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Convexity Conditions for Symmetric Borel Derivatives and Borel Smoothness

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Abstract: *The Borel derivative generalizes classical differentiation by using an integral-based smoothing process. In this paper, monotonicity and convexity conditions for first- and second-order symmetric Borel derivatives are presented. The relation between symmetric Borel derivatives and symmetric Laplace derivatives is also established. Further, Borel smoothness is defined and some basic properties of Borel smooth functions are studied. These results show that several classical shape properties of ordinary derivatives continue to hold in the framework of symmetric Borel derivatives.*

Keywords: Borel derivative, Symmetric Borel derivative, Symmetric Laplace derivative, Borel smoothness, Laplace smoothness.

1 Introduction

The concept of the Borel derivative was introduced by E. Borel as an average derivative. Over the years, Borel derivatives have been extended and investigated in various forms. For instance, first-order Borel derivatives, both unsymmetric and symmetric, were studied by A. Khintchine [3] and by J. Marcinkiewicz and A. Zygmund [4]. Later, W. L. C. Sargent [11] extended the study of first-order unsymmetric Borel derivatives. The subject was further developed by S. N. Mukhopadhyay in connection with generalized higher-order derivatives, including both symmetric and unsymmetric forms, in the book *Higher Order Derivatives* [6]. Since then, several works have examined related properties and applications of these derivatives [7, 8, 10]. In this article, we study first- and second-order symmetric Borel derivatives and Borel smoothness. The main purpose is to give monotonicity and convexity criteria in this setting, establish the relation with symmetric Laplace derivatives, and record some basic properties of Borel smooth functions.

Definitions. Let a function $f(x)$ be special Denjoy integrable in a neighbourhood of $x \in \mathbb{R}$. The right-hand and left-hand Borel derivatives of f at x are defined by

$$BD_1^+ f(x) = \lim_{h\to 0^+} \frac{1}{h}\int_0^h \frac{f(x+t)-f(x)}{t}dt$$

and

$$BD_1^- f(x) = \lim_{h\to 0^+} \frac{1}{h}\int_{-h}^0 \frac{f(x+t)-f(x)}{t}dt,$$

where the improper integrals are assumed to be convergent. That is if

$$\lim_{h\to 0^+}\lim_{\epsilon\to 0^+} \frac{1}{h}\int_\epsilon^h \frac{f(x+t)-f(x)}{t}dt$$

exists, then the value of the limit is $BD_1^+ f(x)$. Similarly,

$$BD_1^- f(x) = \lim_{h\to 0^+}\lim_{\epsilon\to 0^+} \frac{1}{h}\int_{-h}^{-\epsilon} \frac{f(x+t)-f(x)}{t}dt.$$

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If $BD_1^+ f(x) = BD_1^- f(x)$, then f is said to have a Borel derivative at x, denoted by $BD_1 f(x)$. The right upper and lower Borel derivatives of f at x are defined by

$$\overline{BD}_1^+ f(x) = \limsup_{h\to 0^+} \frac{1}{h}\int_0^h \frac{f(x+t)-f(x)}{t}dt$$

and

$$\underline{BD}_1^+ f(x) = \liminf_{h\to 0^+} \frac{1}{h}\int_0^h \frac{f(x+t)-f(x)}{t}dt.$$

Similar definitions apply for left upper and lower Borel derivatives.
The first- and second-order symmetric derivatives of f at x are defined by

$$SD^1 f(x) = \lim_{t\to 0} \frac{f(x+t)-f(x-t)}{2t}$$

and

$$SD^2 f(x) = \lim_{t\to 0} \frac{f(x+t)+f(x-t)-2f(x)}{t^2}.$$

The first- and second-order symmetric Borel derivatives are, respectively

$$SBD^1 f(x) = \lim_{h\to 0} \frac{1}{h}\int_0^h \frac{f(x+t)-f(x-t)}{2t}dt$$

and

$$SBD^2 f(x) = \lim_{h\to 0} \frac{1}{h}\int_0^h \frac{f(x+t)+f(x-t)-2f(x)}{t^2}dt.$$

Changing 'lim' into 'liminf' gives the lower symmetric derivatives and lower symmetric Borel derivatives; changing it into 'limsup' gives the corresponding upper derivatives.

