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Refinements of Generic Algebraic Inequalities with Application

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Abstract: *In this paper, we give alternative proofs and refinements of two generic inequalities due to Bagul and Dhaigude in 2022. As a result, Huygens-type inequalities are established. Illustrations are given to support the main results.*

Keywords: Generic algebraic inequalities, Geometric mean-logarithmic mean inequality, Huygens inequality

1 Introduction

The following fundamental inequality:

$$x + \frac{1}{x} \geq 2, \tag{1}$$

for $x > 0$, with equality at $x = 1$, finds many applications in various branches of mathematics. An elegant proof is based on the following elementary inequality: $[\sqrt{x} - 1/\sqrt{x}]^2 \geq 0$. It also plays a central role in the theory of inequalities, including the determination of power and power-exponential functions. For recent advances in this area, see [1, 3, 6].

In particular, Bagul and Dhaigude [1] gave two generalizations of the inequality in (1). They obtained the result below, which involves a power parameter α that can be tuned.

Proposition 1. *[1, Proposition 2] Let $\alpha > 1$.*

- *If $x \geq 1$, then we have*
$$\alpha x + \frac{1}{x^\alpha} \leq x^\alpha + \frac{\alpha}{x}.$$
- *If $x \in (0, 1]$, then we have*
$$\alpha x + \frac{1}{x^\alpha} \geq x^\alpha + \frac{\alpha}{x}.$$

These two inequalities are reversed if $\alpha \in (0, 1)$, respectively, i.e.,

- *If $x \geq 1$, then we have*
$$\alpha x + \frac{1}{x^\alpha} \geq x^\alpha + \frac{\alpha}{x}.$$
- *If $x \in (0, 1]$, then we have*
$$\alpha x + \frac{1}{x^\alpha} \leq x^\alpha + \frac{\alpha}{x}.$$

The proof in [1, Proposition 2] is based on differentiation, whereas the proof we propose here is completely different, based on the hyperbolic sine function and its associated series expansion. We will see later that our new approach allows for greater flexibility, allowing us to improve on [1, Proposition 2].

Proof of Proposition 1. Let $\alpha > 1$.

- If $x \geq 1$, the desired inequality can be reformulated as

$$\alpha \left(x - \frac{1}{x} \right) \leq x^{\alpha} - \frac{1}{x^{\alpha}}.$$

Based on this, we reconfigure this inequality by introducing $y = \log(x)$, so that y is well defined, with $y \geq 0$ and $x = e^y$. The inequality above is thus rewritten as $\alpha \left(e^y - e^{-y} \right) \leq e^{\alpha y} - e^{-\alpha y}$. Then, by dividing by 2 and introducing the hyperbolic sine function, i.e., $\sinh(t) = (e^t - e^{-t})/2$, $t \in \mathbb{R}$, we obtain the following equivalent inequality:

$$\alpha \sinh(y) \leq \sinh(\alpha y).$$

Taking this new form into account, let us prove it using the classical series expansion of the hyperbolic sine function, i.e., $\sinh(t) = \sum_{k=0}^{\infty} t^{2k+1}/(2k+1)!$, $t \in \mathbb{R}$. We immediately have

$$\sinh(\alpha y) = \sum_{k=0}^{\infty} \frac{(\alpha y)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1} y^{2k+1}}{(2k+1)!}.$$

Since $\alpha > 1$, for any integer $k \geq 0$, we have $\alpha^{2k+1} > \alpha$, and since $y \geq 0$, we have $\alpha^{2k+1} y^{2k+1} \geq \alpha y^{2k+1}$. Thus, we get

$$\sinh(\alpha y) \geq \alpha \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} = \alpha \sinh(y).$$

This demonstrates the desired inequality.

- For the case $x \in (0, 1]$, let us set $z = 1/x$. Then we have $z \geq 1$, and we can apply the previous result, so

$$\alpha z + \frac{1}{z^{\alpha}} \leq z^{\alpha} + \frac{\alpha}{z}.$$

This is equivalent to

$$\alpha x + \frac{1}{x^{\alpha}} \geq x^{\alpha} + \frac{\alpha}{x},$$

which is the desired inequality.

