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Tandem Queue and Its Applications in the Production of Bi-layer Tablets

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Abstract: *Production of bi-layer tablets involves the concept of tandem queue. Tandem queue is a series of queues in which the output of one queue becomes the input to the next queue. In this paper, two tandem queues are employed. Each queue has heterogeneous mean arrival time and service time. These two queues merge into one finite capacity queue in the production process of the bi-layer tablets. The study investigates various performance measures of the queueing model. Mathematical models and simulations are utilized to evaluate system performance under different scenarios.*

Keywords: Tandem, Queue, Arrival, Service

1 Introduction

The concept of a queue has been used from ancient times until now knowingly or unknowingly. In 1909, a Danish engineer A.K. Erlang [8] introduced the first queueing model in the telephone call system. The model $M/G/1$ was solved by F. Pollaczek in 1930. After the 1940s, mathematicians took an interest in queueing theory. Different models in queueing theory are classified by using special notations called Kendall's notations [14] which was initially described by D.G. Kendall in 1953 in the form $a/b/c$. Later A.M. Lee added the notations d and e to the Kendall notation $a/b/c : d/e$ where
a : arrival distribution
b : service distribution
c : number of servers
d : maximum number of customers allowed in the system (in queue in service)
e : queue (in service) discipline

Model $(M/M/1/) : (\infty/FCFS)$ shows the arrival process in Poisson distribution, exponential distribution for service process and single server queueing model where the system capacity is infinite.
The values of a and b are replace by following notations:
M : Markovian (or Exponential) interarrival time or service -time distribution
D : Deterministic (or constant) interarrival time or service time
G : General distribution of service time
GI : General probability distribution-normal or uniform for inter-arrival time
E_k : Erlang-k distribution for interarrival or service time with parameter k

Some of the performance measures are as follows:

i. Average number of customers in the system $\text{Ł}_s = \dfrac{\lambda}{(\mu - \lambda)}$

ii. Average number of customers in the queue $\text{Ł}_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)}$

iii. Average time that a customer spent in the system $W_s = \dfrac{1}{\mu - \lambda}$

iv. Average waiting time for a customer in the queue $W_q = \dfrac{\lambda}{\mu(\mu - \lambda)}$

The tandem queueing system has emphasized much attention over the past few decades due to its applications in real-life. There are numerous academic researches in this domain as well. It is a queueing system organized in a series, where service facilities are arranged in series and customers move through a single path from one server to another. In this process, the arrival rate of the exceeding server depends on the service rate of the preceding server. This type of queueing system is called tandem queueing system. Wireless sensor networks, transportation networks and manufacturing companies are some of the applied fields of tandem queueing systems. In pharmaceutical companies, tandem queues play an important role in producing a complex form of medicine such as bi-layer tablets. These tablets are the formation of two different layers and each layer has different stages involving production processes like mixing, granulation, lubrication, compression and packaging. The tandem queueing system helps to ensure that each stage meets quality standards before passing the product to the next stage. This contributes to the overall quality and consistency of the final product. These tablets are composed of one layer of drug for instant dissolving and the second layer to dissolve after some time. Both layers can be differentiated based on their composition, color and shape. It has several advantages such as reduced number of tablets, reduced side effects and enhanced patient compliance.

We have observed some of the literature useful to our study as follows: Sreekanth et al. [15] analyzed the specific queueing system with three service stages. They used matrix method, successive over relaxation method etc. to find steady-state probabilities and various performance measure. Ritha et al. [12] obtained the performance measure of tandem queue by using the parametric programming technique based on α-cut method. He et al. [7] applied the tandem queue to tollbooth with heterogeneous server and found various results by matrix-analytic method. Xu et al. [17] analyzed the two stage tandem queueing system in which second server has N server and studied various performance pressures with numerical examples. Similarly Dudin et al. [4] studied the multi server model and analyzed the tandem queue. Priya et al. [11] studied the performance measures of parallel four state tandem queue using Burke theorem.

