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Pi-Power Odd Exponentiated G-family of Distributions: Statistical Inferences and Application

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Abstract: *This study presents a new probability distributions family developed using the Pi-power transformation approach. Among this broader class, the study concentrates on one particular distribution whose hazard rate can take on several important shapes, with bathtub, J-shaped, reverse-J, and strictly increasing forms. The main mathematical characteristics of this distribution are discussed and derived, and we estimated its parameters using the method of maximum likelihood. To assess how well the estimation procedure performs, a simulation study is carried out. The results show that both the bias and mean squared error decrease steadily as the sample size grows, and the method performs reasonably well even with relatively small samples. The usefulness of the presented model is further demonstrated through applications to three real datasets drawn from weather and engineering contexts. Two of them are right-skewed, and one data set is symmetrical. Model adequacy measures and goodness-of-fit tests indicate that this distribution offers a better fit than some models taken under study. Overall, the proposed distribution offers a flexible tool for studying hazard behavior and survival-type data, through potential applications across multiple scientific and engineering fields, while also contributing to the broader literature on probability theory, statistical modeling, and inferences.*

Keywords: Moment, Pi-Power Odd, Quantile, Weibull distribution.

1 Introduction

Statistical models play a vital role in describing and analyzing data across a wide range of disciplines. Although classical probability distributions have been widely used in practice, they are often unable to adequately capture the complexity and diversity of real-world datasets. As a result, researchers have devoted considerable effort to developing more flexible and robust distributional models. Many of these advances have been achieved through innovative techniques, such as the T-X family approach, power approach, inverse approach, log-transformation and exponentiation methods, which allow the construction of new distributions with greater adaptability and improved fitting capabilities.

In this study, we have focused on a new method known as the pi-power transformation (PPT) technique, developed by Lone & Jan [15]. The PPT technique provides a unique combination of high flexibility and skewness to the parent distribution. A notable member of this family, the Pi-exponentiated Weibull, was studied by the authors. Before the development of the PPT family, the alpha power transformation (APT) approach was widely recognized in the fields of survival analysis and reliability theory. The APT method has been used by various researchers to propose new flexible and capable G-families and models. For example, Nassar et al. [23] applied the APT method to present a new class of distributions employing log-transformation, while Sapkota et al. [29] have defined the delta power transformation (DPT) technique to define new probability model. Moreover, Nassar et al. [24] derived a novel model based on the APT family using the quantile function, with a specific focus on its cumulative distribution function (CDF) as

$$F(y) = \frac{\log\left[1 + G\left(y;\omega\right)\left(\delta - 1\right)\right]}{\log\left(\delta\right)}; \quad y > 0, \delta > 0, \delta \neq 1,$$

where ω represent the parameter space and $G(y;\omega)$ is the CDF of any parent distribution. Mead et al. (2019) [21] further explored the APT technique, presenting mathematical and statistical characteristics that were not addressed in the earlier work of Mahdavi & Kundu [16]. Additionally, Maruthan & Venkatachalam [18] applied the APT to transform the Lomax distribution.

Other scholars, such as Hozaien et al. [10] and Klakattawi & Aljuhani [12], have developed novel models utilizing both technique APT and TX-transformation. Alotaibi et al. [1] presented a new distribution based on the APT technique using a weighted functional form, while Gomaa et al. [7] developed the new distribution using APT, characterized by a flexible hazard function. This model was applied to model the datasets of COVID-19 from the UK and Italy. The generalized Pareto distribution was introduced by using APT by Bleed et al. [2], and a new power transformed distribution has been developed using the power transformation Gemeay et al. [6] and employed it to model the radiotherapy and environmental datasets. In recent years, several new families of distributions have been developed to better model complex datasets. For example, Elbatal et al. [4] proposed a new distribution family within the APT framework, with a CDF expressed as

$$F\left(x\right)=\frac{G\left(x\right)\alpha^{G\left(x\right)}}{\alpha};\alpha>0,x\in\Re.$$

Similarly, Kyurkchiev [14] introduced another family of distributions based on the logistic function, with its corresponding CDF formulated as:

$$F\left(x\right)=\frac{2G\left(x\right)}{1+G\left(x\right)};x\in\Re.$$

Kavya & Manoharan (2021) [11] presented a new transformation technique, and the CDF corresponding to their method is given by

$$F\left(x\right)=\frac{e}{e-1}\left\{1-e^{-G\left(x\right)}\right\};x\in\Re.$$

Furthermore, Mandouh et al. [17] introduced a two-parameter class of distributions using the APT method, characterized by a CDF presented as

$$F\left(x\right)=\frac{\beta^{kW\{G(x)\}}-1}{\beta-1};\beta>0,\beta\neq1,x\in\Re.$$

Lone & Jan [15] extended the idea of the APT to develop the Pi-Exponentiated Transformed (PET) class of distributions, with its CDF described by

$$F\left(y\right)=\frac{\pi^{\{G(y)\}^{\theta}}-1}{\pi-1};\theta>0,y\in\Re.$$

Further, Sapkota et al. [30] introduced a new class of distribution using the pi-power transformation technique over the Sine function. Further, a new exponential family of distributions was developed by Sapkota et al. [31] and applied it to engineering and medical data.
These contributions reflect an ongoing effort among researchers to create and refine distribution families that effectively capture the characteristics of complex data. In this research, we utilize the Weibull model as the base distribution, leveraging its odd ratio of the CDF. The Weibull family has also been employed by Sapkota [27], and Sapkota & Kumar [27] used the odds ratio approach to define a new family. The PET family has gained popularity for its capability to enhance the skewness and flexibility of the parent distribution.

Building on the PPT concept, we introduce a new technique to improve existing models by incorporating an extra parameter over the odds ratio of the CDF. This approach, which we call the Pi-power odd exponentiated-G (PiPOE-G) family of distributions, provides increased robustness compared to other compound probability models, demonstrating substantial capability for modeling real data.

The two-parameter PiPOE-G family offers enough flexibility to describe datasets with a variety of shapes, including different levels of skewness, tail behavior, and failure rates, yet it remains mathematically manageable. This adaptability makes it particularly useful for modeling data that are strongly skewed, symmetrical, or that show unusual distributional patterns. Using the PiPOE-G family allows researchers and analysts to capture the nuances of real-world data more accurately and ultimately achieve better modeling results.

