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On Certain Statistical Convergence Criteria for Martingale Sequences via Deferred Cesàro Mean with Some Applications

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Abstract: *In this paper, we investigate and study various new notions of statistical convergence criteria of martingale sequences via deferred Cesàro mean. We then establish several results concerning the relation among these beautiful and potentially useful concepts. Moreover, our proposed techniques contributed to finding several new approximation results of the Korovkin-type for martingale sequences over complete normed linear spaces. We also present some concrete examples to establish the stronger side of our results.*

Keywords: Stochastic sequences, Martingale sequences, Statistical uniform integrable, Deferred Cesàro mean, Banach space, Korovkin-type theorem

1 Introduction and Motivation

Let (X_n) be a sequence of random variables relative to the measurable functions $\mathcal{F}_n \subseteq \mathcal{F}$ $(n \in \{0\} \cup \mathbb{N})$ over the probability space $(\Omega, \mathcal{F}, P)$. Then, we adopt a stochastic sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ such that

(i) $\mathbb{E}\{|X_n|\} < +\infty$,

(ii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ almost surely (a.s.) and

(iii) $(\mathcal{F}_n)$ is a measurable sequence of functions;

where $\mathbb{E}$ is the mathematical expectation. Then it is known as a *martingale sequence*.

Now, in view of the convergence of martingale sequences we recall the following definition.

Definition 1.1. A martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ and having probability 1 (that is, $\text{Prob}(X_n) = 1$) is convergent to a finite integrable random variable X_0, if

$$\lim_{n\to\infty} (X_n, \mathcal{F}_n) \longrightarrow X_0 \qquad (\mathbb{E}\{|X_0|\} < +\infty) \quad \text{a.e. (almost everywhere).}$$

Subsequently, we recall sub-martingale sequence, reverse martingale sequence and reverse sub-martingale sequence.

A given stochastic sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ is a *sub-martingale sequence* if

(i) $\mathbb{E}\{|X_n|\} < +\infty$,

(ii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geqq X_n$ almost surely (a.s.) and

(iii) $(\mathcal{F}_n)$ is a measurable sequence of functions;

where $\mathbb{E}$ is the mathematical expectation.

A given stochastic sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ is a *reverse martingale sequence* if

$$\mathbb{E}(\mathcal{F}_n|X_{m+n}) = X_{m+n} \text{ (a.s.)} \quad \text{for } m, n \geqq 1,$$

and a *reverse sub-martingale sequence* if

$$\mathbb{E}(\mathcal{F}_n|X_{m+n}) \geqq X_{m+n} \text{ (a.s.)} \quad \text{for } m, n \geqq 1.$$

Next, we recall the almost sure convergence and rth mean convergence of martingale sequences.

A martingale $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ sequence is *almost sure convergent* (a.s.) to a random variables X_0, if

$$\text{Prob}\left(\left\{\lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0\right\}\right) = 1.$$

We write

$$(X_n, \mathcal{F}_n) \longrightarrow X_0 \quad (a.s.).$$

A martingale $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ sequence is *rth mean convergent* $(r \geqq 1)$ to a random variable X_0, if

$$\mathbb{E}\{|(X_n, \mathcal{F}_n) - X_0|^r\} = 0.$$

Recently, the notion of statistical convergence is very helpful in sequence spaces because it is more general than the classical convergence and such a idea was introduced by two distinguished mathematicians, Fast [5] and Steinhaus [22]. For utilizing this nice idea a few researchers worked in many diversified ways and developed so many interesting results in various fields of pure and applied mathematics, such as summability theory, Soft Computing, Machine Learning, Measure theory and so on. Moreover, the notion of statistical convergence in probability has enhanced the glory of this work. In this direction, interested learners are referred to see [2], [3], [4], [6], [7], [8], [9], [10], [11], [14], [15], [16] and [21].

Let $\eta \subseteq \mathbb{N}$, and also let $\eta_n = \{i : i \leqq n \quad \text{and} \quad i \in \eta\}$. Then the natural density $d(\eta)$ of η is defined by

$$d(\eta) = \lim_{n\to\infty} \frac{|\eta_n|}{n} = k,$$

where k is a real finite number, and $|\eta_n|$ is the cardinality of η_n.

