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Some Fixed Point Results in Fuzzy Strong b-Metric Space

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Abstract: *The theory of fixed points in pure mathematics is the most dynamics and active area of research activities. Also, the fuzzy metric space has been one of the important generalizations of usual metric space. As an extension of fuzzy metric space, the notion of fuzzy strong b-metric space has been introduced by Oner and Sostak in 2020. The purpose of this paper is to study the notions of fuzzy strong b-metric spaces and to establish some fixed point results in complete fuzzy strong b-metric space with examples. Our results extend and improve some well-known results in literature.*

Keywords: Fuzzy strong b-metric space, Uniqueness, Fixed point, Continuous t-norm

1 Introduction

The most active and dynamic area of research in both pure and applied mathematics is the theory of fixed points. The concept of fixed points have already shown to be a useful tool and effective instrument for research in nonlinear analysis. In 1965, Zadeh [13] established the foundation of fuzzy mathematics by introducing the concept of fuzzy sets. In 1975, Kramosil and Michalek [8] introduced a fuzzy metric space as an extension of probabilistic metric space. Later, George and Veeramani [6] in 1994 presented the idea of a fuzzy metric space in modified way. This work lays the foundation for the development of fixed-point theory in fuzzy metric space. Later, Grabiec [5] clarified the fuzzy completeness in metric space and the Banach contraction theorem was extended in fuzzy metric spaces. Also, Fang [4] proved some fixed point theorems for contractive-type mappings in G-complete fuzzy metric space, using the completeness of Grabiec's work.

On the other hand, in 1989, Bakhtin [1] introduced a space in 1989 where a weaker condition is used to generalize the Banach contraction principle [2] in place of the triangle inequality. Similarly, in 1993, Czerwic [3] introduced the notion of b-metric spaces. The idea of fuzzy b-metric space was first proposed by Nadaban [9] in 2016, and also the fuzzy strong b-metric space was defined and described by Oner and Sostak [10] in 2020. Similarly, in 2022, Kanwal et al.[7] explored some new ideas of fixed point results in fuzzy strong b-metric space. It is interesting to see that the class of fuzzy strong b-metric spaces thus distinguishes fuzzy metric spaces from fuzzy b-metric spaces.

The main objective of this paper is to study the notions of fuzzy strong b-metric spaces and to establish some fixed point results in complete fuzzy strong b-metric space with examples in order to justify the main results.

2 Preliminaries

Definition 2.1. *[11] A mapping $*$ defined from $[0,1]\times[0,1]$ to $[0,1]$ is called a continuous triangular norm (t-norm) if it satisfies the following axioms:*

(i) Symmetry: $p*q=q*p$, for $p,q\in[0,1]$.

(ii) Monotonicity: $p*r\leq q*s$ whenever $p\leq q$ and $r\leq s$.

(iii) Associativity: $(p * q) * r) = (p * (q * r))$*, where* $p, q, r \in [0, 1]$.

(iv) Boundary condition: $1 * p = p$*, for all* $p \in [0, 1]$.

Example 2.1. [11] The commonly used triangular norms are given in the following forms
(a) The minimum triangular norm: $p * q = \min\{p, q\}$.
(b) The product triangular norm: $p * q = p.q$.
(c) The Lukasiewicz triangular norm: $p * q = \max\{p + q - 1, 0\}$.

Definition 2.2. *[6] A triplet* $(U, F, *)$ *is known as a fuzzy metric space if* U *is any set,* $*$ *taken as continuous* t*-norm and* F *is a fuzzy set defined on* $U \times U \times (0, \infty) \to [0, 1]$ *if it fulfills the axioms given as below for all* $u, v, w \in U$ *and* $t, s > 0$
(FM-1) $F(u, v, t) > 0$;
(FM-2)$F(u, v, t) = 1$ *for all* $t > 0 \iff u = v$;
(FM-3)$F(u, v, t) = F(v, u, t)$;
(FM-4)$F(u, v, t) * F(v, w, s) \leq F(u, v, t + s)$ *for all* $t, s > 0$;
(FM-5)$F(u, v, .) : (0, \infty) \to [0, 1]$ *is continuous.*
The degree of nearness between u *and* v *with respect to* $t > 0$ *is denoted by* $F(u, v, t)$.

