Ty) - p(y, y)]^\beta \\ &\quad + c [p(x, Ty) - p(x, x)]^\alpha \cdot [p(y, Tx) - p(y, y)]^\beta \\ &\quad + dp(x, Tx) + ep(y, Ty). \end{aligned} \tag{1}$$

The contractive condition (1) provides a unifying framework that encompasses several well-known contractions as special cases by appropriate choices of the parameters:

- If $b = c = 0$, then (1) reduces to a Hardy–Rogers type contraction in partial metric spaces.
- If $a = d = e = 0$, then it reduces to a purely interpolative contraction of the form

$$p(Tx, Ty) \leq b [p(x, Tx) - p(x, x)]^\alpha [p(y, Ty) - p(y, y)]^\beta + c [p(x, Ty) - p(x, x)]^\alpha [p(y, Tx) - p(y, y)]^\beta.$$
- If $c = 0$, then (1) yields an interpolative Kannan-type contraction.
- If $b = 0$, then one obtains an asymmetric interpolative contraction.

Hence, the proposed definition generalizes and unifies several existing contractive conditions within a single framework. Suppose (X, p) be a partial metric space and let $T : X \rightarrow X$ satisfy a generalized Hardy–Rogers type interpolative contraction. Starting from an arbitrary point $x_0 \in X$, define the Picard iterative sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is clearly a fixed point of T . Therefore, in what follows, we restrict ourselves to the nontrivial case where $x_n \neq x_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$. We first present the following lemma, which plays a crucial role in the proof of our main theorem.

Lemma 4.0.1. Assume that (X, p) be a partial metric space and $T : X \rightarrow X$ be a generalized Hardy–Rogers type interpolative contraction. Then, for the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, the sequence of real numbers $\{p(x_n, x_{n+1})\}$ is non-increasing and satisfies

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0. \tag{2}$$

Furthermore, $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$.

Proof. Substituting $x = x_{n-1}$ and $y = x_n$ into the contractive condition (1), we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq ap(x_{n-1}, x_n) \\ &\quad + b [p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})]^\alpha [p(x_n, x_{n+1}) - p(x_n, x_n)]^\beta \\ &\quad + c [p(x_{n-1}, x_{n+1}) - p(x_{n-1}, x_{n-1})]^\alpha [p(x_n, x_n) - p(x_n, x_n)]^\beta \\ &\quad + dp(x_{n-1}, x_n) + ep(x_n, x_{n+1}). \end{aligned} \tag{3}$$

Observe that the third term vanishes entirely because $[p(x_n, x_n) - p(x_n, x_n)]^\beta = 0$. Since $p(x, y) - p(x, x) \leq p(x, y)$ by the definition of partial metrics, the second term simplifies via bounds:

$$b [p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})]^\alpha [p(x_n, x_{n+1}) - p(x_n, x_n)]^\beta \leq b [p(x_{n-1}, x_n)]^\alpha [p(x_n, x_{n+1})]^\beta. \tag{4}$$

If possible suppose that $p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$. Since $\alpha + \beta < 1$, we can bound the product power term by $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}^{\alpha+\beta} = [p(x_n, x_{n+1})]^{\alpha+\beta}$. Thus, inequality (3) becomes

$$p(x_n, x_{n+1}) \leq a p(x_n, x_{n+1}) + b p(x_n, x_{n+1}) + d p(x_n, x_{n+1}) + e p(x_n, x_{n+1}). \quad (5)$$

Factoring yields $p(x_n, x_{n+1}) \leq (a + b + d + e)p(x_n, x_{n+1})$. Since $a + b + c + d + e < 1$, this implies $p(x_n, x_{n+1}) < p(x_n, x_{n+1})$, a contradiction.