Let us consider the case $\alpha \in (0, 1]$, which uses the same mathematical arguments, i.e., the introduction of the sine hyperbolic function and the associated series expansion.

- For $x \geq 1$, by setting $y = \log(x)$ (so $y \geq 0$), the stated inequality

$$\alpha x + \frac{1}{x^{\alpha}} \geq x^{\alpha} + \frac{\alpha}{x},$$

can be reformulated as

$$\alpha \sinh(y) \geq \sinh(\alpha y).$$

With this new form in mind, let us prove it using the classical series expansion of the hyperbolic sine function. We have

$$\sinh(\alpha y) = \sum_{k=0}^{\infty} \frac{(\alpha y)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1} y^{2k+1}}{(2k+1)!}.$$

Since $\alpha \in (0, 1]$, for any integer $k \geq 0$, we have $\alpha^{2k+1} \leq \alpha$, and since $y \geq 0$, we have $\alpha^{2k+1} y^{2k+1} \leq \alpha y^{2k+1}$. Thus, we get

$$\sinh(\alpha y) \leq \alpha \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} = \alpha \sinh(y).$$

The desired inequality is demonstrated.

- For the case $x \in (0, 1]$, let us set $z = 1/x$. Then we have $z \geq 1$, and we can apply the previous result, so
$$\alpha z + \frac{1}{z^\alpha} \geq z^\alpha + \frac{\alpha}{z}.$$
By substitution, we get
$$\alpha x + \frac{1}{x^\alpha} \leq x^\alpha + \frac{\alpha}{x},$$
which is the desired inequality.

This ends the proof. □

This proof is really different from the one in [1], and opens up some room for improvement. The rest of the paper emphasizes this.

2 A Refinement

A refinement of Proposition 1 is possible, due to the new hyperbolic sine approach, as shown below.

Proposition 2. *Let $\alpha > 1$. Then*

- *If $x \geq 1$, we have*
$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \geq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \geq 0.$$
- *If $x \in (0, 1]$, we have*
$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \leq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \leq 0.$$

Let $\alpha \in (0, 1]$. Then

- *If $x \geq 1$, we have*
$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \leq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \leq 0.$$
- *If $x \in (0, 1]$, we have*
$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \geq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \geq 0.$$

Proof. Let $\alpha > 1$.

- Let us consider the case $x \geq 1$. We start with a general hyperbolic sine inequality. Based on the hyperbolic sine series expansion, for $\alpha > 1$ and $y \geq 0$, we have
$$\begin{aligned}\sinh(\alpha y) &= \sum_{k=0}^{\infty} \frac{(\alpha y)^{2k+1}}{(2k+1)!} = \alpha y + \sum_{k=1}^{\infty} \frac{\alpha^{2k+1} y^{2k+1}}{(2k+1)!} \\ &\geq \alpha y + \alpha^3 \sum_{k=1}^{\infty} \frac{y^{2k+1}}{(2k+1)!} = \alpha y + \alpha^3 \left[\sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} - y\right] \\ &= \alpha y + \alpha^3[\sinh(y) - y] = \alpha(1 - \alpha^2)y + \alpha^3 \sinh(y).\end{aligned}$$
By taking $y = \log(x)$ with $x \geq 1$, we get
$$x^\alpha - \frac{1}{x^\alpha} \geq 2\alpha(1 - \alpha^2)\log(x) + \alpha^3\left(x - \frac{1}{x}\right).$$

Therefore, we have

$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \geq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right].$$

By using the well-known logarithmic upper bound [4], [9], $\log(t) \leq (t-1)/\sqrt{t}$, $t \geq 1$, for $x \geq 1$, we get

$$2\log(x) = \log(x^2) \leq \frac{x^2 - 1}{x} = x - \frac{1}{x}.$$

Therefore, we have $x - 1/x \geq 2\log(x)$. Since $\alpha > 1$, we also have $\alpha(\alpha^2 - 1) > 0$, and these results give