Within the PiPOE-G family, the Weibull distribution is particularly noteworthy due to its longstanding use in survival analysis and life-testing for effectively capturing failure rates and survival probabilities McCool [19]. The PiPOE-G framework further adapts and refines the Weibull distribution to align more closely with the exceptional properties observed in various applications.

The main objective of this study is to develop a new and flexible family of probability distributions, called the π-power odd exponentiated (PiPOE-G) family. As a special case of this family, the π-power odd exponentiated Weibull (PiPOEW) distribution is introduced and investigated in detail. The study aims to derive the important statistical and mathematical properties of the proposed distribution, including its density function, moments, quantile function, order statistics, and related measures. The remaining sections of this study are settled as follows. Section 2 introduces the PiPOE-G family, and Section 3 focuses on one of its specific members, the PiPOE–Weibull distribution. Section 4 outlines several of the key statistical properties, while Section 5 covers the inferential procedures for the PiPOEW distribution. Sections 6, 7, and 8 present the simulation study, a real-data application, and the concluding remarks, respectively.

2 Development of a new class of distribution and some statistical properties

Let $Y \sim PiPOE-G$ family, and $U(y;\alpha,\Psi)$ and $u(y;\alpha,\Psi)$ represent the CDF and PDF respectively of this family for $y \in \Re$, and expressed as

$$U(y;\alpha,\Psi) = \begin{cases} 1-\pi^{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)^{\alpha}} & for, \alpha>0, y\in\Re \\ 0 & otherwise \end{cases} \tag{1}$$

and

$$u(y;\alpha,\Psi) = \begin{cases} \alpha(\log\pi)\pi^{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)^{\alpha}}\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)^{\alpha-1}\frac{t(y;\Psi)}{[1-T(y;\Psi)]^2} & for, \alpha>0,\ y\in\Re \\ 0 & otherwise \end{cases}$$

Here $T(y;\Psi)$ and $t(y;\Psi)$ denote the CDF and PDF of the base distribution. Similarly, hazard, survival, and quantile functions of the PiPOE-G family can be presented as

$$h(y;\alpha,\Psi) = \alpha(\log\pi)\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)^{\alpha-1}\frac{t(y;\Psi)}{[1-T(y;\Psi)]^2}\ ;\quad y\in\Re,$$

$$S(y;\alpha,\Psi) = \pi^{-\left(\frac{T(y;\Psi)}{1-T(y;\Psi)}\right)^{\alpha}};\qquad y\in\Re,$$

$$Q_Y(p) = T^{-1}\left[\left\{-\left\{\frac{\log(1-p)}{\log\pi}\right\}\right\}^{1/\alpha}\left\{1+\left\{-\left\{\frac{\log(1-p)}{\log\pi}\right\}\right\}^{1/\alpha}\right\}^{-1}\right];\qquad p\in(0,1). \tag{2}$$

2.1 Linear form of PiPOE family

The linear representation of the PiPOE family is particularly useful because it greatly simplifies the derivation of several properties like moments, generating functions, entropy measures, and order statistics, while also facilitating efficient numerical computation. After some mathematics, the CDF 1 of PiPOE can be obtained in linear form as

$$U(y;\alpha,\Psi) = 1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(\log\pi)^i}{i!}\binom{i\alpha+j-1}{j}T^{i\alpha+j}(y;\Psi);\qquad \alpha>0,\ y\in\Re. \tag{3}$$

Taking differentiation under y, we get

$$u(y;\alpha,\Psi) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+1}(i\alpha+j)(\log\pi)^i}{i!}\binom{i\alpha+j-1}{j}T^{i\alpha+j-1}(y;\Psi)t(y;\Psi);\qquad \alpha>0,\ y\in\Re.$$

Also, we can write a PDF of PiPOE-G as

$$u(y;\alpha,\Psi)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}T^{i\alpha+j-1}(y;\Psi)\ t(y;\Psi);\qquad \alpha>0,\ y\in\Re,$$

where

$$\Delta_{ij}=\frac{(-1)^{i+1}\,(i\alpha+j)\,(\log\pi)^{i}}{i!}\begin{pmatrix} i\alpha+j-1\\ j\end{pmatrix}.$$

3 Pi-Power Odd Exponentiated Weibull (PiPOEW) distribution

In this section, we have derived a particular member of the PiPOE-G family. We have selected the Weibull distribution as a base distribution. Let Y follow the Weibull model, having the following CDF and PDF, respectively, as

$$T(y;\Psi)=1-e^{-\beta y^{\delta}};\qquad \beta,\delta>0,\ y>0 \tag{4}$$

$$t(y;\Psi)=\beta\delta y^{\delta-1}e^{-\beta y^{\delta}};\quad y>0.$$

Now, we have developed the PiPOEW distribution using Equation 4, having CDF as follows

$$U(y;\alpha,\beta,\delta)=1-\pi^{-\left(e^{\beta y^{\delta}}-1\right)^{\alpha}};\alpha,\beta,\delta>0,y>0. \tag{5}$$

Similarly, the PDF of the proposed model is presented as

$$u(y;\alpha,\beta,\delta)=\alpha\beta\delta(\log\pi)y^{\delta-1}e^{\beta y^{\delta}}\left(e^{\beta y^{\delta}}-1\right)^{\alpha-1}\pi^{-\left(e^{\beta y^{\delta}}-1\right)^{\alpha}};y>0. \tag{6}$$

Some important basic functions like survival, hazard, and quantile of the PiPOEW distribution are obtained as follows

$$R(y;\alpha,\beta,\delta)=\pi^{-\left(e^{\beta y^{\delta}}-1\right)^{\alpha}}\qquad ;y>0,$$

$$h(y;\alpha,\beta,\delta)=\alpha\beta\delta(\log\pi)\left(e^{\beta y^{\delta}}-1\right)^{\alpha-1}y^{\delta-1}e^{\beta y^{\delta}}\qquad ;y>0$$

and

$$Q_Y(p)=\left\{\frac{1}{\beta}\log\left\{\left\{-\frac{\log(1-p)}{\log\pi}\right\}^{\frac{1}{\alpha}}+1\right\}\right\}^{1/\delta};p\in(0,1). \tag{7}$$

Further, we can generate the random numbers using the following expression

$$y=\left\{\frac{1}{\beta}\log\left\{\left\{-\frac{\log(1-u)}{\log\pi}\right\}^{\frac{1}{\alpha}}+1\right\}\right\}^{1/\delta};u\in(0,1).$$

The PDF of the PiPOEW distribution can take on several different shapes, ranging from symmetric curves to decreasing forms and various types of right-skewed patterns. A few illustrative examples are shown in Figure 1 (left). In contrast, the HRF for this distribution is able to produce a wide variety of behaviors, including bathtub-shaped curves, steadily increasing patterns, as well as J-shaped and reverse-J forms. These different HRF shapes are illustrated in Figure 1 (right).