Santhi [13] studied a batch service queueing model to evaluate the performance of a cloud computing architecture, specifically focusing on waiting times and compared the results of partial and full batch service model for clients accessing a cloud database. Mitici et al. [10] introduced a new multi-class tandem network with batch service, established its mathematical properties, and applied it to wireless sensor networks. Sridhar [16] used different parameters to find an average queue length and server utilization in both single and batch service, along with an additional service station. Ghimire et al. [5] discussed the idea of serving bulk and individual customers with different service rates and applied real data to verify various performance measure. Bakuli [2] used semi Markov chain technique to analyze two-server model with equal service rates. Masaudi et al. [9] studied the application of queuing network theory to analyze the effects of batch size and throughput on optimizing the utilization of machine resources in a manufacturing system.

In this paper, we take the real data from Arrow Pharmaceuticals Company located in Bhaktapur, Nepal. This company manufactures bi-layer tablets for patient with diabetes. We apply the classical formulas for three servers tandem queueing system. The first server is for dry mixing where all the raw materials are blended carefully. During this stage, the particles of each ingredient are uniformly blend throughout the mixture. The second server is for granulation which helps to improve the flow properties and compressibility of the powder. Finally, the third server is used for lubrication which is the critical process in pharmaceutical tablets. It helps to reduce the friction and enhance the flow of granules as well as maintains the product quality and consistency. The production process is divided into two layers in which output of the first server becomes the input for the second server. The output products from both layers combine for compression and blister packing that follows the finite queueing system with a bulk service rate. We have omitted the quality check stage in our study, as our research solely focuses on queue length analysis and does not pertain to quality assessment. Quality check measures are essential throughout all processes to ensure product consistency. This includes verifying ingredient's weight, checking blend uniformity, and conducting various tests to confirm that the mixture meets specified quality standards.

The remaining part of the paper is organized as follows: Section 2 describes the mathematical model with notations and a flow chart along with some established formulas for tandem queue. Section 3 includes the numerical results and their applications in which validity of the model is verified with the help of MATLAB simulation. Finally, Section 4 concludes the paper with some recommendations.

2 Mathematical Model

Some of the notations used in our study are described as follows:
λ_{ij} : Arrival rate in Poisson distribution where $i = 1, 2$ denotes the number of layers and $j = 1, 2, 3$ are the number of servers
μ_{ij} : Service rate in exponential distribution where $i = 1, 2$ denotes the number of layers and $j = 1, 2, 3$ are the number of servers
λ_C : Arrival rate for compression
μ_C : Service rate for compression
λ_B : Arrival rate for blister packaging
μ_B : Service rate for blister packaging
ρ_C : Utilization factor for compression
ρ_B : Utilization factor for blister packaging
N : Number of tablets formed

In the Figure 1, there are two parallel layers namely, Layer 1 and Layer 2. In each layer, production passes through certain stages. In each stage, different performance measures have been calculated. Both the layers are combined before the final product.

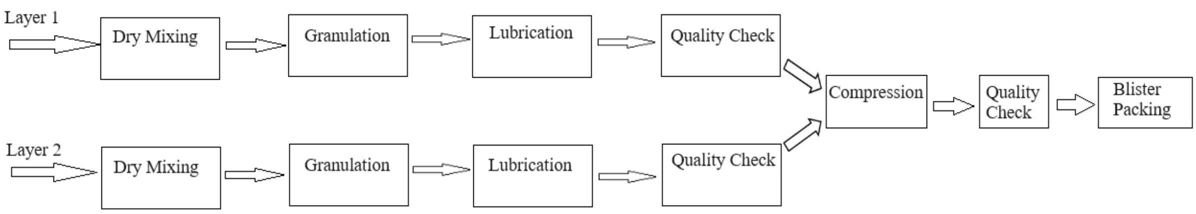

Figure 1: Flow chart of Bi-layer tablet manufacturing.