$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \geq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \geq 0.$$

- For the case $x \in (0, 1]$, let us set $z = 1/x$. Then we have $z \geq 1$, and we can apply the previous result, so

$$z^\alpha - \frac{1}{z^\alpha} - \alpha\left(z - \frac{1}{z}\right) \geq \alpha(\alpha^2 - 1)\left[\left(z - \frac{1}{z}\right) - 2\log(z)\right] \geq 0.$$

Substituting and rearranging (using $\log(z) = -\log(x)$), we get

$$-\left[x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right)\right] \geq -\left\{\alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right]\right\} \geq 0,$$

which is the desired inequality after multiplying both sides by -1.

Let us consider the case $\alpha \in (0, 1]$, which uses the same mathematical arguments, i.e., the introduction of the sine hyperbolic function, the associated series expansion, and a logarithmic inequality.

- Let $x \geq 1$. Based on the hyperbolic sine series expansion, for $\alpha \in (0, 1]$ and $y \geq 0$, we have

$$\begin{aligned}\sinh(\alpha y) &= \sum_{k=0}^{\infty} \frac{(\alpha y)^{2k+1}}{(2k+1)!} = \alpha y + \sum_{k=1}^{\infty} \frac{\alpha^{2k+1} y^{2k+1}}{(2k+1)!} \\ &\leq \alpha y + \alpha^3 \sum_{k=1}^{\infty} \frac{y^{2k+1}}{(2k+1)!} = \alpha y + \alpha^3 \left[\sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} - y\right] \\ &= \alpha y + \alpha^3[\sinh(y) - y] = \alpha(1 - \alpha^2)y + \alpha^3 \sinh(y).\end{aligned}$$

By taking $y = \log(x)$ with $x \geq 1$, we get

$$x^\alpha - \frac{1}{x^\alpha} \leq 2\alpha(1 - \alpha^2)\log(x) + \alpha^3\left(x - \frac{1}{x}\right),$$

which implies that

$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \leq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right].$$

With the same arguments as those in the case $x \geq 1$, the term in the square bracket is positive, and $\alpha(\alpha^2 - 1) \leq 0$. Therefore, we have

$$x^\alpha - \frac{1}{x^\alpha} - \alpha\left(x - \frac{1}{x}\right) \leq \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right] \leq 0.$$

- For the case $x \in (0, 1]$, let us set $z = 1/x$. Then we have $z \geq 1$, and we can apply the previous result, so

$$z^\alpha - \frac{1}{z^\alpha} - \alpha\left(z - \frac{1}{z}\right) \leq \alpha(\alpha^2 - 1)\left[\left(z - \frac{1}{z}\right) - 2\log(z)\right] \leq 0.$$

Substituting and rearranging (again using $\log(z) = -\log(x)$), we get

$$-\left[x^{\alpha} - \frac{1}{x^{\alpha}} - \alpha\left(x - \frac{1}{x}\right)\right] \leq -\left\{\alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right]\right\} \leq 0,$$

which is the desired inequality after multiplying both sides by -1.

This ends the proof. □

Proposition 2 is a clear improvement over Proposition 1, mainly due to the intermediate term $\alpha(\alpha^2 - 1)\left[(x - 1/x) - 2\log(x)\right]$ that governs the sign of the main term at the heart of the inequalities, i.e., $x^{\alpha} - 1/x^{\alpha} - \alpha(x - 1/x)$.
Let us now illustrate this proposition. Figure 1 displays the following difference function:

$$m(x) = x^{\alpha} - \frac{1}{x^{\alpha}} - \alpha\left(x - \frac{1}{x}\right) - \alpha(\alpha^2 - 1)\left[\left(x - \frac{1}{x}\right) - 2\log(x)\right],$$

for $x \in (0.3, 4)$, and $\alpha = 1.5$ on the one hand, and $\alpha = 0.5$ on the other hand.