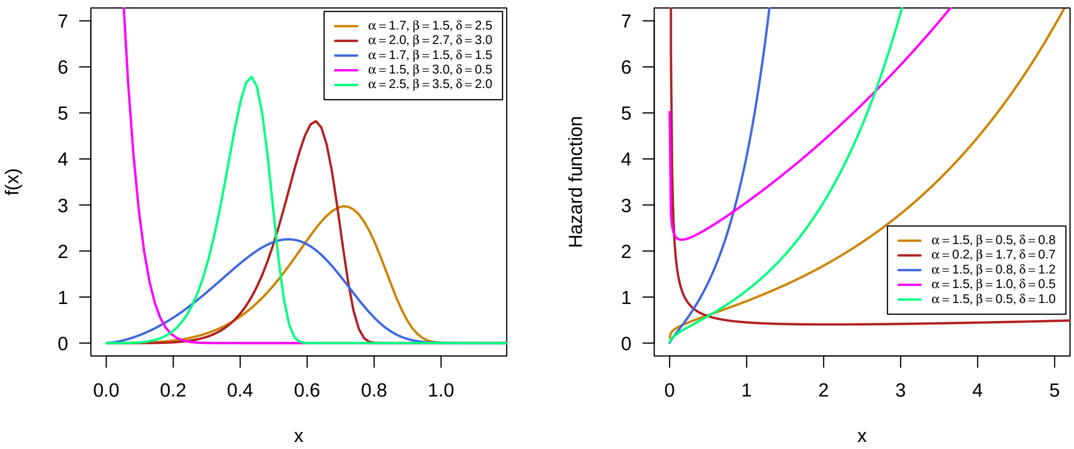

Figure 1: Several curves of PDF and HRF of PiPOEW model.

4 Mathematical properties of the proposed model

4.1 Linear representation

The linear form of any probability distribution is needed to obtain the expression for the mean, variance, and various orders of moments. The linear arrangement of the PDF of the PiPOEW model can be presented as

$$u(y;\alpha,\beta,\delta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^{*}_{ijk}y^{\delta-1}e^{-(k+1)\beta y^{\delta}};\ \alpha,\beta,\delta>0,\ y>0, \tag{8}$$

where

$$\Delta^{*}_{ijk}=(-1)^{k}\beta\delta\binom{i\alpha+j-1}{k}\Delta_{ij},$$

and

$$\Delta_{ij}=\frac{(-1)^{i+1}(i\alpha+j)(\log\pi)^{i}}{i!}\binom{i\alpha+j-1}{j}.$$

4.2 Moments

The moments of a probability distribution play a fundamental role in statistical analysis, as they describe important features of the distribution. In particular, moments are used to derive measures of central tendency, dispersion, asymmetry, and peakedness, including the mean, variance, skewness, and kurtosis. The r^{th} raw moment μ'_r of the PiPOEW model can be expressed as

$$\begin{aligned}E\left[Y^{r}\right]&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^{*}_{ijk}\int_{0}^{\infty}y^{\delta+r-1}e^{-(k+1)\beta y^{\delta}}dy\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^{*}_{ijk}\int_{0}^{\infty}t^{\frac{r}{\delta}+1-1}e^{-(k+1)\beta t}dt\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^{*}_{ijk}\frac{\delta^{-1}\Gamma\left(\frac{r}{\delta}+1\right)}{\{(k+1)\beta\}^{\frac{r}{\delta}+1}}.\end{aligned} \tag{9}$$

The first and second order moments of the PiPOEW model are obtained using Equation 9. Table 1 reports the calculated values of the mean, variance, and the coefficient of variation (CV) for the proposed distribution. The results show that the parameters provide substantial flexibility in controlling the location and dispersion characteristics of the PiPOEW distribution. In particular, increases in α, β, or δ tend to reduce the mean and variance, while larger values of α and δ also lead to lower relative variability as measured by the coefficient of variation.

$$E\left[Y\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ijk}^{*}\frac{\delta^{-1}\Gamma\left(\frac{1}{\delta}+1\right)}{\left\{\left(k+1\right)\beta\right\}^{\frac{1}{\delta}+1}},$$

and

$$E\left[Y^{2}\right]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ijk}^{*}\frac{\delta^{-1}\Gamma\left(\frac{2}{\delta}+1\right)}{\left\{\left(k+1\right)\beta\right\}^{\frac{2}{\delta}+1}},$$

and

$$V[Y]=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ijk}^{*}\frac{\delta^{-1}\Gamma\left(\frac{2}{\delta}+1\right)}{\left\{(k+1)\beta\right\}^{\frac{2}{\delta}+1}}-\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta_{ijk}^{*}\frac{\delta^{-1}\Gamma\left(\frac{1}{\delta}+1\right)}{\left\{(k+1)\beta\right\}^{\frac{1}{\delta}+1}}\right]^{2}.$$