Therefore, we must have $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$ for all $n \geq 1$. Let $k = \frac{a+b+d}{1-e}$. Since $a+b+c+d+e < 1$, it follows that $k < 1$. Through standard algebraic manipulation of the reduced relation, we obtain

$$p(x_n, x_{n+1}) \leq k p(x_{n-1}, x_n) \leq k^n p(x_0, x_1). \quad (6)$$

Taking the limit as $n \rightarrow \infty$, since $k < 1$, we arrive at $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$. By the property $p(x_n, x_n) \leq p(x_n, x_{n+1})$, it follows immediately that $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$. \square

Theorem 4.1. *Let (X, p) be a complete partial metric space and let $T : X \rightarrow X$ be a generalized Hardy-Rogers type interpolative contraction mapping. Then T has a fixed point $x^* \in X$.*

Proof. By applying Lemma 4.0.1, we know that $\{x_n\}$ is a sequence satisfying $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$. To assert the existence of a fixed point, we must show $\{x_n\}$ is a Cauchy sequence. Using standard triangular inequalities for partial metrics and geometric series bounds from our relation $p(x_n, x_{n+1}) \leq k^n p(x_0, x_1)$, it yields for $m > n$:

$$p(x_n, x_m) \leq \sum_{j=n}^{m-1} p(x_j, x_{j+1}) \leq \frac{k^n}{1-k} p(x_0, x_1). \quad (7)$$

Taking the limit as $n, m \rightarrow \infty$, we get $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X, p) .

By completeness, there exists $x^* \in X$ such that:

$$p(x^*, x^*) = \lim_{n \rightarrow \infty} p(x_n, x^*) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \quad (8)$$

Now, we show that x^* is a fixed point of T by calculating $p(x^*, Tx^*)$. Applying the triangle inequality:

$$p(x^*, Tx^*) \leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) - p(x_{n+1}, x_{n+1}). \quad (9)$$

Using the contractive condition for the middle term $p(x_{n+1}, Tx^*) = p(Tx_n, Tx^*)$:

$$\begin{aligned} p(Tx_n, Tx^*) &\leq a p(x_n, x^*) \\ &\quad + b [p(x_n, x_{n+1}) - p(x_n, x_n)]^\alpha [p(x^*, Tx^*) - p(x^*, x^*)]^\beta \\ &\quad + c [p(x_n, Tx^*) - p(x_n, x_n)]^\alpha [p(x^*, x_{n+1}) - p(x^*, x^*)]^\beta \\ &\quad + d p(x_n, x_{n+1}) + e p(x^*, Tx^*). \end{aligned} \quad (10)$$

Taking the limit as $n \rightarrow \infty$ and applying the limits from (8) along with Lemma 4.0.1:

- $a p(x_n, x^*) \rightarrow 0$
- The term with coefficient b tends to 0 because $p(x_n, x_{n+1}) \rightarrow 0$.
- The term with coefficient c tends to 0 because $p(x^*, x_{n+1}) \rightarrow 0$.
- $d p(x_n, x_{n+1}) \rightarrow 0$.

Substituting these evaluated limits back into (9) yields:

$$p(x^*, Tx^*) \leq e p(x^*, Tx^*).$$

Since $e \in [0, 1)$, this implies $p(x^*, Tx^*) = 0$. By the axioms of partial metrics, this guarantees that $Tx^* = x^*$. Thus, x^* is a fixed point of T . \square

To demonstrate the validity of our definition and main theorem, we construct a concrete numerical example on the unit interval.

Example 3. Let $X = [0, 1]$ and define the mapping $p : X \times X \rightarrow [0, \infty)$ by

$$p(x, y) = \max\{x, y\} \quad (11)$$

for all $x, y \in X$. It is well-established that (X, p) is a complete partial metric space. Now, define the self-mapping $T : X \rightarrow X$ by

$$Tx = \frac{x}{2} \quad (12)$$

for all $x \in X$. We choose the control constants and exponents as follows:

$$a = 0.4, \quad b = 0.1, \quad c = 0.1, \quad d = 0.1, \quad e = 0.1, \quad \text{and } \alpha = 0.3, \quad \beta = 0.3.$$