A given sequence (X_n) of random variables is statistically convergent to X_0 if, for each $\epsilon > 0$,

$$\eta_\epsilon = \{i : i \in \mathbb{N} \quad \text{and} \quad |X_i - X_0| \geqq \epsilon\}$$

has natural density zero ([5] and [22]). Thus, for each $\epsilon > 0$,

$$d(\eta_\epsilon) = \lim_{n\to\infty} \frac{|\eta_\epsilon|}{n} = 0.$$

We write,

$$\text{stat} \lim_{n\to\infty} X_n = X_0.$$

We now introduce the definitions of statistical convergence of martingale and sub-martingale sequences.

Definition 1.2. A martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ and having probability 1 is statistically convergent to a finite integrable random variable X_0 $(\mathbb{E}\{|X_0|\} < +\infty)$ if, for all $\epsilon > 0$,

$$\eta_\epsilon = \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

has zero natural density, with

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$$

almost surely. That is, for every $\epsilon > 0$,

$$d(\eta_\epsilon) = \lim_{n\to\infty} \frac{|\eta_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat}_{\text{mart}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

Definition 1.3. A sub-martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ and having probability 1 is statistically convergent to a finite integrable random variable X_0 ($\mathbb{E}\{|X_0|\} < +\infty$) if, for all $\epsilon > 0$,

$$\eta_\epsilon = \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

has zero natural density, with

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geqq X_n$$

almost surely. That is, for every $\epsilon > 0$,

$$d(\eta_\epsilon) = \lim_{n\to\infty} \frac{|\eta_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat}_{\text{submart}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

Based on the above Definitions, we establish the following theorem.

Theorem 1.1. (see [17]) If a martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is convergent to a finite integrable random variable X_0 with $\mathbb{E}\{|X_0|\} < +\infty$, then it is statistically convergent to the finite integrable random variable X_0. However, it not generally true in the converse sense.

The example below illustrates the non-validity of converse part of the Theorem 1.1.

Example 1.1. Let $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ be a sequence (monotonically increasing) of 0-mean independent random variables over the σ-fields, and $(X_n) \in \mathcal{F}_n$ be such that

$$X_n = \begin{cases} 1 & (n = 2^m; m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

Here we observe that, the martingale $(X_n, \mathcal{F}_n)$ sequence is convergent statistically to zero but not simply martingale convergent.

Next, in view of discussing the better behavior of statistical convergence of martingale sequence, we introduce the notions of statistical reverse martingale convergence and statistical reverse sub-martingale convergence.

Definition 1.4. A reverse martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ and having probability 1 is statistically convergent to an integrable random variable X_0 almost surely if, for all $\epsilon > 0$,

$$\eta_\epsilon = \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

has zero natural density, with

$$\mathbb{E}(\mathcal{F}_n|X_{m+n}) = X_{m+n} \quad \text{for } m, n \geqq 1$$

almost surely. That is, for every $\epsilon > 0$,

$$d(\eta_\epsilon) = \lim_{n\to\infty} \frac{|\eta_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat}_{\text{rev-mart}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

Definition 1.5. A reverse sub-martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ with $\mathbb{E}\{|X_n|\} < +\infty$ and having probability 1 is statistically convergent to an integrable random variable X_0 almost surely if, for all $\epsilon > 0$,

$$\eta_\epsilon = \{i : i \leqq n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

has zero natural density, with

$$\mathbb{E}(\mathcal{F}_n | X_{m+n}) \geqq X_{m+n} \quad \text{for } m, n \geqq 1$$

almost surely. That is, for every $\epsilon > 0$,

$$d(\eta_\epsilon) = \lim_{n \to \infty} \frac{|\eta_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat}_{\text{rev}-\text{submart}} \lim_{n \to \infty} (X_n, \mathcal{F}_n) = X_0.$$

Based on the above Definition 1.5 we establish the following theorem, where the reverse sub-martingale sequence converges almost surely to an infinite integrable random variable.

Theorem 1.2. Let $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ be a statistical reverse sub-martingale sequence, then there exists an infinite integrable random variable X_0 such that

$$\text{stat}_{\text{rev}-\text{submart}} \lim_{n \to \infty} (X_n, \mathcal{F}_n) = X_0 \ (a.s.).$$

Proof. Let $\mathfrak{A}$ denotes the event and let the martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ does not statistically converge. Suppose we consider a set

$$\mathfrak{A}_{s,t} = \{\text{stat} \liminf (X_n, \mathcal{F}_n) \leqq s < t \leqq \limsup X_n\}.$$