In the following subsections, we discuss some of the mathematical models and performance measures relating to tandem queue.

2.1 $M/M/1$ Tandem queueing model

This is the model in which arrivals occur in Poisson distribution with mean arrival time $\frac{1}{\lambda_{11}}, \frac{1}{\lambda_{12}}, \frac{1}{\lambda_{13}}$ in layer 1 and $\frac{1}{\lambda_{21}}, \frac{1}{\lambda_{22}}, \frac{1}{\lambda_{23}}$ in layer 2 . There are three servers in each layer to provide the better services with mean service time $\frac{1}{\mu_{11}}, \frac{1}{\mu_{12}}, \frac{1}{\mu_{13}}$ in Layer 1 and $\frac{1}{\mu_{21}}, \frac{1}{\mu_{22}}, \frac{1}{\mu_{23}}$ in Layer 2. To observe the performance measures in the different layers following formulas are used.

Average number of customers in the system $$L_{si} = \sum_{j=1}^{3} \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}}$$

Average number of customers in the queue $$L_{qi} = \sum_{j=1}^{3} \frac{\lambda_{ij}^2}{\mu_{ij}(\mu_{ij} - \lambda_{ij})}$$

Average time that a customer spent in the system $$W_{si} = \sum_{j=1}^{3} \frac{1}{\mu_{ij} - \lambda_{ij}}$$

Average waiting time for a customer in the queue $$W_{qi} = \sum_{j=1}^{3} \frac{\lambda_{ij}}{\mu_{ij}(\mu_{ij} - \lambda_{ij})}$$

2.2 $M/M/1$ finite capacity model

After completing the production process from Layer 1 and Layer 2 the production materials go through a finite capacity stage for compression and packaging where the service rate are in bulk for both the stages. In this case, following formulas are used to find the performance measures.
Average number of customers in compression

$$L_{sC} = \frac{\rho_C}{1-\rho_C} - \frac{(N+1)\rho_C^{N+1}}{1-\rho_C^{N+1}}, \qquad \text{where} \qquad \rho_C = \frac{\lambda_C}{27\mu_C}$$

Average number of customers in Blister packaging

$$L_{sB} = \frac{\rho_B}{1-\rho_B} - \frac{(N+1)\rho_B^{N+1}}{1-\rho_B^{N+1}}, \qquad \text{where} \qquad \rho_B = \frac{\lambda_B}{10\mu_B}$$

3 Numerical Results and Interpretations

In the study of queueing theory performance measures are calculated specially in terms of queue length and waiting time. Validity of these performance measures are checked by means of simulation. We have used MATLAB to verify the results we have obtained. This section presented tables and graphs based on the data collected at Arrow Pharmaceutical Company Pvt. Ltd., Bhaktapur to validate the data with a visual representation. Here, L_{q1} and L_{q2} are the number of customers in the queue of Layer 1 and Layer 2. Likewise L_{s1} and L_{s2} represent the number of customers in the system of Layer 1 and Layer 2 respectively. Waiting time for a customer in the queue and in the system of Layer 1 and Layer 2 are denoted by W_{q1}, W_{q2} and W_{s1}, W_{s2} respectively. Weight of the raw materials for Layer 1 is 47 kg and that for Layer 2 is 7.95 kg.

Table 1: Performance Measure of Layer 1
$\frac{1}{\mu_{11}} = 12, \frac{1}{\mu_{12}} = 14, \frac{1}{\mu_{13}} = 12$

$\frac{1}{\lambda_{11}}$	L_{q1}	L_{s1}	W_{q1}	W_{s1}
15	0.17	3.00	22.00	60.00
17	0.82	3.66	35.33	79.33
19	1.47	4.33	51.33	101.33
21	2.13	5.00	70.00	126.00

The Table 1 provides the queue length and the waiting time for different arrival rates. Based on the data provided by the pharmaceutical company, Table 1 shows the number of arriving customers and waiting time for a customer in the queue and in the system of layer 1. The mean arrival time of 15 minutes and the service time of 12 minutes for dry mixing has been observed. Similarly, the arrival time for granulation is 14 minutes and for lubrication is 12 minutes. We have considered some other mean arrival time as well whose performance measures are also shown on the table.