Example 2.2. [6] Let (F, d) be a metric space and the continuous t-norm is denoted as $p * q = p.q$ for all $p, q \in [0, 1]$ and F be a fuzzy set on $U \times U \times (0, \infty)$ defined as below
$F(u, v, t) = \dfrac{kt^n}{kt^n + md(u, v)}$, where $k, m, n \in \mathbb{R}^+$ and $u, v \in F$.
Then $(U, F, *)$ is a fuzzy metric space.

In the above example, if we take $k = m = n = 1$, we get $F(u, v, t) = \dfrac{t}{t + d(u, v)}$.

So that $(U, F, *)$ is a fuzzy metric space and it is said to be the standard fuzzy metric space induced by the metric d. It is important to note that the above example also holds if we take the t-norm defined as $p * q = min\{p, q\}$, so that F is a fuzzy metric with respect to this continuous t-norm.

In 2016, Nadaban [9] explored the idea of fuzzy b-metric space to generalize the notion of the fuzzy metric spaces introduced by Kramosil and Michalek. Also, Sedghi and Shobe [12] extended this work by introducing fuzzy b-metric spaces, which are more general than fuzzy metric spaces, using a weaker form of the triangle inequality.

Definition 2.3. *[9] Let* U *be a nonempty set, let* $b \geq 1$ *be a given real number and* $*$ *is a continuous* t*-norm, and a fuzzy set* F *on* $U \times U \times \mathbb{R}^+$*. If the following axioms are satisfied then the order tuple* $(U, F, *)$ *is known as the fuzzy* b*-metric space*
$\forall\ u, v, w \in U$ *and* $t, s > 0$
(Fb-i) $F(u, v, t) > 0$;
(Fb-ii) $F(u, v, t) = 1$ *for all* $t > 0$ *if and only if* $u = v$;
(Fb-iii) $F(u, v, t) = F(v, u, t)$;
(Fb-iv) $F(u, v, t) * F(v, w, s) \leq F(u, w, b(t + s))$ *for all* $t, s > 0$;
(Fb-v) $F(u, v, .) : (0, \infty) \to [0, 1]$ *is left continuous;*
(Fb-vi)$\lim_{t\to\infty} F(u, v, t) = 1$.

In fuzzy b-metric space if we put $b = 1$ then it becomes a fuzzy metric space.

We give the following new example of fuzzy b-metric space.

Example 2.3 Let $U = [0, 1)$. We consider a function $F : U \times U \times [0, 1) \to [0, 1]$ defined by

$$F(u, v, t) = e^{-\frac{d(u,v)}{t}} \quad \forall u, v \in U,$$

where the t-norm is taken in terms of product , i.e., $u \cdot v = uv$, and the metric d is a b-metric given as

$$d(u, v) = \begin{cases} 0 & \text{if } u = v, \\ 3|u - v| & \text{if } u, v \in [0, 1), \\ \frac{1}{3}|u - v| & \text{otherwise.} \end{cases}$$

To verify this, we only show the axiom (Fb-iv). For this let $u, v, w \in U$ and $t, s > 0$, then by definition, we get

$$\begin{aligned} F(u,w,t+s) =& e^{-\frac{d(u,w)}{t+s}} \\ &\geq e^{-\frac{b(d(u,v)+d(v,w))}{t+s}} \\ &= e^{-\frac{b\cdot d(u,v)}{t+s}} \cdot e^{-\frac{b\cdot d(v,w)}{t+s}} \\ &\geq e^{-\frac{b\cdot d(u,v)}{t}} \cdot e^{-\frac{b\cdot d(v,w)}{s}} \\ &\geq F\left(u,v,\frac{t}{b}\right) \cdot F\left(v,w,\frac{s}{b}\right). \end{aligned}$$

Thus, $(U, F, *)$ is a fuzzy b-metric space.