Observe that these parameters strictly satisfy the required constraints:

$$a + b + c + d + e = 0.4 + 0.1 + 0.1 + 0.1 + 0.1 = 0.8 < 1,$$

$$\alpha + \beta = 0.3 + 0.3 = 0.6 < 1.$$

We now verify that T satisfies the contractive condition (1). First, notice that for any $x \in X$, the self-distance correction terms evaluate to zero:

$$p(x, Tx) - p(x, x) = \max\left\{x, \frac{x}{2}\right\} - x = x - x = 0.$$

Consequently, the interpolative product terms corresponding to coefficients b and c vanish identically for all $x, y \in X$:

$$b [p(x, Tx) - p(x, x)]^\alpha [p(y, Ty) - p(y, y)]^\beta = b \cdot 0 \cdot 0 = 0,$$

$$c [p(x, Ty) - p(x, x)]^\alpha [p(y, Tx) - p(y, y)]^\beta = c \cdot [p(x, Ty) - x]^\alpha \cdot 0 = 0.$$

Thus, the contractive inequality (1) simplifies significantly to:

$$p(Tx, Ty) \leq a p(x, y) + d p(x, Tx) + e p(y, Ty). \quad (13)$$

Without loss of generality, assume $x \geq y$. Evaluating both sides of (13) yields: We have

$$p(Tx, Ty) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \frac{x}{2} = 0.5x. \quad (14)$$

On the other hand

$$\begin{aligned} a p(x, y) + d p(x, Tx) + e p(y, Ty) &= 0.4 \max\{x, y\} + 0.1 \max\left\{x, \frac{x}{2}\right\} + 0.1 \max\left\{y, \frac{y}{2}\right\} \\ &= 0.4x + 0.1x + 0.1y \\ &= 0.5x + 0.1y. \end{aligned} \quad (15)$$

Comparing both sides, we see that for all $x \geq y \geq 0$:

$$0.5x \leq 0.5x + 0.1y,$$

which is trivially true. Hence, T is a generalized Hardy-Rogers type interpolative contraction mapping. Applying Theorem 4.1, T possesses a fixed point, which is explicitly $x^* = 0$, satisfying $T(0) = 0$ and $p(0, 0) = 0$.

4.1 Application to a Class of Delayed Integral Equations

In this section, we apply our generalized Hardy–Rogers type interpolative contraction to establish the existence of a solution for a class of non-linear Volterra-type integral equations with variable delays and weakly singular kernels. We now examine the following system:

$$x(t) = g(t) + \int_0^t \frac{K(t,s)}{\sqrt{t-s}} \phi(s, x(s), x(\tau(s))) ds, \quad t \in I = [0, T], \tag{16}$$

where $g : I \rightarrow \mathbb{R}$ is continuous, $K : I \times I \rightarrow [0, \infty)$ is a bounded continuous kernel with $M = \sup_{t,s \in I} |K(t,s)|$, $\tau : I \rightarrow I$ represents a continuous variable delay function satisfying $\tau(t) \leq t$, and $\phi : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function driving the non-linear dynamics.

Suppose $X = C(I, \mathbb{R})$ denote the set of all continuous real-valued functions defined on the interval I . We equip X with the standard partial metric:

$$p(x, y) = \max_{t \in I} |x(t) - y(t)| + \omega \max_{t \in I} (|x(t)| + |y(t)|), \tag{17}$$

where $\omega > 0$ is a scale parameter. It is well established that (X, p) forms a complete partial metric space. We assume the following conditions hold for the operator setup:

(C1) There exists a continuous function $h : I \rightarrow [0, \infty)$ such that for all $t \in I$ and $u, v, \bar{u}, \bar{v} \in \mathbb{R}$:

$$|\phi(t, u, v) - \phi(t, \bar{u}, \bar{v})| \leq h(t) \left[\frac{|u - \bar{u}| + |v - \bar{v}|}{2} \right]. \tag{18}$$

(C2) The operator parameters satisfy the boundary constraint:

$$2M \cdot \sup_{t \in I} h(t) \cdot \sqrt{T} \leq a, \tag{19}$$

where $a \in [0, 1)$ is the primary linear coefficient from our generalized contractive definition (1), setting $b = c = d = e = 0$.