The first- and second-order symmetric Laplace derivatives of f at x are defined by

$$SLD^1 f(x) = \lim_{s\to\infty} s^2 \int_0^\delta e^{-st}\left[\frac{f(x+t)-f(x-t)}{2}\right] dt$$

and

$$SLD^2 f(x) = \lim_{s\to\infty} s^3 \int_0^\delta e^{-st}\left[\frac{f(x+t)+f(x-t)-2f(x)}{2}\right] dt.$$

The function f is said to be Borel smooth at x if

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \frac{f(x+t)+f(x-t)-2f(x)}{t}dt = 0$$

and the function f is called Laplace smooth at x if

$$\lim_{s\to\infty} s^2 \int_0^\delta e^{-st}[f(x+t)+f(x-t)-2f(x)]dt = 0.$$

The function f is said to be symmetrically Borel continuous at x of even order (odd order) if

$$\lim_{h\to 0} \frac{1}{h}\int_0^h \frac{f(x+t)+f(x-t)}{2}dt = f(x) \quad \left(\lim_{h\to 0} \frac{1}{h}\int_0^h \frac{f(x+t)-f(x-t)}{2}dt = 0\right).$$

A function f is said to be Baire$_1^*$ on a set E if every nonempty perfect set contained in E contains a portion on which the restriction of f is continuous. If f is Baire$_1^*$ on the set E, we write $f \in \mathcal{B}_1^*$.

2 Main Results

Lemma 2.0.1. *If p, q are positive integers and $\delta > 0$, then*

$$s^q \int_0^\delta e^{-st} t^p dt = p! s^{q-p-1} + o(1) \qquad \text{as } s \to \infty.$$

Proof. By putting $u = st$, we obtain

$$s^q \int_0^\delta e^{-st} t^p dt = s^{q-p-1} \int_0^{s\delta} e^{-u} u^p du = p! s^{q-p-1} + o(1), \quad s \to \infty.$$

□

Lemma 2.0.2. *If f is Borel smooth at x, then*

$$\overline{BD}_1^+ f(x) = \overline{BD}_1^- f(x) \qquad \text{and} \quad \underline{BD}_1^- f(x) = \underline{BD}_1^+ f(x).$$

Proof. Let

$$g(h) = \frac{1}{h} \int_0^h \frac{f(x+t) + f(x-t) - 2f(x)}{t} dt.$$

Since f is Borel smooth at x,

$$\lim_{h \to 0} g(h) = 0.$$

We can write

$$g(h) = \frac{1}{h} \int_0^h \frac{f(x+t) - f(x)}{t} dt + \frac{1}{h} \int_0^h \frac{f(x-t) - f(x)}{t} dt,$$

so

$$g(h) + \frac{1}{h} \int_h^0 \frac{f(x-t) - f(x)}{t} dt = \frac{1}{h} \int_0^h \frac{f(x+t) - f(x)}{t} dt. \tag{1}$$

Putting $-t = z$ for the integral in the left side of (1) we get,

$$g(h) + \frac{1}{h} \int_{-h}^0 \frac{f(x+z) - f(x)}{z} dz = \frac{1}{h} \int_0^h \frac{f(x+t) - f(x)}{t} dt. \tag{2}$$

Taking lim sup as $h \to 0^+$,

$$\overline{BD}_1^- f(x) \geq \overline{BD}_1^+ f(x). \tag{3}$$

Again putting $h = -k$ in (2) we get,

$$g(-k) + \frac{1}{k} \int_0^k \frac{f(x+z) - f(x)}{z} dz = \frac{1}{k} \int_{-k}^0 \frac{f(x+t) - f(x)}{t} dt.$$

Taking lim sup as $k \to 0^+$,

$$\overline{BD}_1^+ f(x) \geq \overline{BD}_1^- f(x). \tag{4}$$

From (3) and (4) we get $\overline{BD}_1^+ f(x) = \overline{BD}_1^- f(x)$. Similarly taking lim inf we get, $\underline{BD}_1^+ f(x) = \underline{BD}_1^- f(x)$. □

Lemma 2.0.3. *If f has a local maximum at x and Borel smoothness at x, then $BD_1 f(x)$ exists and $BD_1 f(x) = 0$.*