Figure 2 displays the graph of queue length vs arrival time for Table 1. In either of the graphs, we can observe that queue length is shorter for small arrival times and longer for greater arrival times. The red coloured graph represents the number of customers in the queue whereas the blue coloured graph indicates the number of customers in the system. There are always more customers in the system than in the queue. This queueing phenomenon has been satisfied in both the graphs of Figure 2. Similarly, Table 2 shows the number of arriving customers and waiting time for a customer in the queue and in the system of Layer 2 with the mean arrival time 4 minutes. The service time of first, second and third servers are respectively 2, 3 and 2.5 minutes. All the corresponding values for the mean arrival time 5, 6 and 7 minutes are also shown in the table.

Figure 3 is the graphs of queue length vs arrival times for Layer 2. These graphs are also based on the arrival time number of customer are greater in the system in comparison to the number of customers in

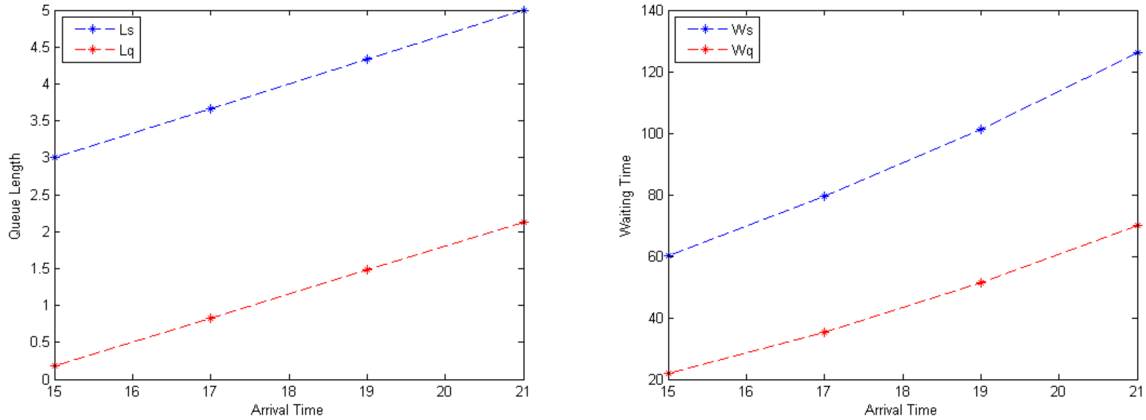

Figure 2: Graph of Layer 1.

Table 2: Performance measure of Layer 2
$\frac{1}{\mu_{21}} = 2, \frac{1}{\mu_{22}} = 3, \frac{1}{\mu_{23}} = 2.5$

$\frac{1}{\lambda_{21}}$	L_{q2}	L_{s2}	W_{q2}	W_{s2}
4	0.16	3.00	5.50	13.00
5	1.69	4.50	13.00	23.50
6	3.18	6.00	23.50	37.00
7	4.66	7.50	37.00	53.50

the queue. In the real life scenario, we have experienced that number of customers is always greater as the arrival rate increases. These realistic situations have been satisfied in each of the graphs.