Table 1: Mean, variance, and CV of PiPOEW distribution

α	β	δ	Mean	Variance	CV
0.25	1.05	0.50	1.9850	29.6152	2.7415
0.25	1.05	1.00	0.7341	1.4462	1.6382
0.25	1.05	0.75	0.9708	3.7050	1.9828
0.25	1.00	0.50	2.1885	35.9976	2.7415
0.25	1.00	1.00	0.7703	1.5951	1.6396
0.25	1.00	0.75	1.0359	4.2202	1.9832
0.25	1.50	0.50	0.9727	7.1106	2.7415
0.25	1.50	1.00	0.5134	0.7091	1.6403
0.25	1.50	0.75	0.6032	1.4314	1.9833
0.35	1.05	0.50	1.5806	18.3041	2.7069
0.35	1.05	1.00	0.7418	1.0303	1.3683
0.35	1.05	0.75	0.8890	2.5479	1.7956
0.35	1.00	0.50	1.7426	22.2487	2.7069
0.35	1.00	1.00	0.7793	1.1353	1.3673
0.35	1.00	0.75	0.9455	2.9079	1.8035
0.35	1.50	0.50	0.7744	4.3948	2.7070
0.35	1.50	1.00	0.5199	0.5042	1.3659
0.35	1.50	0.75	0.5510	0.9859	1.8021
0.45	1.05	0.50	0.8639	2.6211	1.8740
0.45	1.05	1.00	0.6852	0.3944	0.9166
0.45	1.05	0.75	0.6738	0.8552	1.3724
0.45	1.00	0.50	0.9527	3.1854	1.8733
0.45	1.00	1.00	0.7192	0.4355	0.9175
0.45	1.00	0.75	0.7041	0.9945	1.4163
0.45	1.50	0.50	0.4229	0.6297	1.8765
0.45	1.50	1.00	0.4713	0.2007	0.9506
0.45	1.50	0.75	0.4246	0.3254	1.3436

4.3 Moment Generating Function (MGF) of PiPOEW distribution

The MGF is also used to compute various orders of moments. For PiPOEW distribution, it can be attained as follows

$$\begin{aligned} M_Y(t) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^l}{l!}\Delta^*_{ijk}\int_0^{\infty} y^{\delta+l-1}e^{-(k+1)\beta y^{\delta}}dy \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^l}{l!}\Delta^*_{ijk}\int_0^{\infty} t^{\frac{l}{\delta}+1-1}e^{-(k+1)\beta t}dt \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^l}{l!}\Delta^*_{ijk}\frac{\delta^{-1}\Gamma\left(\frac{l}{\delta}+1\right)}{\{(k+1)\beta\}^{\frac{l}{\delta}+1}}. \end{aligned}$$

4.4 Characteristic Function (CF) of PiPOEW distribution

The CF is an important mathematical tool in probability theory that uniquely determines the distribution of a random variable. It provides valuable information about the distribution and can be used to derive several statistical properties, including moments and limiting distributions. The CF for the PiPOEW model is computed as

$$\begin{aligned} \Phi_Y(t) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(vt)^l}{l!}\Delta^*_{ijk}\int_0^{\infty} y^{\delta+l-1}e^{-(k+1)\beta y^{\delta}}dy \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(vt)^l}{l!}\Delta^*_{ijk}\int_0^{\infty} t^{\frac{l}{\delta}+1-1}e^{-(k+1)\beta t}dt \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(vt)^l}{l!}\Delta^*_{ijk}\frac{\delta^{-1}\Gamma\left(\frac{l}{\delta}+1\right)}{\{(k+1)\beta\}^{\frac{l}{\delta}+1}}, \end{aligned}$$

where $v = \sqrt{-1}$.

4.5 Incomplete moment (IM) of PiPOEW distribution

IMs are useful extensions of moments that provide information about the characteristics of a distribution over a restricted range. They are particularly helpful in reliability analysis and risk assessment, as they capture the partial contribution of a random variable to the probability of being below or above a specified threshold. The IM of PiPOEW distribution can be computed as

$$\begin{aligned} M_r(z) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^*_{ijk}\int_0^{z} y^{\delta+r-1}e^{-(k+1)\beta y^{\delta}}\,dy \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Delta^*_{ijk}\,\frac{\delta^{-1}\,\gamma\left(\frac{\delta+r}{\delta},(k+1)\beta z^{\delta}\right)}{\left\{(k+1)\beta\right\}^{\frac{\delta+r}{\delta}}}, \end{aligned}$$

where $\gamma(.)$ incomplete gamma function.

4.6 Mean Residual Life (MRL) of PiPOEW distribution

The MRL function describes the expected remaining lifetime of a unit or system, given that it has survived up to a certain time. It is widely used in reliability theory and survival analysis to assess the aging behavior of a distribution. The MRL for PiPOEW distribution can be derived as

$$\begin{aligned} \overline{M}(z) &= \frac{1}{F(z)}\left[\mu - \sum_{i,j,k=0}^{\infty}\Delta^*_{ijk}\int_0^{z} y^{\delta}e^{-(k+1)\beta y^{\delta}}\,dy\right] - z \\ &= \frac{1}{F(z)}\left[\mu - \sum_{i,j,k=0}^{\infty}\Delta^*_{ijk}\,\frac{\delta^{-1}\,\gamma\left(\frac{\delta+1}{\delta},(k+1)\beta z^{\delta}\right)}{\left\{(k+1)\beta\right\}^{\frac{\delta+1}{\delta}}}\right] - z. \end{aligned}$$

5 Parameter estimation

To estimate the proposed model's parameters, we have used the maximum likelihood method (MLE). Let $y_i(i = 1, ..., n) \sim PiPOEW(y_i; \alpha, \beta, \delta)$ having PDF defined in equation 6, then the log-likelihood density can be presented as

$$\begin{aligned}\ell(\underline{y}; \alpha, \beta, \delta) = n\log(\alpha\beta\delta) + n\log(\log\pi) - \log\pi\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log\left(e^{\beta y_i{}^{\delta}} - 1\right) \\ + (\delta-1)\sum_{i=1}^{n}\log y_i + \beta\sum_{i=1}^{n} y_i{}^{\delta}.\end{aligned} \tag{10}$$

Differentiating equation 10 under the associated parameters, we get

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \log\pi\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha}\log\left(e^{\beta y_i{}^{\delta}} - 1\right) + \sum_{i=1}^{n}\log\left(e^{\beta y_i{}^{\delta}} - 1\right),$$