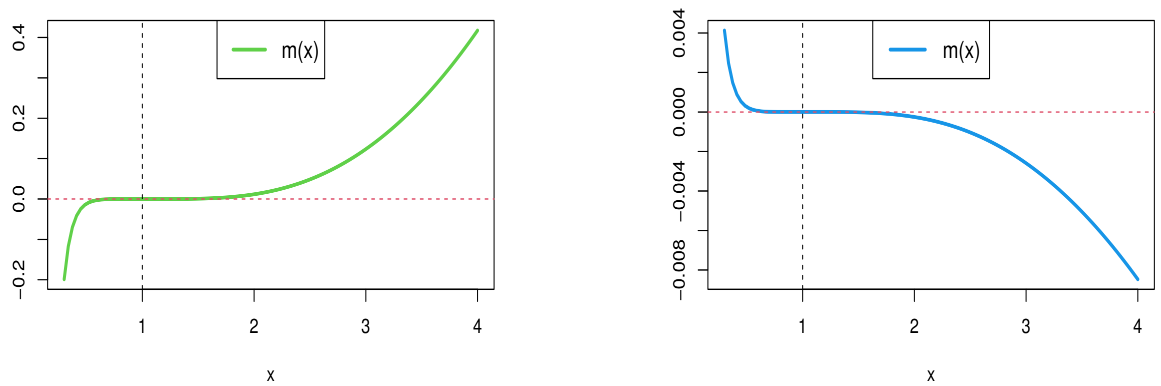

Figure 1: Illustration of $m(x)$ for $x \in (0.3, 4)$ and $\alpha = 1.5$ (left) and $\alpha = 0.5$ (right)

This figure fully illustrates the fact that, for the setting under consideration, we have $m(x) \geq 0$ or $m(x) \leq 0$, depending on whether $x \in (0, 1)$ or $x \geq 1$, under the rigorous conditions of Proposition 2.
In fact, this result can be applied in many mathematical circumstances. In the remainder of the paper, we support this claim by improving one of the most famous trigonometric inequalities in the literature.

3 Application to Huygens-Type Inequality

The Huygens inequality [2, 5, 8, 10] is known as

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 3,$$

for $x \in (0, \pi/2)$. In 2010, this inequality was refined by Neuman and Sándor [8] as follows:

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 2\frac{x}{\sin(x)} + \frac{x}{\tan(x)} > 3, \tag{2}$$

for $x \in (0, \pi/2)$. We call these inequalities Huygens-type inequalities. In the result below, we propose a refinement of the left inequality of (2).

Proposition 3. *If $x \in (0, \pi/2)$, we have*

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 2\frac{x}{\sin(x)} + \frac{x}{\tan(x)} + 6\left\{\frac{x}{\sin(x)} - \frac{\sin(x)}{x} + 2\log\left[\frac{\sin(x)}{x}\right]\right\}$$
$$> 2\frac{x}{\sin(x)} + \frac{x}{\tan(x)} > 3.$$

Proof. It is well known that $0 < \sin(x)/x < 1$ if $x \in (0, \pi/2)$. Therefore, for $\alpha = 2$, replacing x by $\sin(x)/x$ in the second inequality of Proposition 2 and using the inequality $[x/\sin(x)]^2 < \tan(x)/x$ (see [7]), we get

$$\left[\frac{\sin(x)}{x}\right]^2 + 2\frac{x}{\sin(x)} \leq \left[\frac{x}{\sin(x)}\right]^2 + 2\frac{\sin(x)}{x} + 6\left\{\frac{\sin(x)}{x} - \frac{x}{\sin(x)} - 2\log\left[\frac{\sin(x)}{x}\right]\right\}$$
$$< \frac{\tan(x)}{x} + 2\frac{\sin(x)}{x} + 6\left\{\frac{\sin(x)}{x} - \frac{x}{\sin(x)} - 2\log\left[\frac{\sin(x)}{x}\right]\right\}.$$