Proof. Since f has a local maximum at x, there exists $r > 0$ such that
$f(x+t) \leq f(x)$ and $f(x-t) \leq f(x)$ for all $0 \leq t < r$. Hence, for $0 < h < r$,

$$\frac{1}{h}\int_0^h \frac{f(x+t)-f(x)}{t}dt \leq 0.$$

Taking lim sup as $h \to 0^+$, $\overline{BD}_1^+ f(x) \leq 0$.
Again

$$\frac{1}{h}\int_0^h \frac{f(x-t)-f(x)}{t}dt \leq 0.$$

Therefore,

$$-\frac{1}{h}\int_0^h \frac{f(x-t)-f(x)}{t}dt \geq 0.$$

Taking lim inf as $h \to 0^+$, $\underline{BD}_1^- f(x) \geq 0$.
So,

$$\overline{BD}_1^+ f(x) \leq 0 \leq \underline{BD}_1^- f(x).$$

Since f is Borel smooth, by Lemma 2.0.2

$$\overline{BD}_1^- f(x) = \overline{BD}_1^+ f(x) \leq 0 \leq \underline{BD}_1^- f(x) \leq \overline{BD}_1^- f(x).$$

Therefore, $BD_1 f(x)$ exists and equals zero. □

Next we will use the relation of derivatives to prove the monotonicity property of symmetric Borel derivative. A general relation between higher-order derivatives is proved on page 118 of [6]. However, we give here a proof for the sake of completeness.

Theorem 2.1. *Let f be special Denjoy integrable in some neighbourhood of x. Then*

$$\underline{SD}^1 f(x) \leq \underline{SBD}^1 f(x) \leq \overline{SBD}^1 f(x) \leq \overline{SD}^1 f(x) \tag{5}$$

and

$$\underline{SD}^2 f(x) \leq \underline{SBD}^2 f(x) \leq \overline{SBD}^2 f(x) \leq \overline{SD}^2 f(x) \tag{6}$$

Proof. We only prove the last part of (5) and the rest can be treated in a similar manner. We may suppose that $\overline{SD}^1 f(x) < \infty$ and let $\overline{SD}^1 f(x) < M < \infty$. Then there is a $\delta > 0$ such that

$$\frac{f(x+t)-f(x-t)}{2t} < M$$

for $0 < |t| < \delta$. Hence for $0 < \epsilon < h < \delta$,

$$\begin{aligned}\frac{1}{h}\int_\epsilon^h \frac{f(x+t)-f(x-t)}{2t}dt &< \frac{1}{h}\int_\epsilon^h M dt \\ &= M\frac{h-\epsilon}{h}.\end{aligned}$$

Letting $\epsilon \to 0$ and then $h \to 0$ we have $\overline{SBD}^1 f(x) \leq M$. Since M is arbitrary, this completes the proof that $\overline{SBD}^1 f(x) \leq \overline{SD}^1 f(x)$. □

Theorem 2.2. *Let $f : (a,b) \to \mathbb{R}$ be such that* *(i) f is continuous on (a,b);*

(ii) $\overline{SBD}^2 f(x) \geq 0$ for all $x \in (a,b) \setminus N$, where N is a countable subset of (a,b);

(iii) f is Borel smooth on N.

Then f is convex on (a,b).

Proof. We first prove the result under the stronger assumption

$$\overline{SBD}^2 f(x) > 0 \quad \text{for all } x \in (a,b) \setminus N.$$

Suppose that f is not convex on (a,b). Then there are numbers m_0 and c such that $g_{m_0}(x) = f(x)+m_0x+c$ has a local maximum at some point $z \in (a,b)$. For sufficiently small $\epsilon > 0$, every $m \in (m_0-\epsilon, m_0+\epsilon)$ gives a function $g_m(x) = f(x) + mx + c$ having a local maximum at some point $z_m \in (a,b)$. Hence, for small $t > 0$,

$$g_m(z_m + t) + g_m(z_m - t) - 2g_m(z_m) \leq 0.$$

It follows that

$$\overline{SBD}^2 g_m(z_m) \leq 0,$$

and therefore $\overline{SBD}^2 f(z_m) \leq 0$. By the strict positivity assumption, this implies $z_m \in N$. Since f is Borel smooth at z_m, so is g_m. By Lemma 2.0.3, $BD_1 g_m(z_m) = 0$, and hence

$$BD_1 f(z_m) = -m.$$

If $m_1 \neq m_2$, then $z_{m_1} \neq z_{m_2}$, because otherwise the same value of $BD_1 f$ would be both $-m_1$ and $-m_2$. Thus the interval $(m_0-\epsilon, m_0+\epsilon)$ gives uncountably many points of the countable set N, a contradiction. Hence the theorem is true in the strict case. For the general case, put $f_\eta(x) = f(x) + \eta x^2$, where $\eta > 0$. Then f_η is continuous, Borel smooth on N, and

$$\overline{SBD}^2 f_\eta(x) = \overline{SBD}^2 f(x) + 2\eta > 0$$

for $x \in (a,b) \setminus N$. By the strict case, f_η is convex on (a,b). Letting $\eta \downarrow 0$, we conclude that f is convex on (a,b). □