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} - \log\pi\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha}\left\{\log\left(e^{\beta y_i{}^{\delta}} - 1\right)\right\}^2,$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} - \log\pi\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha-1}e^{\beta y_i{}^{\delta}}y_i{}^{\delta} + (\alpha-1)\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{-1}e^{\beta y_i{}^{\delta}}y_i{}^{\delta} + \sum_{i=1}^{n}y_i{}^{\delta},$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta^2} = -\frac{n}{\beta^2} - \log\pi\sum_{i=1}^{n}\left\{(\alpha-1)\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha-2}e^{2\beta y_i{}^{\delta}}y_i{}^{\delta} + \left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha-1}e^{\beta y_i{}^{\delta}}y_i{}^{\delta}\right\}y_i{}^{\delta} \\ + (\alpha-1)\sum_{i=1}^{n}\left\{\left(e^{\beta y_i{}^{\delta}} - 1\right)^{-1}e^{\beta y_i{}^{\delta}}y_i{}^{\delta} - e^{2\beta y_i{}^{\delta}}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{-2}y_i{}^{\delta}\right\}y_i{}^{\delta},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} - \log\pi\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha-1}e^{\beta y_i{}^{\delta}}\beta y_i{}^{\delta}\log y_i + (\alpha-1)\sum_{i=1}^{n}\left(e^{\beta y_i{}^{\delta}} - 1\right)^{-1}e^{\beta y_i{}^{\delta}}\beta y_i{}^{\delta}\log y_i \\ + \sum_{i=1}^{n}\log y_i + \beta\sum_{i=1}^{n}y_i{}^{\delta}\log y_i\,,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\ell}{\partial\delta^2} = -\frac{n}{\delta^2} - \log\pi\beta\sum_{i=1}^{n}\frac{\partial}{\partial\delta}\left\{\left(e^{\beta y_i{}^{\delta}} - 1\right)^{\alpha-1}e^{\beta y_i{}^{\delta}}y_i{}^{\delta}\right\}\log y_i \\ + (\alpha-1)\sum_{i=1}^{n}\frac{\partial}{\partial\delta}\left\{\left(e^{\beta y_i{}^{\delta}} - 1\right)^{-1}e^{\beta y_i{}^{\delta}}\beta y_i{}^{\delta}\right\}\log y_i + \beta\sum_{i=1}^{n}y_i{}^{\delta}(\log y_i)^2.\end{aligned}$$

To obtain estimates using the MLE method, the three non-linear expressions can be solved with the help of appropriate optimization software.

6 Simulation

In this section, we have used the `maxLik` package in R Henningsen & Toomet [9] to draw simulated data from the PiPOEW distribution, relying on the quantile function given in equation 7 and selecting several combinations of parameter values. For each simulated dataset, the MLEs were obtained with the function `maxLik()` using the BFGS optimization method. This procedure permitted us to explore how well the parameters could be recovered and to check whether the estimators consistently overshot or fell short of the true parameter values, giving us a clear picture of both the direction and size of any estimation bias. To carry out our study, we showed a simulation experiment with sample sizes extending from 10 to 250 in increments of 10. Each simulation was repeated 500 times to yield steady estimates of MLEs. The outcomes bias and mean square error (MSE) are displayed in Figures 2–3, which display MSEs for each parameter along with 95% confidence intervals. Our results indicate that for two different sets of parameters (plan I: $\alpha = 1.25$ $\beta = 0.75$ and $\delta = 0.50$ and plan II: $\alpha = 1.5$ $\beta = 0.5$ and $\delta = 0.75$), the MSE showed a consistent decline as the size of the sample increased. This outcome recommends that the MLE technique demonstrates properties of asymptotic efficiency, consistency, and invariance.

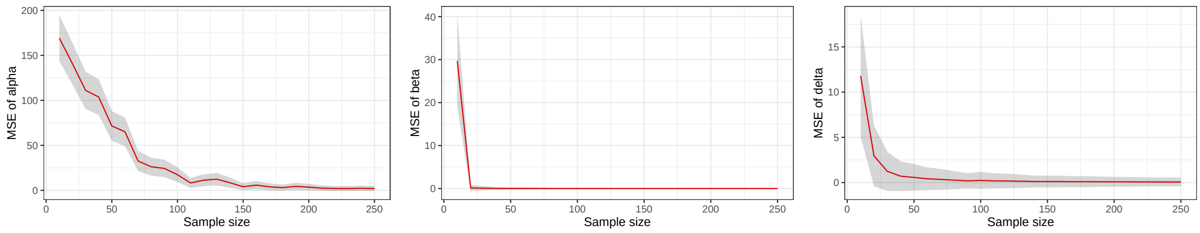

Figure 2: Graphs of MSE for plan I

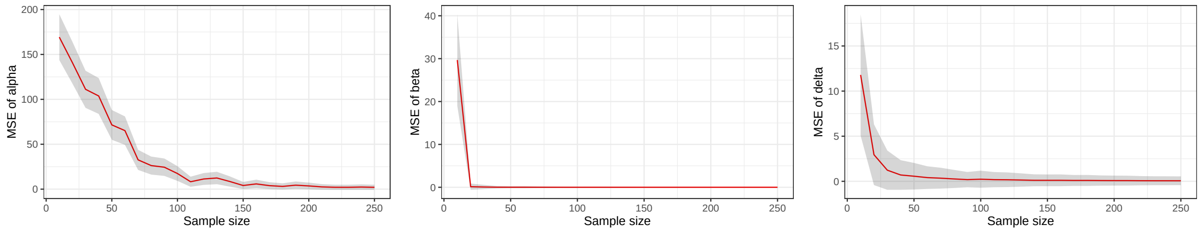

Figure 3: Graphs of MSE for plan II

7 Real data application

Here, we have demonstrated how the PiPOEW distribution can be applied to real-life data by analyzing three empirical datasets. A brief description of each dataset is given below, highlighting their role in assessing the practical significance and effectiveness of the presented model.

Dataset – I
The first dataset comprises 30 observations focusing on the intervals between failures in repairable items reported by Murthy et al. [22].
"1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17".

