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Few Theorems on an Extension of Bailey's Formula Involving Product of Two Generalized Hypergeometric Functions

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Abstract: *This paper presents an extensive study of the product two generalized hypergeometric functions. Particularly motivated by the papers of Kim et al. and Rakha et al., our aim of this note is to provide two interesting extensions of the well-known Bailey's formula involving product of two generalized hypergeometric functions. The special cases of our main findings are some well known results.*

Keywords: Generalized hypergeometric series, Kummer's type II transformation, Bailey's identity, Contiguous results

1 Introduction

The theory of generalized hypergeometric functions plays a fundamental role in mathematical analysis due to its extensive applications in pure and applied mathematics [1, 8, 9]. Among the many remarkable contributions to this field, Bailey's formula has been a cornerstone, providing elegant identities that connect hypergeometric series to various special functions. Extensions and generalizations of Bailey's formula offer deeper insights into the structural properties and interrelationships of hypergeometric functions, which have applications in combinatorics, number theory, and physics [1, 14, 20]. This research paper focuses on a class of results extending Bailey's formula to encompass the product of two generalized hypergeometric functions [2, 8, 10]. These extensions not only broaden the scope of Bailey's original work but also highlight novel interconnections between higher-order hypergeometric functions. By examining these theorems, we aim to develop a more unified framework for understanding the convergence, summation, and transformation properties of hypergeometric series in the context of their products [8, 9, 10, 15].

The generalized hypergeometric function ${}_pF_q$ characterized by p numerator parameters and q denominator parameters, is defined as follows [14]

$$ {}_pF_q\left[\begin{array}{cccc} h_1, & \ldots, & h_p & \\ & & & ; \quad z \\ k_1, & \ldots, & k_q & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(h_1)_n \ldots (h_p)_n}{(k_1)_n \ldots (k_q)_n} \frac{z^n}{n!} \tag{1} $$

Here, $(u)_n$ represents the Pochhammer symbol, which is commonly expressed in terms of the Gamma function, defined as follows;

$$ \begin{aligned} (u)_n &= \frac{\Gamma(u+n)}{\Gamma(u)} \\ &= \begin{cases} u.(u+1)...(u+n-1) & (n \in \mathbb{N}, u \in \mathbb{C}) \\ 1 & (n=0; u \in \mathbb{C}\backslash\{0\}) \end{cases} \end{aligned} \tag{2} $$

Through the theory of differential equations, Kummer [4, 7, 14] obtained the following results:

$$ e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{array}{c} \alpha \\ \\ 2\alpha \end{array} ; \quad x\right] = {}_0F_1\left[\begin{array}{c} - \\ \\ \alpha+\frac{1}{2} \end{array} ; \quad \frac{x^2}{16}\right] \tag{3} $$

The following two results contiguous to that of Kummer's second theorem, established by Rathie and Nagar [20] in 1995

$$e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha+1\end{matrix};x\right]={}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-\frac{x}{2(2\alpha+1)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right] \tag{4}$$

$$e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha-1\end{matrix};x\right]={}_0F_1\left[\begin{matrix}- \\ \alpha-\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]+\frac{x}{2(2\alpha-1)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right] \tag{5}$$

In 2023, Kim [2] developed the following theorems

$$\begin{aligned}e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha+2\end{matrix};x\right]\\ &={}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-\frac{x}{2(\alpha+1)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\\ &+\frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{5}{2}\end{matrix};\frac{x^2}{16}\right]\end{aligned} \tag{6}$$

and

$$\begin{aligned}e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha-2\end{matrix};x\right]\\ &={}_0F_1\left[\begin{matrix}- \\ \alpha-\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]-\frac{x}{2(\alpha-1)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha-\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\\ &+\frac{(\alpha-2)x^2}{4(\alpha-1)(2\alpha-1)(2\alpha-3)}\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\end{aligned} \tag{7}$$