Corollary 1. *Let $f : (a,b) \to \mathbb{R}$ be such that*

(i) f is continuous on (a,b);

(ii) $SBD^2 f(x)$ exists and $SBD^2 f(x) = 0$ for all $x \in (a,b) \setminus N$, where N is a countable subset of (a,b);

(iii) f is Borel smooth on N.

Then f is linear on (a,b).

Proof. By Theorem 2.2, f is convex. Since $SBD^2(-f)(x) = 0$ on $(a,b) \setminus N$ and $-f$ is also Borel smooth on N, Theorem 2.2 also gives that $-f$ is convex. Hence f is both convex and concave, and therefore f is linear. □

Theorem 2.3. *Let $F : (a,b) \to \mathbb{R}$ be such that*

(i) F is continuous on (a,b);

(ii) $\overline{SD}^2 F(x) \geq 0$ for all $x \in (a,b) \setminus N$, where N is a countable subset of (a,b);

(iii) F is smooth on N.

Then F is convex on (a,b).

Proof. This is the standard convexity criterion for upper second symmetric derivatives; see Lemma 3.20, p. 328, of Zygmund [14]. The smoothness assumption on N gives

$$\frac{F(x+h)+F(x-h)-2F(x)}{h} \to 0$$

at every $x \in N$, which is the exceptional-set condition required in that lemma. Hence F is convex. □

Theorem 2.4. *Let $f : (a,b) \to \mathbb{R}$ be continuous and suppose that $\underline{SBD}^1 f(x) \geq 0$ for all $x \in (a,b) \setminus N$, where N is a countable subset of (a,b). Then f is non-decreasing in (a,b).*

Proof. Let $c \in (a,b)$ and let

$$F(x) = \int_c^x f(t)dt.$$

Therefore, $F'(x)$ exists on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$. Let $h > 0$ be such that $a < x-h < x+h < b$. We have by integration by parts,

$$\begin{aligned}
&\frac{1}{h}\int_\epsilon^h \frac{F(x+t)+F(x-t)-2F(x)}{t^2}dt \\
&= \frac{1}{h}\left[\frac{F(x+t)+F(x-t)-2F(x)}{-t}\bigg|_\epsilon^h + \int_\epsilon^h \frac{f(x+t)-f(x-t)}{t}dt\right] \\
&= \frac{1}{h}\left[\frac{F(x+h)+F(x-h)-2F(x)}{-h} + \frac{F(x+\epsilon)+F(x-\epsilon)-2F(x)}{\epsilon}\right] \\
&\qquad + \frac{2}{h}\int_\epsilon^h \frac{f(x+t)-f(x-t)}{2t}dt
\end{aligned}$$

Letting $\epsilon \to 0^+$, we get

$$\frac{1}{h}\int_0^h \frac{F(x+t)+F(x-t)-2F(x)}{t^2}dt$$

$$= \frac{2}{h}\int_0^h \frac{f(x+t)-f(x-t)}{2t}dt - \frac{F(x+h)+F(x-h)-2F(x)}{h^2}$$

Hence

$$\limsup_{h\to 0} \frac{1}{h}\int_0^h \frac{F(x+t)+F(x-t)-2F(x)}{t^2}dt$$

$$\geq \limsup_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{-h^2} + \liminf_{h\to 0} \frac{2}{h}\int_0^h \frac{f(x+t)-f(x-t)}{2t}dt$$

Therefore

$$\begin{aligned}
&\overline{SBD}^2 F(x) \geq -\underline{SD}^2 F(x) + 2\underline{SBD}^1 f(x) \\
\implies &\underline{SD}^2 F(x) + \overline{SBD}^2 F(x) \geq 2\underline{SBD}^1 f(x)
\end{aligned}$$

Since $\underline{SD}^2 F(x) \leq \overline{SD}^2 F(x)$ and by Theorem 2.1, $\overline{SBD}^2 F(x) \leq \overline{SD}^2 F(x)$, the above inequality gives $2\overline{SD}^2 F(x) \geq 2\underline{SBD}^1 f(x)$ and hence, by the given condition, $\overline{SD}^2 F(x) \geq 0$ for $x \in (a,b) \setminus N$. Since f is continuous, F is smooth and continuous in (a,b). Hence, by Theorem 2.3, F is convex. Thus $F' = f$ is non-decreasing in (a,b). □

Remark. Theorem 2.4 shows that the non-negativity of the lower first symmetric Borel derivative gives the usual monotonicity conclusion, up to a countable exceptional set.