Definition 2.4. *[6] Assume $(U, F, *)$ be a fuzzy b-metric space. A sequence $\{u_n\}$ in U is said to be convergent in U if $\lim_{n\to\infty} F(u_n, u, t) = 1$ for each $t > 0$.*

Definition 2.5. *[6] A sequence $\{u_n\}$ in U is said to be Cauchy sequence in U if $\lim_{n\to\infty} F(u_n, u_{m+n}, t) = 1$ where $t > 0$ and $m, n > 0$. If every Cauchy sequence is convergent in a fuzzy b-metric space the it is called the complete fuzzy b-metric space.*

Definition 2.6. *[10] Let U be a non-empty set, $b \geq 1$, $*$ be a continuous t-norm, and F be a fuzzy set on $U \times U \times (0, \infty)$ such that for all $u, v, w \in U$ and $t, s > 0$, we have*
(i) $F(u, v, t) > 0$;
(ii) $F(u, v, t) = 1$ if and only if $u = v$;
(iii) $F(u, v, t) = F(v, u, t)$;
(iv) $F(u, v, t) * F(v, w, s) \leq F(u, z, t + bs)$;
(v) $F(u, v, \cdot) : (0, \infty) \to [0, 1]$ is continuous.
Then F is called a fuzzy strong b-metric on U and $(U, F, *)$ is called a fuzzy strong b-metric space.

Example 2.4. [10] Let (U, d) be a strong b-metric space.
Define

$$F(u,v,t) = \frac{t}{t + d(u,v)}$$

for $t > 0$ and $u, v \in U$. Then $(U, F, *)$ is a fuzzy strong b-metric space which is called standard fuzzy strong b-metric space induced by d. Thus $(U, F, *)$ satisfies all the properties to a fuzzy strong b-metric space.

Clearly, the class of strong b-metric space lies between the fuzzy metric space and the fuzzy b-metric space.
We have introduced the following useful lemma in fuzzy strong b-metric space:

Lemma 2.1. *Let $(U, F, *)$ be a fuzzy strong b-metric space with $b \geq 1$. Then $F(u, v, \cdot) : (0, \infty) \to [0, 1]$ is non-decreasing for all $u, v \in U$ and $k \in \left(0, \frac{1}{b}\right)$.*

Proof. Let us consider $F(u, v, t) > F(u, v, s)$, where $s, t > 0$. Then, we get

$$F(u,v,t) * U\left(v,v,\frac{s-t}{k}\right) \leq F(u,v,s) < F(u,v,t).$$

Since $F(u, v, s - t) = 1$, we have $F(u, v, t) < F(u, v, t)$, which is a contradiction. □

3 Main Results

Now, we establish some unique fixed point results in complete fuzzy strong b-metric space for single self mappings with suitable example to justify the main results with the help of following lemmas.

Lemma 3.1. *If $(U, F, *)$ is a fuzzy strong b-metric space, then for a given $b \geq 1$ and for some $k \in \left(0, \frac{1}{b}\right)$ and for $u, v \in U$ such that*

$$\frac{1}{F(u,v,t)} \leq \frac{1}{F\left(u,v,\frac{k}{t}\right)}, \quad \forall t > 0,$$

we have $u = v$.

Proof. Using the hypothesis and successive iteration, we get

$$\begin{aligned}\frac{1}{F(u,v,t)} &\leq \frac{1}{F\left(u,v,\frac{k}{t}\right)}, \quad \forall t > 0.\\ &\leq \frac{1}{F\left(u,v,\frac{k}{t^2}\right)}\\ &\cdots\\ &\leq \frac{1}{F\left(u,v,\frac{k}{t^n}\right)}\end{aligned}$$

Using the above condition, we get

$$F(u,v,t) \geq F\left(u,v,\frac{k}{t^n}\right), \quad n \in \mathbb{N},\ t > 0.$$

Taking the limit as $n \to \infty$, we get

$$F(u,v,t) \geq \lim_{n\to\infty} F\left(u,v,\frac{k}{t^n}\right) = 1, \quad t > 0,$$

so, by using the definition of fuzzy strong b-metric space, we have $u = v$. This completes the proof. ☐

Lemma 3.2. *If $(U, F, *)$ is a complete fuzzy strong b-metric space, then for a given $b \geq 1$ and*

$$F(u,v,kt) \geq F(u,v,t), \quad \textit{for all } u,v \in U,\ t > 0 \textit{ and } k \in \left(0, \frac{1}{b}\right),$$

we have $u = v$.