The identity (3) was also derived by Rathie and Choi [19] by using the Guass's summation theorem. Through the work of Rathie and Pogany [20] and Bailey [1] by employing Guass's second summation theorem and by generalized Kummer's second theorem (3) that is ,

$$e^{-\frac{x}{2}}{}_2F_2\left[\begin{matrix}\alpha, & d+1 \\ 2\alpha+1, & d\end{matrix};x\right]={}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]+\frac{(2\alpha-d)}{2d(2\alpha+1)}x\times{}_0F_1\left[\begin{matrix}- \\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right] \tag{8}$$

For $d=2\alpha$, we, at once get (3). Several studies are done in the product of generalized hypergeometric functions [13]. Bailey [1] has derived the product of two ${}_0F_1$ functions as the identity given below;

$${}_0F_1\left[\begin{matrix}- \\ \alpha\end{matrix};x\right]\times{}_0F_1\left[\begin{matrix}- \\ \beta\end{matrix};x\right]={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) & \\ \alpha, & \beta, & \alpha+\beta-1\end{matrix};4x\right] \tag{9}$$

For the product of the generalized hypergeometric series, Preece [12] established the following identity through the theory of differential equations, which is called the Preece's identity.

$$
{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;x\right]\times{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;-x\right]={}_1F_2\left[\begin{matrix}\alpha,\\ \\ \alpha+\frac{1}{2},\quad 2\alpha\end{matrix}\;;\frac{x^2}{4}\right] \tag{10}
$$

Rathie [17] provided a concise proof (11) and derived two contiguous relations. Bailey [1] extended Preece's identity (11) by utilizing the Watson's summation theorem of ${}_3F_2$, leading to the formulation of Bailey's identity.

$$
{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;x\right]\times{}_1F_1\left[\begin{matrix}\beta\\ \\ 2\beta\end{matrix}\;;\;-x\right]={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\quad\frac{1}{2}(\alpha+\beta+1)\\ \\ \alpha,\quad\beta,\quad\alpha+\beta\end{matrix}\;;\frac{x^2}{4}\right] \tag{11}
$$

Rathie and Choi [18] gave a very short proof of (10) . The result (11) can be written as

$$
\begin{aligned}
e^{-x}{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;x\right]\times{}_1F_1\left[\begin{matrix}\beta\\ \\ 2\beta\end{matrix}\;;\;x\right]\\
={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\quad\frac{1}{2}(\alpha+\beta)\\ \\ \alpha+\frac{1}{2},\quad\beta+\frac{1}{2},\quad\alpha+\beta\end{matrix}\;;\frac{x^2}{4}\right]
\end{aligned} \tag{12}
$$

Very recently Kim et al. [4] generalized Bailey's result (13) in the following form:

$$
\begin{aligned}
e^{-x}{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;x\right]\times{}_2F_2\left[\begin{matrix}\beta,\quad d+1\\ \\ 2\beta+1,\quad d\end{matrix}\;;x\right]\\
={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\quad\frac{1}{2}(\alpha+\beta)\\ \\ \alpha+\frac{1}{2},\quad\beta+\frac{1}{2},\quad\alpha+\beta\end{matrix}\;;\frac{x^2}{4}\right]\\
+\frac{x(2\beta-d)}{2d(2\beta+1)}{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2),\quad\frac{1}{2}(\alpha+\beta+1)\\ \\ \alpha+\frac{1}{2},\quad\beta+\frac{3}{2},\quad\alpha+\beta+1\end{matrix}\;;\frac{x^2}{4}\right]
\end{aligned} \tag{13}
$$

Clearly for $d=2\beta$ we get (12) Also in 2022, Poudel et al. [11] evaluated the product of

$$
e^{-x}{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\end{matrix}\;;\;x\right]\times{}_2F_2\left[\begin{matrix}\beta,\quad d+n\\ \\ 2\beta+n,\quad d\end{matrix}\;;x\right] for\quad n=2
$$

In this paper, we shall establish the results for the product

$$
e^{-x}{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha\pm n\end{matrix}\;;\;x\right]\times{}_2F_2\left[\begin{matrix}\beta,\quad d+1\\ \\ 2\beta+1,\quad d\end{matrix}\;;x\right] for\quad n=1=2
$$

2 Main Results

In this section, we will prove the claims made in the theorems below.