Corollary 2. *Let $f : (a,b) \to \mathbb{R}$ be continuous and suppose that $SBD^1 f(x)$ exists and $SBD^1 f(x) = 0$ for all $x \in (a,b) \setminus N$, where N is a countable subset of (a,b). Then f is constant on (a,b).*

Proof. Since $SBD^1 f(x) = 0$ on $(a,b) \setminus N$, Theorem 2.4 applied to f gives that f is non-decreasing. Applying the same theorem to $-f$ gives that $-f$ is non-decreasing. Hence f is both non-decreasing and non-increasing, and therefore f is constant. □

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose that $SBD^1 f(x)$ exists in (a,b). Then there are points ξ_1 and ξ_2 in (a,b) such that*

$$SBD^1 f(\xi_1) \le \frac{f(b) - f(a)}{b - a} \le SBD^1 f(\xi_2).$$

Proof. Let $k = \frac{f(b)-f(a)}{b-a}$. Suppose, on the contrary, that there is no $\xi \in (a,b)$ such that $SBD^1 f(\xi) \le k$. Let $g(x) = f(x) - k(x-a)$. Then $SBD^1 g(x)$ exists and $SBD^1 g(x) > 0$ on (a,b). By Theorem 2.4, g is non-decreasing. Moreover, g cannot be constant on any subinterval; otherwise $SBD^1 g$ would be zero at every interior point of that subinterval. Hence g is strictly increasing, so $g(b) > g(a)$. But by the definition of k, $g(b) = g(a)$. This contradiction proves the existence of $\xi_1 \in (a,b)$ such that

$$SBD^1 f(\xi_1) \le \frac{f(b) - f(a)}{b - a}.$$

The second inequality follows similarly by applying the same argument to $-f$. □

Theorem 2.5. *Let f be continuous in $[a,b]$.*

(i) If $SBD^1 f$ exists and is continuous on (a,b), then f' exists on (a,b) with the same value.

(ii) If $SBD^2 f$ exists and is continuous on (a,b), then f'' exists on (a,b) with the same value.

Proof. Let $F(x) = \displaystyle\int_a^x SBD^1 f(t)dt$. Then $F'(x) = SBD^1 f(x)$ for all $x \in (a,b)$. Let $\psi = f - F$. Then $SBD^1 \psi(x) = SBD^1 f(x) - F'(x) = 0$ for all $x \in (a,b)$. Thus, by Corollary 2, ψ is constant. Since $F'(x)$ exists in (a,b), $f'(x)$ exists on (a,b) and $f'(x) = F'(x) = SBD^1 f(x)$ for all $x \in (a,b)$.

To prove the second part, take $F(x) = \displaystyle\int_a^x (x-t) SBD^2 f(t)dt$. Since $SBD^2 f(x)$ is continuous in (a,b), $F''(x)$ exists and $F''(x) = SBD^2 f(x)$. Proceeding similarly to the first case, we get $SBD^2 \psi(x) = SBD^2 f(x) - F''(x) = 0$. Hence by Corollary 1, ψ is linear in (a,b) and hence f'' exists in (a,b) and $F''(x) = f''(x)$ for all $x \in (a,b)$. Thus $SBD^2 f(x) = f''(x)$ in (a,b). □

Theorem 2.6. *Let f be special Denjoy integrable in some neighbourhood of x. If $SBD^2 f(x)$ exists, then $SLD^2 f(x)$ exists with the same value.*

Proof. Suppose $SBD^2 f(x)$ exists and is finite, so for any $\epsilon > 0$ there exists $\delta > 0$ such that, for $0 < h < \delta$,

$$\left| \frac{1}{h} \int_0^h \frac{f(x+t) + f(x-t) - 2f(x) - t^2 SBD^2 f(x)}{t^2} dt \right| < \epsilon. \tag{7}$$