Dataset – II
We are utilizing an authentic dataset originally employed by Hinkley [8] for our data analysis. This dataset comprises thirty consecutive measurements of March precipitation (in inches) specifically recorded for Minneapolis/St Paul.
"0.77, 1.74, 1.20, 0.47, , 0.81, 1.20, 1.95, 1.43, 3.37, 1.51, 2.10, 2.20, 3.00, 3.09, 0.52, 1.62, 2.81, 1.87, 1.18, 1.35, 1.31, 0.32, 0.59, 0.81, 1.89, 0.90, 2.05, 4.75, 2.48, 0.96"]

Dataset – III
The following 60 observations represent the breaking stress (in GPa) of carbon fibers studied by Nichols & Padgett [25]. A preliminary examination suggests that the data follow an approximately symmetric trend. The recorded values are given below: "3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53"

In Figure 4, we have displayed the boxplots of all three datasets and indicate that datasets I and II are right-skewed and III is symmetrical.

7.1 Model Analysis

To examine how well the models fit data sets I, II, and III, we have computed several standard goodness-of-fit measures. Model adequacy was judged using a combination of likelihood-based and distributional criteria,

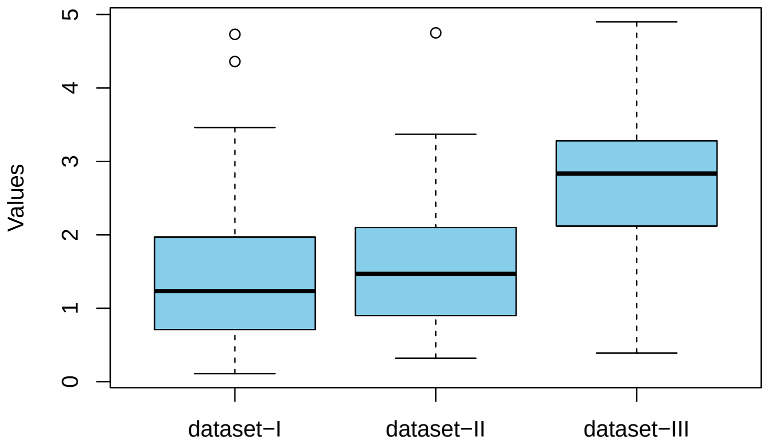

Figure 4: Boxplots for all three datasets

namely the negative twice log-likelihood (-2logL), Akaike Information Criterion (AIC), Hannan–Quinn Information Criterion (HQIC), Anderson–Darling (AD) statistic, Cramér–von Mises (CVM) statistic, and the Kolmogorov–Smirnov (KS) test together with their associated p-values. All analyses were implemented in the R statistical environment McElreath [20], R Core Team [26] and an R package `NeuDist` developed by Kumar et al., [13]. The fitting capability of the proposed PiPOEW model was then benchmarked against several competing distributions, including the inverse Weibull (IW), the alpha power inverse exponential (APIE) distribution introduced by Ceren et al. [3], the KM-inverse Weibull (KMIW) model proposed by Gauthami et al. [5] using Kavya and Mariyamma (KM) transformation technique, Pi-Exponentiated Weibull (PEW) Lone & Jan [15] and Exponential Power (ExpP) Srivastava & Kumar [32].

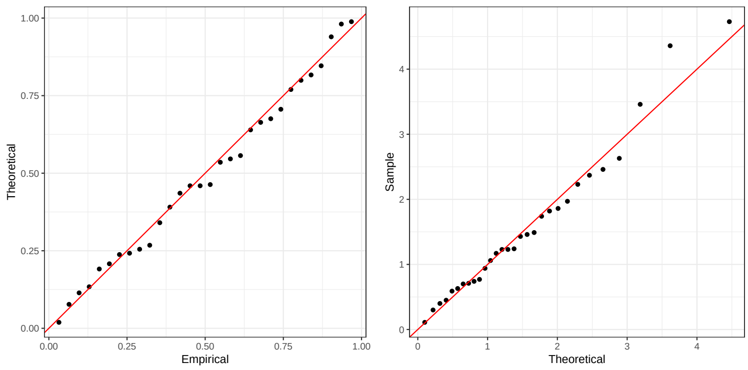

Figure 5: Graphs PP and QQ of PiPOEW distribution (dataset-I).

The MLEs of the model parameters and their corresponding standard errors (SE) for the three datasets are presented in Tables 2–4. The comparative performance of the fitted models is summarized in Tables 5–7, which report the information criteria ($-2\log L$, AIC, and HQIC) together with the goodness-of-fit statistics, namely the KS, AD, and CVM statistics and their associated p-values. In addition, the Probability–Probability (PP) and Quantile–Quantile (QQ) plots shown in Figures 5–7 provide a visual assessment of the adequacy of the proposed model for the datasets under consideration.
The results in Tables 5–7 indicate that the proposed PiPOEW distribution provides an excellent fit to all three datasets. For Dataset-I, the PiPOEW model yielded KS = 0.0766 ($p = 0.9946$), AD = 0.2158 ($p = 0.9855$), and CVM = 0.0286 ($p = 0.9822$), demonstrating a close agreement between the observed and fitted values. Similarly, for Dataset-II, the model produced KS = 0.0697 ($p = 0.9986$), AD = 0.1674

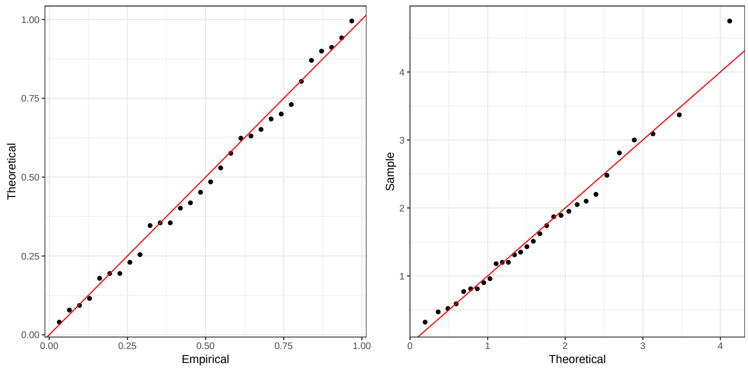

Figure 6: Graphs PP and QQ of PiPOEW distribution (dataset-II).