Using the inequality $[x/\sin(x)]^2 < \tan(x)/x$ once again and rearranging the terms, we find that

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 2\frac{x}{\sin(x)} + \frac{x}{\tan(x)} - 6\left\{\frac{\sin(x)}{x} - \frac{x}{\sin(x)} - 2\log\left[\frac{\sin(x)}{x}\right]\right\}.$$

This gives the desired inequality. Note that the term $x/\sin(x) - \sin(x)/x + 2\log[\sin(x)/x]$ can also be shown to be positive by using well-known geometric mean-logarithmic mean inequality $\sqrt{y_1 y_2} \leq (y_1 - y_2)/[\log(y_1) - \log(y_2)]$, by setting $y_1 = \sin(x)/x$ and $y_2 = x/\sin(x)$. The desired inequalities are obtained. □

Proposition 3 is a clear improvement of double inequality (2). The appearance of the logarithmic term can be seen as surprising, but it makes a strong positive difference with Equation (2). Thus, this enhances our understanding of this notable inequality and how it can be improved.

Let us now illustrate it. Figure 2 displays the following difference function:

$$n(x) = 2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} - \left[2\frac{x}{\sin(x)} + \frac{x}{\tan(x)} + 6\left\{\frac{x}{\sin(x)} - \frac{\sin(x)}{x} + 2\log\left[\frac{\sin(x)}{x}\right]\right\}\right],$$

for $x \in (0.1, 0.8)$ for zoom reasons on the one hand, and $x \in (0.1, 1.5)$ for a global view on the other hand.

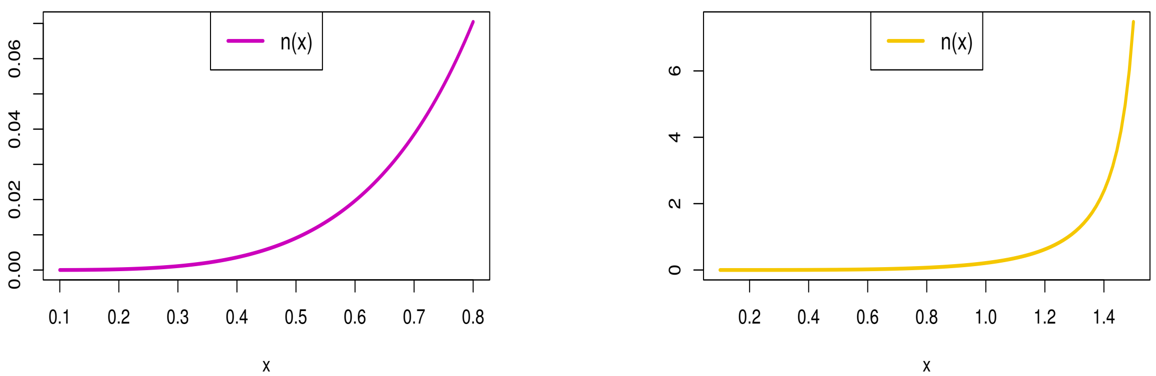

Figure 2: Illustration of $n(x)$ for $x \in (0.1, 0.8)$ (left) and $x \in (0.1, 1.5)$ (right)

It is obvious that, in the considered setting, $n(x) > 0$. Furthermore, a closer look at the y axis of the case $x \in (0.1, 0.8)$ reveals that the obtained inequality is "very sharp" for small and reasonable values of x. This trigonometric inequality is just one example of the application of Proposition 2. We believe that this will lay the foundation for more, in various mathematical fields, beyond the scope of the inequality framework.

4 Conclusion

In this paper, we derived several novel inequalities involving power functions by employing series expansion techniques They extend and refine two generic inequalities due to Bagul and Dhaigude in 2022. Huygens-type inequalities are derived, as a notable application. A graphical analysis is performed to support the theoretical results. We believe that some of the techniques used may be of independent interest for other inequalities, which we leave for a future work.

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