Proof. By using the axiom (v) of the definition of fuzzy strong b-metric space, we have

$$\lim_{t\to\infty} F(u,v,t) = 1 \implies F(u,v,kt) = 1.$$

Then from the axiom (ii) of fuzzy strong b-metric space, we obtain $u = v$.
We the following main results. ☐

Theorem 3.1. *Let $(U, F, *)$ be a complete fuzzy strong b-metric space, $b \geq 1$ a given real number and $g : U \to U$ a mapping. If there exists $k \in \left(0, \frac{1}{b}\right)$ such that*

$$F(gu, gv, t) \geq F\left(u,v,\frac{t}{k}\right), \textit{where } u,v \in U, t > 0 \tag{1}$$

, and also for $u_0 \in U$ and $v \in (0,1)$ such that

$$\lim_{n\to\infty} F\left(u_0, f(u_0), \frac{t}{v^n}\right) = 1, t > 0. \tag{2}$$

Then g has a unique fixed point in U.

Proof. Let $u_0 \in U$ and $u_{n+1} = gu_n$, $n \in \mathbb{N}$. If we take $u = u_n$ and $v = u_{n-1}$ in (1), then we have

$$F(u_n, u_{n+1}, t) \geq F\left(u_{n-1}, u_n, \frac{t}{k}\right), \quad n \in \mathbb{N},\ t > 0,$$

Then we clearly see that $\{u_n\}$ is a Cauchy sequence.
As $(U, F, *)$ is complete, there exists $u \in U$ such that

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} F(u, u_n, t) = 1, \quad t > 0. \tag{3}$$

Using conditions (1) and axiom (iv) of definition, we get

$$\begin{aligned} F(g(u), u, t) &\geq F\left(g(u), u_n, \frac{t}{2k}\right) * F\left(u_n, u, \frac{t}{2k}\right) \\ &\geq F\left(u, u_{n-1}, \frac{t}{2k}\right) * F\left(u_n, u, \frac{t}{2k}\right) \end{aligned}$$

for all $t > 0$. By (3), as $n \to \infty$, we have

$$F(g(u), u, t) \geq 1 * 1 = 1.$$

This implies that $g(u) = u$ and so u is a fixed point of g.
For uniqueness, if possible, we assume that u and v are any two fixed points for g. Then, from (1), we get

$$F(u, v, t) = F(g(u), g(v), t) \geq F\left(u, v, \frac{t}{k}\right), \quad t > 0,$$

Hence we have $u = v$. □

Theorem 3.2. *Let $(U, F, *)$ be a complete fuzzy strong b-metric space, $*$ be a continuous t-norm with $b \geq 1$ and*

$$\lim_{u\to\infty} F(u, v, t) = 1, \quad \forall u, v \in U. \tag{4}$$

If $f : U \to U$ is a mapping satisfying $F(fu, fv, kt) \geq F(u, v, t)$, for all $u, v \in U$, where $0 < k < \frac{1}{b}$. Then, the function f has a unique fixed point.

Proof. Let $u_0 \in U$. For $n \in \mathbb{N}$, we define $u_n = f^n u_0$. Then, for n , $t > 0$ we have

$$\begin{aligned} F(u_n, u_{n+1}, t) &\leq F\left(u_{n-1}, u_n, \frac{t}{k}\right) \\ &\leq F\left(u_{n-2}, u_{n-1}, \frac{t}{k^2}\right) \\ &\cdots \\ &\leq F\left(u_0, u_1, \frac{t}{k^n}\right). \end{aligned}$$