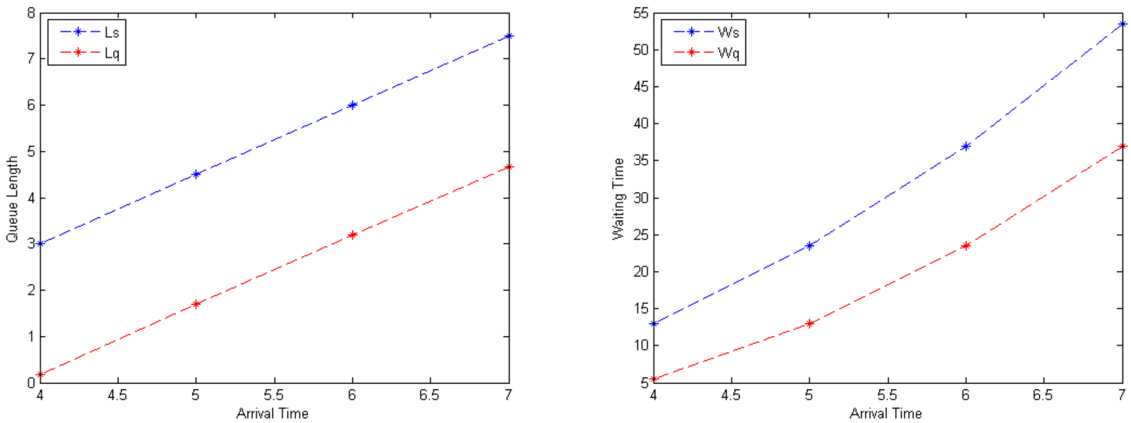

Figure 3: Graph of Layer 2.

Components from Layer 1 and Layer 2 are sent to the compression server from two separate hoppers to create bi-layer tablets. In the compression process, the two layers are compressed together into a single tablet. This can be done using a tablet compression machine. After compression, some of the tablets undergo a quality check to ensure that they meet the required specifications for weight, hardness, thickness, and content uniformity. Dissolution testing is also conducted to assess the release profile of the active ingredients in each layer. In the final stage, blister packing takes place for distribution and consumption.

For the calculation of compression and blister packaging, total weight of 60 kg is formulated from Layer 1 and Layer 2 in which weight of 1 tablet is 1200mg.
Followings are the number of tablets, arrival rates, service rates, traffic intensity and the number of customers in the compression.
$N = 49924$, $\lambda_C = 172.15/mins.$, $\mu_C = 171.56/mins.$, $\rho_C = 0.0383$, $L_{sC} = 0.0398$
Similarly, arrival rates, service rates, traffic intensity and the number of customers in the blister packaging are listed below.
$N = 43200$, $\lambda_B = 169.41/mins.$, $\mu_B = 168.09/mins.$, $\rho_B = 0.1008$, $L_{sB} = 0.1121$

4 Conclusion and Recommendations

In this paper, we analyzed two tandem queues in which each queue has heterogeneous mean arrival times and service times. By applying formulas of the $M/M/1$ queue, we have calculated various performance measures with varying arrival times and service times. Our research findings show that when arrival time increases, both the queue length and the waiting time increase. Finally, total number of customers in the system was calculated. This calculation follows a bulk queueing system for both compression and packaging stages. A bulk size of 27 tablets and 10 tablets was considered for the compression process and the blister packaging, respectively. This provides valuable insights into the overall workload and efficiency of the system, helping to make the appropriate decisions in process optimization and resource allocation for bi-layer tablet manufacturing.
Based on our study and the data from the pharmaceutical company, we would like to make the following recommendations:

i. Management should invest in modern and suitable handling equipment for the movement of raw materials.

ii. Time and cost can be reduced by implementing a continuous manufacturing process.

iii. Blister packaging is preformed using machines for improved hygiene and faster production.

iv. Motivation and training the staff in using modern equipment is essential for the overall improvement of the company.

Besides these conclusions and recommendations there are limitations in the production of by-layer tablets.

i. Exact production time can not be calculated due to different sample sizes for quality check.

ii. Cost constraints are not taken into account.

iii. Bi-layer tablets are produced seasonally.

5 Manufacturing Equipment

The following are pictures of some of the equipment used in the production of bi-layer tablets at Arrow Pharmaceutical Company Pvt. Ltd.

6 Acknowledgment

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Granulation Machine

Quality Check Machine

Bi-Layer Tablets

Compression Machine

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