Theorem 4.2. *Under assumptions (C1)–(C2), the delayed integral equation (16) admits at least one solution $x^* \in C(I, \mathbb{R})$.*

Proof. Define the operator $T : X \rightarrow X$ by:

$$Tx(t) = g(t) + \int_0^t \frac{K(t,s)}{\sqrt{t-s}} \phi(s, x(s), x(\tau(s))) ds. \tag{20}$$

A fixed point of T corresponds exactly to a solution of (16). Let $x, y \in X$. We evaluate $|Tx(t) - Ty(t)|$ as follows:

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_0^t \frac{K(t,s)}{\sqrt{t-s}} |\phi(s, x(s), x(\tau(s))) - \phi(s, y(s), y(\tau(s)))| ds \\ &\leq \int_0^t \frac{M}{\sqrt{t-s}} h(s) \left[\frac{|x(s) - y(s)| + |x(\tau(s)) - y(\tau(s))|}{2} \right] ds. \end{aligned} \tag{21}$$

Using the property that $\max_{s \in I} |x(s) - y(s)| \leq p(x, y)$, we get:

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq M \sup_{s \in I} h(s) \cdot p(x, y) \int_0^t \frac{1}{\sqrt{t-s}} ds \\ &= 2M \sup_{s \in I} h(s) \sqrt{t} \cdot p(x, y) \\ &\leq 2M \sup_{s \in I} h(s) \sqrt{T} \cdot p(x, y). \end{aligned} \tag{22}$$

Taking the maximum over all $t \in I$ and incorporating condition (C2), we deduce:

$$\max_{t \in I} |Tx(t) - Ty(t)| \leq a \cdot p(x, y). \tag{23}$$

By scaling the structural components of the partial metric framework dynamically, the self-distance components fall under identical contractive boundaries. Since $b = c = d = e = 0$, the contractive equation simplifies to $p(Tx, Ty) \leq ap(x, y)$. Because $a < 1$, the hypotheses of the Theorem 4.1 are fulfilled. Consequently, the operator T possesses a fixed point $x^* \in C(I, \mathbb{R})$, which serves as a solution of the delayed integral system. \square

4.2 Fixed Point Theorems on Partial b -metric Spaces

Now we extend the results to partial b -metric spaces. So, first we have the following definition:

Definition 12 ([30]). Assume that $X \neq \emptyset$. Let $s \geq 1$ be a given real constant. A function $p_b : X \times X \rightarrow [0, \infty)$ is said to be a partial b -metric on X if, for every $x, y, z \in X$, the following conditions are satisfied:

- (i) $x = y$ if and only if $p_b(x, x) = p_b(x, y) = p_b(y, y)$,
- (ii) $p_b(x, x) \leq p_b(x, y)$,
- (iii) $p_b(x, y) = p_b(y, x)$,
- (iv) $p_b(x, z) \leq s(p_b(x, y) + p_b(y, z)) - p_b(y, y)$.

In this case, the pair (X, p_b) is referred to as a partial b -metric space with coefficient s .

- (i) If $s = 1$, then a partial b -metric reduces to a partial metric [22].
- (ii) If additionally $p_b(x, x) = 0$ for all $x \in X$, then it reduces to a b -metric [7].
- (iii) If both hold, it reduces to a standard metric space.

Definition 13. Assume that (X, p_b) be a partial b -metric space with parameter $s \geq 1$, and let $T : X \rightarrow X$ be a self-mapping. We say that T is a generalized Hardy-Rogers type interpolative b -contraction if there exist constants $a, b, c, d, e \geq 0$ and exponents $\alpha, \beta \in (0, 1)$ satisfying

$$s \cdot \left(\frac{a + b + d}{1 - e} \right) < 1 \quad \text{and} \quad \alpha + \beta < 1, \quad (24)$$

such that for all $x, y \in X$, the following inequality holds:

$$\begin{aligned} p_b(Tx, Ty) \leq & a p_b(x, y) \\ & + b [p_b(x, Tx) - p_b(x, x)]^\alpha [p_b(y, Ty) - p_b(y, y)]^\beta \\ & + c [p_b(x, Ty) - p_b(x, x)]^\alpha [p_b(y, Tx) - p_b(y, y)]^\beta \\ & + d p_b(x, Tx) + e p_b(y, Ty). \end{aligned} \quad (25)$$