Theorem 2.1. *The following identity holds true*

$$\begin{aligned}
{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha+1\end{matrix};\ x\right]\times{}_2F_2\left[\begin{matrix}\beta, & d+1\\ & \\ 2\beta+1, & d\end{matrix};x\right]
&= e^x{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{matrix};\frac{x^2}{4}\right]\\
&- e^x c_1 x\quad {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1\end{matrix};\frac{x^2}{4}\right]\\
&+ e^x c_2 x\quad {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix};\frac{x^2}{4}\right]\\
&- e^x c_1 c_2 x^2\quad {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+3) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix};\frac{x^2}{4}\right]
\end{aligned}\tag{14}$$

where

$$c_1=\frac{1}{2(2\alpha+1)},\quad c_2=\frac{(2\beta-d)}{2d(2\beta+1)}$$

Proof. To prove the theorem (i.e. result 14), let us consider the sum

$$S=e^{-x}\left\{{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha+1\end{matrix};\ x\right]\times{}_2F_2\left[\begin{matrix}\beta, & d+1\\ & \\ 2\beta+1, & d\end{matrix};x\right]\right\}$$

$$=e^{-\frac{x}{2}}{}_1F_1\left[\begin{matrix}\alpha\\ \\ 2\alpha+1\end{matrix};\ x\right]\times e^{-\frac{x}{2}}{}_2F_2\left[\begin{matrix}\beta, & d+1\\ & \\ 2\beta+1, & d\end{matrix};x\right]$$

On using (4) and (8), we get

$$\begin{aligned}
S=&\left\{{}_0F_1\left[\begin{matrix}-\\ \\ \alpha+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-c_1x\times{}_0F_1\left[\begin{matrix}-\\ \\ \alpha+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\right\}\\
&\times\left\{{}_0F_1\left[\begin{matrix}-\\ \\ \beta+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]-c_2x\times{}_0F_1\left[\begin{matrix}-\\ \\ \beta+\frac{3}{2}\end{matrix};\frac{x^2}{16}\right]\right\}
\end{aligned}$$

$$= \left\{ {}_0F_1\left[\begin{matrix} - \\ \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] \times {}_0F_1\left[\begin{matrix} - \\ \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$- c_1 x \left\{ {}_0F_1\left[\begin{matrix} - \\ \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] \times {}_0F_1\left[\begin{matrix} - \\ \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$+ c_2 x \left\{ {}_0F_1\left[\begin{matrix} - \\ \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] \times {}_0F_1\left[\begin{matrix} - \\ \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$- c_1 c_2 x^2 \left\{ {}_0F_1\left[\begin{matrix} - \\ \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] \times {}_0F_1\left[\begin{matrix} - \\ \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$= {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right]$$

$$- c_1 x \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+ c_2 x \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$- c_1 c_2 x^2 \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+3) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]$$

By shifting the term e^{-x} of the left hand side to right hand side we get,

$${}_1F_1\left[\begin{matrix} \alpha \\ \\ 2\alpha+1 \end{matrix} ; \quad x\right] \times {}_2F_2\left[\begin{matrix} \beta, & d+1 \\ & \\ 2\beta+1, & d \end{matrix} ; x\right]$$

$$= e^x {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right]$$

$$- e^x c_1 x \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+ e^x c_2 x \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$- e^x c_1 c_2 x^2 \quad {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+3) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]$$

where

$$c_1 = \frac{1}{2(2\alpha+1)}, \quad c_2 = \frac{(2\beta-d)}{2d(2\beta+1)}$$

This is the required right hand side. This proves the Theorem 2.1. □

We prove the upcoming theorems in the similar way.