Let $0 < \delta_1 < \delta$. Then integrating by parts and using (7) we get,

$$\begin{aligned}
&\left| \int_0^{\delta_1} e^{-st} \left[\frac{f(x+t) + f(x-t) - 2f(x) - t^2 SBD^2 f(x)}{t^2} \right] t^2 dt \right| \\
&= \Bigg| e^{-s\delta_1} \delta_1^2 \int_0^{\delta_1} \frac{f(x+t) + f(x-t) - 2f(x) - t^2 SBD^2 f(x)}{t^2} dt \\
&- \int_0^{\delta_1} \left[(2te^{-st} - st^2 e^{-st}) \int_0^t \frac{f(x+\xi) + f(x-\xi) - 2f(x) - \xi^2 SBD^2 f(x)}{\xi^2} d\xi \right] dt \Bigg| \\
&\le \epsilon e^{-s\delta_1} \delta_1^3 + 2\epsilon \int_0^{\delta_1} t^2 e^{-st} dt + s\epsilon \int_0^{\delta_1} t^3 e^{-st} dt
\end{aligned}$$

Multiplying by $\dfrac{s^3}{2}$ and using Lemma 2.0.1 we have,

$$\begin{aligned}&\left|s^3\int_0^{\delta_1} e^{-st}\left[\frac{f(x+t)+f(x-t)}{2}-f(x)-\frac{t^2}{2}SBD^2f(x)\right]dt\right|\\ &\leq \frac{e^{-s\delta_1}\delta_1^3 s^3}{2}\epsilon+\epsilon s^3\int_0^{\delta_1} t^2e^{-st}dt+\frac{s^4\epsilon}{2}\int_0^{\delta_1} t^3e^{-st}dt\\ &=\frac{e^{-s\delta_1}\delta_1^3 s^3}{2}\epsilon+2\epsilon+3\epsilon+o(1)\ \text{ as } s\to\infty.\end{aligned}$$

Also, for any fixed σ with $\delta_1<\sigma<\delta$, put

$$H(u)=\int_{\delta_1}^{u}\left[\frac{f(x+t)+f(x-t)}{2}-f(x)-\frac{t^2}{2}SBD^2f(x)\right]dt.$$

Since H is bounded on $[\delta_1,\sigma]$, integration by parts gives

$$s^3\int_{\delta_1}^{\sigma} e^{-st}\left[\frac{f(x+t)+f(x-t)}{2}-f(x)-\frac{t^2}{2}SBD^2f(x)\right]dt=o(1)$$

as $s\to\infty$. Therefore, since ϵ is arbitrary,

$$\lim_{s\to\infty} s^3\int_0^{\sigma} e^{-st}\left[\frac{f(x+t)+f(x-t)}{2}-f(x)-\frac{t^2}{2}SBD^2f(x)\right]dt=0.$$

Finally, by Lemma 2.0.1,

$$\frac{s^3}{2}\int_0^{\sigma} e^{-st}t^2dt\to 1.$$

Hence $SLD^2f(x)$ exists and $SLD^2f(x)=SBD^2f(x)$. □

Lemma 2.6.1. *If $\phi(t)\to 0$ as $t\to 0$, then*

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \phi(t)dt=0.$$

Proof. Since $\phi(t)\to 0$ as $t\to 0$, for every $\epsilon>0$ there is $\delta>0$ such that $|\phi(t)|<\epsilon$ whenever $|t|<\delta$. If $0<|h|<\delta$, then

$$\left|\int_0^h \phi(t)dt\right|\leq \epsilon|h|.$$

Therefore

$$\left|\frac{1}{h}\int_0^h \phi(t)dt\right|<\epsilon,$$

and the result follows. □

Theorem 2.7. *If f is symmetrically Borel continuous of odd order on $[a,b]$, then $F(x)=\int_a^x f(t)dt$ is Borel smooth on $[a,b]$.*

Proof. We have, for any $x\in(a,b)$,

$$F(x+t)-F(x)=\int_x^{x+t} f(\xi)d\xi=\int_0^t f(z+x)dz$$

and

$$F(x)-F(x-t)=\int_{x-t}^{x} f(\xi)d\xi=\int_0^t f(x-z)dz.$$

Therefore

$$\begin{aligned}&\frac{1}{h}\int_0^h \frac{F(x+t)+F(x-t)-2F(x)}{t}dt\\ =&\frac{1}{h}\int_0^h \frac{1}{t}\int_0^t \big[f(x+z)-f(x-z)\big]dz\,dt.\end{aligned} \tag{8}$$