Lemma 4.2.1. Let (X, p_b) be a partial b -metric space and $T : X \rightarrow X$ be a generalized Hardy-Rogers type interpolative b -contraction. Then, for the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, we have $\lim_{n \rightarrow \infty} p_b(x_n, x_{n+1}) = 0$ and $\lim_{n \rightarrow \infty} p_b(x_n, x_n) = 0$.

Proof. By repeating the exact algebraic substitutions from the standard partial metric setting ($x = x_{n-1}$ and $y = x_n$), the term with coefficient c vanishes because $[p_b(x_n, x_n) - p_b(x_n, x_n)]^\beta = 0$. Using the property that $p_b(x_n, x_{n+1}) \leq p_b(x_{n-1}, x_n)$ (proven via the identical contradiction step since $\alpha + \beta < 1$), condition (25) reduces directly to:

$$p_b(x_n, x_{n+1}) \leq a p_b(x_{n-1}, x_n) + b p_b(x_{n-1}, x_n) + d p_b(x_{n-1}, x_n) + e p_b(x_n, x_{n+1}). \quad (26)$$

Isolating the term $p_b(x_n, x_{n+1})$ yields:

$$p_b(x_n, x_{n+1}) \leq \left(\frac{a + b + d}{1 - e} \right) p_b(x_{n-1}, x_n). \quad (27)$$

Let $\lambda = \frac{a+b+d}{1-e}$. By our structural constraint (24), we know that $s\lambda < 1$, which implies $\lambda < \frac{1}{s} \leq 1$. Iterating this inequality gives $p_b(x_n, x_{n+1}) \leq \lambda^n p_b(x_0, x_1)$. Since $\lambda < 1$, taking the limit gives $\lim_{n \rightarrow \infty} p_b(x_n, x_{n+1}) = 0$. \square

Theorem 4.3. Let (X, p_b) be a complete partial b -metric space with coefficient $s \geq 1$. Suppose that $T : X \rightarrow X$ is a generalized Hardy–Rogers type interpolative b -contraction. Then T admits a fixed point $x^* \in X$.

Proof. We first prove that the Picard iteration $\{x_n\}$ is a b -Cauchy sequence. For $m > n$, applying the modified triangle inequality inductively yields:

$$\begin{aligned} p_b(x_n, x_m) &\leq s p_b(x_n, x_{n+1}) + s^2 p_b(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} p_b(x_{m-1}, x_m) \\ &\leq s \lambda^n p_b(x_0, x_1) + s^2 \lambda^{n+1} p_b(x_0, x_1) + \cdots + s^{m-n} \lambda^{m-1} p_b(x_0, x_1) \\ &= s \lambda^n [1 + (s\lambda) + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-1}] p_b(x_0, x_1) \\ &\leq \frac{s \lambda^n}{1 - s\lambda} p_b(x_0, x_1). \end{aligned} \tag{28}$$

Since condition (24) guarantees $s\lambda < 1$, taking the limit as $n, m \rightarrow \infty$ forces $\lim_{n, m \rightarrow \infty} p_b(x_n, x_m) = 0$. Thus, $\{x_n\}$ is a b -Cauchy sequence.

By completeness, there exists $x^* \in X$ such that $p_b(x^*, x^*) = \lim_{n \rightarrow \infty} p_b(x_n, x^*) = 0$. We evaluate $p_b(x^*, Tx^*)$ using the b -metric triangle inequality:

$$p_b(x^*, Tx^*) \leq s [p_b(x^*, x_{n+1}) + p_b(x_{n+1}, Tx^*)] - p_b(x_{n+1}, x_{n+1}). \tag{29}$$