Theorem 2.2. *The following relation holds true*

$$
\begin{aligned}
{}_1F_1\left[\begin{array}{c}\alpha\\ \\ 2\alpha-1\end{array};\ x\right]\times{}_2F_2\left[\begin{array}{cc}\beta, & d+1\\ & \\ 2\beta+1, & d\end{array};x\right]
&= e^x{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) & \\ & & \\ \alpha-\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta-1\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_1x\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_2x\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha-\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_1c_2x^2\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{array};\frac{x^2}{4}\right]
\end{aligned}
$$

where

$$c_1=\frac{1}{2(2\alpha-1)},\quad c_2=\frac{(2\beta-d)}{2d(2\beta+1)}$$

Theorem 2.3. *The following relation holds true*

$$
\begin{aligned}
{}_1F_1\left[\begin{array}{c}\alpha\\ \\ 2\alpha+2\end{array};\ x\right]\times{}_2F_2\left[\begin{array}{cc}\beta, & d+1\\ & \\ 2\beta+1, & d\end{array};x\right]
&= e^x{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{array};\frac{x^2}{4}\right]\\
&- e^xc_1x\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_2x^2\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+3) & \\ & & \\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_3x\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{array};\frac{x^2}{4}\right]\\
&- e^xc_1c_3x^2\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+3) & \\ & & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{array};\frac{x^2}{4}\right]\\
&+ e^xc_2c_3x^3\quad{}_2F_3\left[\begin{array}{ccc}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+4) & \\ & & \\ \alpha+\frac{5}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+3\end{array};\frac{x^2}{4}\right]
\end{aligned}
$$

where

$$c_1 = \frac{1}{2(\alpha+1)}, \quad c_2 = \frac{\alpha}{4(\alpha+1)(2\alpha+1)(2\alpha+3)} \quad \textit{and} \quad c_3 = \frac{(2\beta-d)}{2d(2\beta+1)}$$

Theorem 2.4. *The following relation holds true*

$$\begin{aligned}
{}_1F_1\left[\begin{array}{c} \alpha \\ \\ 2\alpha-2 \end{array} ;\; x\right] \times {}_2F_2\left[\begin{array}{cc} \beta, & d+1 \\ & \\ 2\beta+1, & d \end{array} ; x\right]
&= e^x\,{}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta-1), & \frac{1}{2}(\alpha+\beta-2) & \\ & & \\ \alpha-\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta-2 \end{array} ; \frac{x^2}{4}\right] \\
&- e^x c_1 x \;\; {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) & \\ & & \\ \alpha-\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta-1 \end{array} ; \frac{x^2}{4}\right] \\
&+ e^x c_2 x^2 \;\; {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta \end{array} ; \frac{x^2}{4}\right] \\
&+ e^x c_3 x \;\; {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) & \\ & & \\ \alpha-\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta-1 \end{array} ; \frac{x^2}{4}\right] \\
&- e^x c_1 c_3 x^2 \;\; {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta+1) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta \end{array} ; \frac{x^2}{4}\right] \\
&+ e^x c_2 c_3 x^3 \;\; {}_2F_3\left[\begin{array}{ccc} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta+2) & \\ & & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{array} ; \frac{x^2}{4}\right]
\end{aligned}$$

where

$$c_1 = \frac{1}{2(\alpha-1)}, \quad c_2 = \frac{\alpha-2}{4(\alpha-1)(2\alpha-1)(2\alpha-3)} \quad \textit{and} \quad c_3 = \frac{(2\beta-d)}{2d(2\beta+1)}$$

3 Conclusion

In this paper we established the results on

$${}_1F_1\left[\begin{array}{c} \alpha \\ \\ 2\alpha\pm n \end{array} ;\; x\right] \times {}_2F_2\left[\begin{array}{cc} \beta, & d+1 \\ & \\ 2\beta+1, & d \end{array} ; x\right]$$

for $n = 1$ and 2. These results may be useful in mathematics, engineering, and some other branches of sciences. Future work may involve applying these extensions to solve practical problems, exploring their connections with orthogonal polynomials, and investigating analogous results in q-series and basic hypergeometric functions. This continuous expansion of Bailey's legacy demonstrates the enduring importance of hypergeometric series in mathematical analysis and its potential for inspiring further advancements in the field.

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