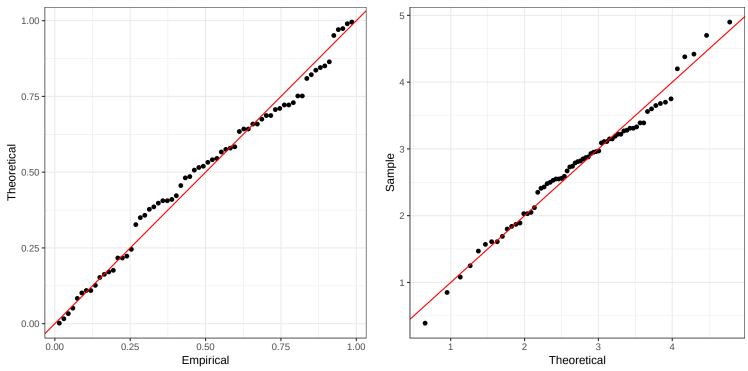

Figure 7: Graphs PP and QQ of PiPOEW distribution (dataset-III).

($p = 0.9970$), and CVM = 0.0212 ($p = 0.9964$), indicating an even stronger fit. For Dataset-III, the PiPOEW distribution also showed satisfactory performance with KS = 0.0819 ($p = 0.7684$), AD = 0.0774 ($p = 0.7085$), and CVM = 0.4676 ($p = 0.7795$). The consistently small goodness-of-fit statistics and large p-values across all datasets confirm the flexibility and effectiveness of the proposed distribution in modeling diverse real-world data.

A comparison with the competing IW, APIE, KMIW, PEW, and ExpP distributions further demonstrates the superiority of the PiPOEW model. Overall, the proposed distribution attains the smallest goodness-of-fit statistics and generally lower model selection criteria, together with comparatively larger p-values. Although the PEW distribution achieves slightly smaller values of $-2\log L$, AIC, and HQIC in some cases, its relatively poorer KS, AD, and CVM statistics indicate a less adequate fit to the data. Therefore, the PiPOEW distribution provides the most balanced and reliable representation of the datasets. This conclusion is further supported by the fitted density and distribution curves in Figures 8–10, which visually illustrate the improved fitting performance of the proposed PiPOEW model compared with the alternative distributions.

Table 2: Parameter estimates with standard errors for Dataset-I

Model	MLE(α)	SE(α)	MLE(β)	SE(β)	MLE(δ)	SE(δ)
PiPOEW	21.7154	9.4924	0.6721	0.0107	0.0487	0.0224
IW	1.0730	0.1314	0.7517	0.1570	–	–
APIE	0.7511	1.7550	0.8583	0.5583	–	–
KMIW	0.9447	0.1201	1.0714	0.2109	–	–
PEW	42.9893	4.8910	0.3304	0.0416	3.9817	0.1613
ExpP	0.9125	0.1350	0.4146	0.0789	–	–

Table 3: Parameter estimates with standard errors for Dataset-II

Model	MLE(α)	SE(α)	MLE(β)	SE(β)	MLE(δ)	SE(δ)
PiPOEW	46.4493	19.8759	0.6794	0.0066	0.0281	0.0126
IW	1.5496	0.2026	1.0253	0.1978	–	–
APIE	0.0100	0.0159	2.5075	0.4538	–	–
KMIW	1.3516	0.1838	1.2704	0.1778	–	–
PEW	72.4983	22.4212	0.3813	0.0413	4.1573	0.2445
ExpP	0.9664	0.1442	0.3916	0.0751	–	–

Table 4: Parameter estimates with standard errors for Dataset-III

Model	MLE(α)	SE(α)	MLE(β)	SE(β)	MLE(δ)	SE(δ)
PiPOEW	46.4493	19.8759	0.6794	0.0066	0.0281	0.0126
IW	1.5496	0.2026	1.0253	0.1978	–	–
APIE	0.0100	0.0159	2.5075	0.4538	–	–
KMIW	1.3516	0.1838	1.2704	0.1778	–	–
PEW	12.6982	7.3209	0.9499	0.1916	1.215	0.4515
ExpP	0.8627	0.1000	0.3187	0.0448	–	–

Table 5: Various statistics for model selection and goodness-of-fit (dataset-I)

Model	$-2\log L$	AIC	HQIC	KS	p(KS)	AD	p(AD)	CVM	p(CVM)
PiPOEW	79.9321	85.9321	87.2769	0.0766	0.9946	0.2158	0.9855	0.0286	0.9822
IW	92.7512	96.7512	97.6477	0.1338	0.6559	1.2296	0.2563	0.1830	0.3040
APIE	93.0534	97.0534	97.9499	0.1504	0.5064	1.2768	0.2398	0.1864	0.2964
KMIW	92.9769	96.9769	97.8734	0.9731	0.0000	1.3443	0.2184	0.2466	0.1929
PEW	53.6769	59.6769	61.0216	0.1516	0.4958	0.2067	0.2554	1.2622	0.2448
ExpP	82.3679	86.3679	87.2644	0.1199	0.7819	0.0865	0.6586	0.6185	0.6293

Table 6: Various statistics for model selection and goodness-of-fit (dataset-II)

Model	$-2\log L$	AIC	HQIC	KS	p(KS)	AD	p(AD)	CVM	p(CVM)
PiPOEW	77.3566	83.3566	84.7014	0.0697	0.9986	0.1674	0.9970	0.0212	0.9964
IW	83.8340	87.8340	88.7305	0.1523	0.4893	0.7597	0.5098	0.1202	0.4971
APIE	82.1052	86.1052	87.0017	0.1273	0.7155	0.6015	0.6452	0.0898	0.6403
KMIW	84.5777	88.5777	89.4742	0.9959	0.0000	1.4320	0.1936	0.2503	0.1882
PEW	48.2655	54.2655	55.6103	0.1427	0.5746	0.1910	0.2864	1.2653	0.2437
ExpP	82.6729	86.6729	87.5694	0.1659	0.3810	0.1773	0.3172	1.1304	0.2951

Table 7: Various statistics for model selection and goodness-of-fit (dataset-III)