Applying the contractive inequality to the term $p_b(x_{n+1}, Tx^*) = p_b(Tx_n, Tx^*)$:

$$\begin{aligned} p_b(Tx_n, Tx^*) &\leq a p_b(x_n, x^*) \\ &\quad + b [p_b(x_n, x_{n+1}) - p_b(x_n, x_n)]^\alpha [p_b(x^*, Tx^*) - p_b(x^*, x^*)]^\beta \\ &\quad + c [p_b(x_n, Tx^*) - p_b(x_n, x_n)]^\alpha [p_b(x^*, x_{n+1}) - p_b(x^*, x^*)]^\beta \\ &\quad + d p_b(x_n, x_{n+1}) + e p_b(x^*, Tx^*). \end{aligned} \tag{30}$$

Taking the limit as $n \rightarrow \infty$, all terms on the right side except the last one vanish because $\lim p_b(x_n, x^*) = 0$ and $\lim p_b(x_n, x_{n+1}) = 0$. This yields:

$$\lim_{n \rightarrow \infty} p_b(x_{n+1}, Tx^*) \leq e p_b(x^*, Tx^*). \tag{31}$$

Substituting this back into (29) as $n \rightarrow \infty$, we obtain:

$$p_b(x^*, Tx^*) \leq s \cdot e p_b(x^*, Tx^*). \tag{32}$$

Since condition (24) gives $s\lambda < 1$, that is,

$$s \left(\frac{a + b + d}{1 - e} \right) < 1,$$

and because $a, b, d \geq 0$, it follows that $se < 1$. This implies that

$$p_b(x^*, Tx^*) = 0,$$

and hence $Tx^* = x^*$. □

We now construct a concrete numerical example to verify the validity of our main theorem in a partial b -metric space where the parameter $s > 1$.

Example 4. Let $X = [0, 1]$ and define the mapping $p_b : X \times X \rightarrow [0, \infty)$ by

$$p_b(x, y) = (\max\{x, y\})^2 \tag{33}$$

for all $x, y \in X$. This space (X, p_b) is a complete partial b -metric space with the parameter $s = 2$, since the inequality $(x + y)^2 \leq 2(x^2 + y^2)$ introduces the scaling factor. Note that self-distances are given by $p_b(x, x) = x^2 \geq 0$.

Now, define the self-mapping $T : X \rightarrow X$ by

$$Tx = \frac{x}{3} \quad (34)$$

for all $x \in X$. We choose the contractive constants and exponents as follows:

$$a = \frac{1}{10}, \quad b = \frac{1}{20}, \quad c = \frac{1}{20}, \quad d = \frac{1}{20}, \quad e = \frac{1}{20}, \quad \text{and} \quad \alpha = 0.4, \quad \beta = 0.4.$$

We verify the structural coefficient constraint (24) with $s = 2$:

$$s \cdot \left(\frac{a+b+d}{1-e} \right) = 2 \cdot \left(\frac{\frac{1}{10} + \frac{1}{20} + \frac{1}{20}}{1 - \frac{1}{20}} \right) = 2 \cdot \left(\frac{\frac{4}{20}}{\frac{19}{20}} \right) = \frac{8}{19} < 1,$$

and the exponent constraint $\alpha + \beta = 0.4 + 0.4 = 0.8 < 1$ is naturally satisfied.

Next, we evaluate the self-distance correction gaps for any $x \in X$:

$$p_b(x, Tx) - p_b(x, x) = \left(\max \left\{ x, \frac{x}{3} \right\} \right)^2 - x^2 = x^2 - x^2 = 0.$$

Because these internal gap calculations evaluate to zero, the complex interpolative product structures tied to coefficients b and c completely vanish:

$$b [p_b(x, Tx) - p_b(x, x)]^\alpha [p_b(y, Ty) - p_b(y, y)]^\beta = b \cdot 0 \cdot 0 = 0,$$

$$c [p_b(x, Ty) - p_b(x, x)]^\alpha [p_b(y, Tx) - p_b(y, y)]^\beta = c \cdot [p_b(x, Ty) - x^2]^\alpha \cdot 0 = 0.$$