Since f is symmetrically Borel continuous of odd order at x,

$$\frac{1}{t}\int_0^t \Big[f(x+z)-f(x-z)\Big]dz \to 0 \quad \text{as} \quad t\to 0.$$

Now using Lemma 2.6.1 and equation (8)

$$\frac{1}{h}\int_0^h \frac{F(x+t)+F(x-t)-2F(x)}{t}dt \to 0 \quad \text{as} \quad h\to 0.$$

Thus F is Borel smooth at x. □

Lemma 2.7.1. *If f is Borel smooth at x, then f is Laplace smooth at x.*

Proof. Put

$$A(t)=f(x+t)+f(x-t)-2f(x).$$

Borel smoothness means

$$\frac{1}{h}\int_0^h \frac{A(t)}{t}dt \to 0 \quad (h\to 0).$$

Repeating the Abelian integration-by-parts argument used in Theorem 2.6, with $A(t)/t$ in place of the second-order quotient, gives

$$s^2\int_0^\sigma e^{-st}A(t)dt \to 0 \quad (s\to\infty),$$

for sufficiently small $\sigma>0$; the part of the integral away from zero is $o(1)$ by the same tail estimate. Hence f is Laplace smooth at x. □

Theorem 2.8. *Let f be Borel smooth in (a,b). Then f is of class $\mathcal{B}_1^*$ in (a,b).*

Proof. By Lemma 2.7.1, f is Laplace smooth in (a,b). The corresponding result for Laplace smooth functions is Theorem 4.4.2 of [9]. Therefore f is of class $\mathcal{B}_1^*$ in (a,b). □

Corollary 4. *If f is Borel smooth in (a,b), then f is continuous on a dense open set in (a,b).*

Proof. By Lemma 2.7.1, f is Laplace smooth in (a,b). The conclusion follows from Theorem 4.4.3 of [9]. □

3 Conclusion

We have obtained monotonicity and convexity criteria for symmetric Borel derivatives and related these derivatives to symmetric Laplace derivatives. The results show that symmetric Borel derivatives preserve several classical consequences of ordinary derivatives, such as monotonicity and convexity, under suitable hypotheses. The discussion of Borel smoothness also connects this notion with known properties of Laplace smooth functions. Further study may consider higher-order analogues and additional examples illustrating the sharpness of the assumptions.

References

[1] Bullen, P. S., and Mukhopadhyay, S. N., 1974, Integration by parts formulae for some trigonometric integrals. Proc. London Math. Soc., 3(1), 159-173.

[2] Garai, A., and Ray, S., 2011, On the symmetric Laplace derivative. Acta Math. Hungar., 133(1-2), 166-184.

[3] Khintchine, A., 1927, Recherches sur la structure des fonctions mesurables. Fund. Math., 9, 212-279.

[4] Marcinkiewicz, J., and Zygmund, A., 1936, On the differentiability of functions and summability of trigonometric series. Fund. Math., 26, 1-43.

[5] Mukhopadhyay, S. N., and Ray, S., 2010, On Laplace derivative. Anal. Math., 36(2), 131-153.

[6] Mukhopadhyay, S. N., 2012, Higher Order Derivatives. Chapman and Hall/CRC Press, Boca Raton.

[7] Mukhopadhyay, S. N., and Ray, S., 2016, Relation between L_p-derivatives and Peano, approximate Peano and Borel derivatives of higher-order. Real Anal. Exchange, 41(1), 1-22.

[8] Ray, S., and Ghosh, S., 2014, On Borel derivative. Bull. Calcutta Math. Soc., 106(4), 273-280.

[9] Ray, S., and Ghosh, S., 2016, On Laplace smooth functions. Bull. Allahabad Math. Soc., 31(2), 155-165.

[10] Ray, S., and Ghosh, S., 2016, On the higher-order Borel derivative. Proc. IMBIC, 5, 94-99.

[11] Sargent, W. L. C., 1935, The Borel derivatives of a function. Proc. London Math. Soc., s2-38(1), 180-196.

[12] Svetic, R. E., 2001, The Laplace derivative. Comment. Math. Univ. Carolin., 42(2), 331-343.

[13] Thomson, B. S., 1994, Symmetric Properties of Real Functions. Marcel Dekker, Inc., New York.

[14] Zygmund, A., 2002, Trigonometric Series. 3rd ed., Cambridge University Press, Cambridge.