Model	$-2\log L$	AIC	HQIC	KS	p(KS)	AD	p(AD)	CVM	p(CVM)
PiPOEW	171.6294	177.6294	180.2251	0.0819	0.7684	0.0774	0.7085	0.4676	0.7795
IW	242.3898	246.3898	248.1203	0.2303	0.0018	1.1562	0.0010	6.5040	0.0006
APIE	201.6224	205.6224	207.3529	0.1725	0.0395	0.4599	0.0501	2.4408	0.0534
KMIW	240.7990	244.7990	246.5294	0.9634	0.0000	1.0889	0.0014	6.0555	0.0009
PEW	134.9861	140.9861	143.5818	0.1779	0.0307	0.4283	0.0606	2.7694	0.0361
ExpP	226.8008	230.8008	232.5313	0.3757	0.0000	3.3658	0.0000	15.6706	0.0000

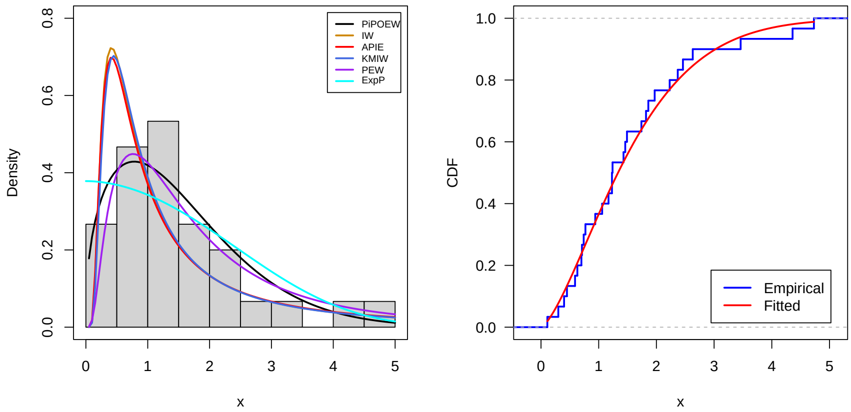

Figure 8: Fitted models (left) and fitted CDF of PiPOEW (right) (dataset-I)

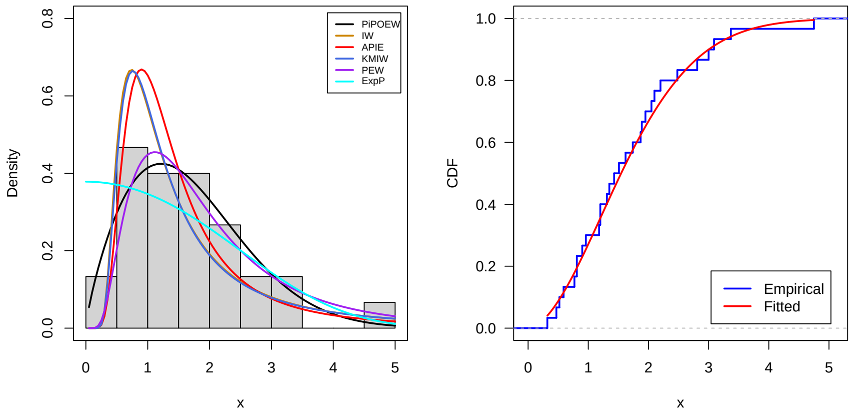

Figure 9: Fitted models (left) and fitted CDF of PiPOEW (right) (dataset-II)

8 Conclusion

In conclusion, this study proposed a new π-power odd exponentiated family of distributions, termed the PiPOE-G family, and introduced the PiPOEW distribution as a special case based on the Weibull distribution. The statistical properties of the proposed model were investigated, and the performance of the maximum likelihood estimators was assessed through a comprehensive Monte Carlo simulation study. The simulation results demonstrated that both the bias and mean squared error decrease as the sample size increases, indicating the consistency and reliability of the estimation procedure. Furthermore, the estimators showed satisfactory performance even for relatively small sample sizes, highlighting the practical usefulness of the proposed distribution for real-world applications.

The usefulness of the PiPOEW distribution in practical settings was further illustrated by applying it to three real datasets. Its capability was evaluated against several existing distributions using standard model comparison criteria and goodness-of-fit measures. The results clearly establish that the PiPOEW distribution provides a good fit relative to the competing models, suggesting that it can serve as a flexible and effective tool in a wide range of applications like medical research, economics, reliability analysis, insurance studies, and survival analysis.

Moreover, this study provides a foundation for the development of the π-power odd exponentiated transformation family of distributions and opens new avenues for constructing more flexible probability models in future research. The proposed framework can be extended to generate a variety of new distributions with improved modeling capabilities. The findings of this study may assist researchers and practitioners

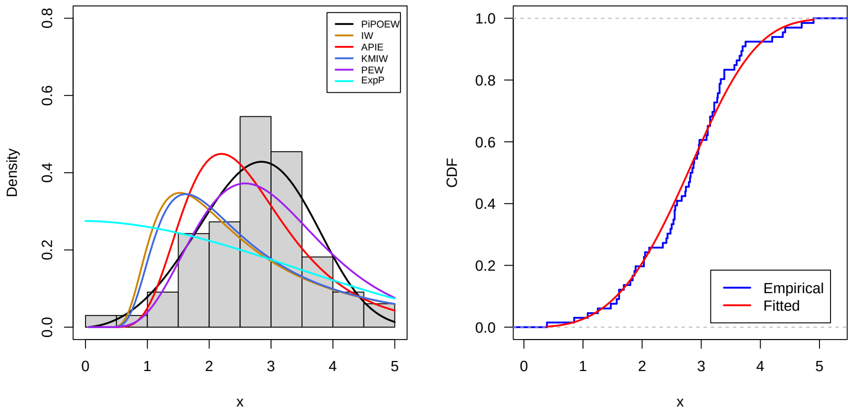

Figure 10: Fitted models (left) and fitted CDF of PiPOEW (right) (dataset-III)

in better capturing complex data characteristics and obtaining more accurate statistical inferences across a wide range of application areas.

One limitation of the present study is that the proposed model was evaluated using only three real-world datasets exhibiting right-skewed and approximately symmetric characteristics. Therefore, its performance for other types of data distributions has not been fully explored. Future research may focus on developing additional models within the proposed family by employing different baseline distributions to further enhance its flexibility and applicability. In addition, Bayesian estimation methods, censored data analysis, regression extensions, or applications to larger datasets can be investigated as an alternative inferential framework for the proposed model and assessed using a wider range of real-life datasets from different fields.

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