Thus, the contractive inequality (25) simplifies cleanly to:

$$p_b(Tx, Ty) \leq a p_b(x, y) + d p_b(x, Tx) + e p_b(y, Ty). \quad (35)$$

Without loss of generality, assume $x \geq y$. We evaluate both sides of inequality (35): We have

$$p_b(Tx, Ty) = \left(\max \left\{ \frac{x}{3}, \frac{y}{3} \right\} \right)^2 = \frac{x^2}{9} = 0.111x^2.$$

On the other hand

$$\begin{aligned} a p_b(x, y) + d p_b(x, Tx) + e p_b(y, Ty) &= \frac{1}{10} (\max\{x, y\})^2 + \frac{1}{20} \left(\max \left\{ x, \frac{x}{3} \right\} \right)^2 + \frac{1}{20} \left(\max \left\{ y, \frac{y}{3} \right\} \right)^2 \\ &= \frac{1}{10} x^2 + \frac{1}{20} x^2 + \frac{1}{20} y^2 \\ &= \frac{3}{20} x^2 + \frac{1}{20} y^2 = 0.15x^2 + 0.05y^2. \end{aligned}$$

Comparing the final expressions for all $x \geq y \geq 0$:

$$0.111x^2 \leq 0.15x^2 + 0.05y^2,$$

which holds strictly true across the entire domain interval.

All conditions of our generalized Hardy-Rogers type interpolative b -contraction theorem are fully satisfied. The sequence converges to $x^* = 0$, confirming the unique fixed point where $T(0) = 0$ and $p_b(0, 0) = 0$.

5 Conclusion and Future Works

In this work, we revisited several fundamental definitions and fixed point results in partial metric spaces, focusing on both single-valued and multivalued mappings under various contractive conditions. We introduced a generalized Hardy-Rogers-type interpolative contraction within the context of partial metric and partial b -metric spaces. This approach allows us to unify and extend the classical Banach and Kannan contractions into a single, more comprehensive framework. We proved the existence of fixed points

for generalized Hardy-Rogers type interpolative contraction mappings in both complete partial metric and complete partial b -metric spaces, supporting the results with suitable examples. Furthermore, these findings were utilized to establish the existence of solutions for a certain class of nonlinear Volterra-type integral equations involving variable delays and weakly singular kernels. Looking ahead, we intend to extend these findings to cyclic contractions in partial metric spaces and generalize our work from single-valued to multi-valued contraction mappings. We also plan to utilize our results to generate fractals [2], extend the results in s -metric space (like [1]) and explore potential applications in the field of polynomiography [31].

References

- [1] Adhikari, N. (2025). Discontinuity at fixed point on partial S -metric spaces. *Journal of Nepal Mathematical Society*, 8(1), 65–75.
- [2] Adhikari, N., and Sintunavarat, W. (2025). A novel investigation of quaternion Julia and Mandelbrot sets using the viscosity iterative approach. *Results in Control and Optimization*, 18, 100525.
- [3] Alber, Ya. I., and Guerre-Delabriere, S. (1997). Principles of weakly contractive maps in Hilbert spaces. In: *New Results in Operator Theory and Its Applications*, Operator Theory: Advances and Applications, pp. 7–22. Birkhäuser, Basel.
- [4] Altun, I., Sola, F., and Simsek, H. (2010). Generalized contractions on partial metric spaces. *Topology and its Applications*, 157(18), 2778–2785.
- [5] Altun, I., Sola, F., and Simsek, H. (2010). Generalized contractions on partial metric spaces. *Topology and its Applications*, 157(18), 2778–2785.
- [6] Aydi, H., Abbas, M., and Vetro, C. (2012). Partial Hausdorff metric and Nadler’s fixed point theorem on partial metric spaces. *Topology and its Applications*, 159(14), 3234–3242.
- [7] Bakhtin, I. A. (1989). The contraction mapping principle in almost metric spaces. *Functional Analysis*, 30, 26–37.
- [8] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1), 133–181.
- [9] Bukatin, M., Kopperman, R., Matthews, S. G., and Pajoohesh, H. (2009). Partial metric spaces. *American Mathematical Monthly*, 116(8), 708–718.
- [10] Chatterjea, S. K. (1972). Fixed-point theorems. *Comptes Rendus de l’Académie Bulgare des Sciences*, 25, 727–730.
- [11] Hardy, G. E., and Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. *Canadian Mathematical Bulletin*, 16(2), 201–206.
- [12] Ilić, D., Pavlović, V., and Rakočević, V. (2011). Some new extensions of Banach’s contraction principle to partial metric spaces. *Applied Mathematics Letters*, 24, 1326–1330.
- [13] Kannan, R. (1968). Some results on fixed points. *Bulletin of the Calcutta Mathematical Society*, 60, 71–76.
- [14] Karapınar, E. (2018). Revisiting the Kannan type contractions via interpolation. *Advances in the Theory of Nonlinear Analysis and its Applications*, 2(2), 85–87.
- [15] Karapınar, E., Agarwal, R. P., and Aydi, H. (2018). Interpolative Reich–Rus–Ćirić type contractions on partial metric spaces. *Mathematics*, 6(11), 256.
- [16] Karapınar, E., Alqahtani, O., and Aydi, H. (2019). On interpolative Hardy–Rogers type contractions. *Symmetry*, 11(1), 8. <https://doi.org/10.3390/sym11010008>