For any $m \in \mathbb{N}$, we get

$$\begin{aligned} F(u_n, u_{n+m}, t) &\leq F\left(u_n, u_{n+1}, \frac{t}{3b}\right) * F\left(u_{n+1}, u_{n+2}, \frac{t}{(3b)^2}\right) * \cdots * F\left(u_{n+m-1}, u_{n+m}, \frac{t}{(3b)^m}\right), \\ &\leq F\left(u_0, u_1, \frac{t}{3bk^n}\right) * F\left(u_0, u_1, \frac{t}{(3b)^2 k^{n+1}}\right) * \cdots * F\left(u_0, u_1, \frac{t}{(3b)^m k^{n+m-1}}\right). \\ &= F\left(u_0, u_1, \frac{t}{(3bk)^m k^{n-1}}\right) * F\left(u_0, u_1, \frac{t}{(3bk)^m k^{n-1}}\right) * \cdots * F\left(u_0, u_1, \frac{t}{(3bk)^m k^{n-1}}\right) \\ &\leq 1 * 1 * \cdots * 1 = 1. \end{aligned}$$

So, we have

$$\lim_{n\to\infty} F(u_n, u_{n+m}, t) = 1.$$

Hence $\{u_n\}$ is a Cauchy sequence. As $(U, F, *)$ is a complete fuzzy strong b-metric space, then there exists a point $u \in U$ such that

$$\lim_{n\to\infty} u_n = u.$$

Now, we prove that u is a fixed point of f. For this,

$$\begin{aligned} F(fu, u, t) &\leq F\left(fu, fu_n, \frac{t}{3b}\right) * F\left(u_{n+1}, u, \frac{t}{3b}\right) \\ &\leq F\left(u, u_n, \frac{t}{3bk}\right) * F\left(u_{n+1}, u, \frac{t}{3b}\right) \to 1 * 1 = 1, \end{aligned}$$

which shows that $fu = u$. Thus u is a fixed point of f.
For uniqueness, if possible, for some $v \in U$, we assume that $fv = v$, then

$$\begin{aligned} F(u, v, t) =& F(fu, fv, t) \leq F\left(u, v, \frac{t}{k}\right) \\ &= F\left(fu, fv, \frac{t}{k}\right) \\ &\leq F\left(u, v, \frac{t}{k^2}\right) \\ &\dots \\ &\leq F\left(u, v, \frac{t}{k^n}\right) \to 1 \text{ as } n \to \infty. \end{aligned}$$

This implies $u = v$. This proves that the fixed point is unique. □

Now we give the following example to verify the above results.

Example 3.1. Let $U = [0, 1] \subset \mathbb{R}$ and $F : U \times U \times (0, \infty) \to [0, 1]$ be a function defined as

$$F(u, v, t) = \begin{cases} \dfrac{t}{t + |u - v|} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

For a function $f : U \to U$ defined by $f(u) = \frac{u}{8}$ and $k = \frac{1}{4}$, and $*$ as the minimum t-norm, we have

$$\begin{aligned} F(f(u), f(v), kt) =& \frac{\frac{t}{4}}{\frac{t}{4} + |f(u) - f(v)|} \\ &= \frac{\frac{t}{4}}{\frac{t}{4} + |\frac{u-v}{8}|} \\ &= \frac{t}{t + |\frac{u-v}{2}|} \\ &\geq \frac{t}{t + |u - v|}, \end{aligned}$$

which gives $F(fu, fv, kt) \geq F(u, v, t)$. So, f has a unique fixed point.

Corollary 3.2.1. *Let $(U, F, *)$ be a complete fuzzy strong b-metric space, where $*$ is a continuous t-norm and for a given $b \geq 1$, such that*

$$u_1 * u_2 = \min\{u_1, u_2\}$$

and $F(u, v, t)$ is strictly increasing fuzzy strong b-metric space and

$$\lim_{t\to\infty} F(u, v, tk) = 1, \quad \forall u, v \in U.$$

Let $f : U \to U$ be a self-mapping which satisfies the given condition $\forall u, v \in U$:

$$F(fu, fv, kt) \geq (u, fu, t) * (v, fv, t),$$

where $t \geq 0$ and $0 < k < \frac{1}{b}$. Then f has a unique fixed point.

4 Conclusion

We have established two fixed point results in complete fuzzy strong b-metric space for single self mapping with a suitable example. Our results extends the results of Oner and Sostak [10], Sedghi and Shobe [12] in fuzzy strong b-metric space. Also, since the class of strong b-metric space lies between the fuzzy metric space and the fuzzy b-metric space, our results improves other similar results in these spaces for single self mappings.

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