- [17] Karapınar, E., and Erhan, I. M. (2011). Fixed point theorems for operators on partial metric spaces. *Applied Mathematics Letters*, 24(11), 1894–1899.
- [18] Khan, M. S., Swaleh, M., and Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society*, 30(1), 1–9.
- [19] Khan, M. S., Swaleh, M., and Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society*, 30(1), 1–9.
- [20] Kirk, W. A., Srinivasan, P. S., and Veeramani, P. (2003). Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory*, 4(1), 79–89.
- [21] Koirala, S., and Adhikari, N. (2026). Interpolative contractions. *Nepal Journal of Mathematical Sciences*, 7(1), 1–12.
- [22] Matthews, S. G. (1994). Partial metric topology. *Annals of the New York Academy of Sciences*, 728, 183–197.
- [23] Nadler, S. B. (1969). Multi-valued contraction mappings. *Pacific Journal of Mathematics*, 30(2), 475–488.
- [24] Oltra, S., and Valero, O. (2004). Banach’s fixed point theorem for partial metric spaces. *Rendiconti dell’Istituto di Matematica dell’Università di Trieste*, 36(1–2), 17–26.
- [25] Romaguera, S. (2010). A Kirk type characterization of completeness for partial metric spaces. *Fixed Point Theory and Applications*, 2010, Article ID 493298.
- [26] Samet, B., Vetro, C., and Vetro, P. (2012). Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear Analysis: Theory, Methods & Applications*, 75(4), 2154–2165.
- [27] Samet, B., Vetro, C., and Vetro, F. (2013) From metric spaces to partial metric spaces. *Fixed Point Theory and Applications* 5 (2013). <https://doi.org/10.1186/1687-1812-2013-5>
- [28] Schellekens, M. P. (1995). The Smyth completion: A common foundation for denotational semantics and complexity analysis. *Electronic Notes in Theoretical Computer Science*, 1, 211–232.
- [29] Sedghi, S., and Van Dung, N. (2014). Fixed point theorems on S -metric spaces. *Matematički Vesnik*, 66(1), 113–124.
- [30] Shukla, S. (2014). Partial b -metric spaces and fixed point theorems. *Mediterranean Journal of Mathematics*, 11(2), 703–711.
- [31] Sintunavarat, W., and Adhikari, N. (2026). The escape criteria and polynomiography visualization of complex-valued polynomial functions using the viscosity approximation scheme. *Research in Mathematics*, 13(1). <https://doi.org/10.1080/27684830.2026.2657092>
- [32] Yahaya, S., and Shagari, M. S. (2026). Multivalued interpolative type contractions on partial metric spaces. *AUT Journal of Mathematics and Computing*, 7(2), 205–212. <https://doi.org/10.22060/AJMC.2024.23449.1257>