It is easy to verify that

$$\mathfrak{A} = \cup \{\mathfrak{A}_{s,t} \mid s < t \quad (s, t \text{ are finite rational})\}.$$

Let $\mathfrak{H}_n$ denotes the number of up-crossings martingale sequences over $[s, t]$, and let

$$\mathfrak{H}_n = \{i : i \leqq n \quad \text{and} \quad | [(X_n, \mathcal{F}_n), (X_{n-1}, \mathcal{F}_{n-1}), \cdots, (X_0, \mathcal{F}_0)] - X_0 | \geqq \epsilon\}.$$

As $\mathfrak{H}_n$ is non-negative, finite, so we immediately get

$$\text{stat}_{\text{rev}-\text{submart}} \mathbb{E} \mid \mathfrak{H}_n \mid \leqq \frac{\mathbb{E}\{(X_0, \mathcal{F}_0) - s\}}{t - s} \leqq \frac{\mathbb{E}\{| (X_0, \mathcal{F}_0) |\} + | s |}{t - s}.$$

Since there is an $(\mathcal{F}_n)$- measurable infinite function $\mathfrak{H}_0$ over $(\Omega, \mathcal{F}, P)$ such that

$$\text{stat}_{\text{rev}-\text{submart}} \mathfrak{H}_n \to \mathfrak{H}_0 \ (a.s.),$$

and by monotone convergence theorem, we have

$$\text{stat}_{\text{rev}-\text{submart}} \int_\Omega \mathfrak{H}_0 dP = \text{stat}_{\text{rev}-\text{submart}} \sup \mathbb{E}\{\mathfrak{H}_n\} \leqq \frac{\mathbb{E}\{| (X_0, \mathcal{F}_0) |\} + | s |}{t - s};$$

which is finite. Therefore, $\mathfrak{H}_0$ is finite almost surely and $\mathfrak{A}_{s,t} \subset \mathfrak{H}_0$ and $\text{Prob}(\mathfrak{H}_0) = 0$, then $\text{Prob}(\mathfrak{A}_{s,t}) = 0$. Thus,

$$\text{stat}_{\text{rev}-\text{submart}} \lim_{n \to \infty} (X_n, \mathcal{F}_n) = X_0 \ (a.s.).$$

□

Inspired basically by the aforesaid investigations, here we investigate and study various notions of new statistical convergence criteria of martingale sequences via deferred Cesàro mean. We then establish several new elementary theorems concerning the relation between these beautiful and potentially useful concepts. Moreover, in view of our proposed techniques, we find several new approximation of the Korovkin-type results with algebraic test functions (with probability one) for a martingale sequence over a complete norm linear space. Finally, we present some specific examples associated with the generalized Bernstein polynomials of the martingale sequence, which shows that our established results are much more stronger than the classical and statistical versions of some earlier existing results.

2 Deferred Cesàro Statistical Convergence of Martingale Sequence

Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$ such that $u_n < v_n$ and $\lim_{n\to\infty} v_n = +\infty$. Then the deferred Cesàro mean for the martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ is defined by

$$\mathcal{D}(X_n, \mathcal{F}_n) = \frac{(X_{u_n+1}, \mathcal{F}_{u_n+1}) + (X_{u_n+2}, \mathcal{F}_{u_n+2}) + \cdots + (X_{v_n}, \mathcal{F}_{v_n})}{v_n - u_n}$$
$$= \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} (X_k, \mathcal{F}_k).$$

Now, we propose the following definitions.

Definition 2.1. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ having probability 1 is deferred Cesàro statistically convergent to a finite integrable random variable X_0 with $\mathbb{E}|X_0| < \infty$ if, for all $\epsilon > 0$, the natural density

$$\mathfrak{U}_\epsilon = \{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

is zero. That is, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|}{u_n - v_n} = 0.$$

Here, we write

$$\mathrm{DMC_{stat}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

Definition 2.2. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is statistically deferred Cesàro summable to a finite integrable random variable X_0 with $\mathbb{E}|X_0| < \infty$ if, for all $\epsilon > 0$,

$$\mathfrak{U}_\epsilon = \{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathcal{D}(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}$$

has zero natural density. That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathcal{D}(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|}{u_n - v_n} = 0.$$

Here, we write

$$\mathrm{stat_{DMC}} \lim_{n\to\infty} \mathcal{D}(X_i, \mathcal{F}_i) = X_0.$$

Now we demonstrate an inclusion theorem concerning these two new interesting concepts.

Theorem 2.1. (see [17]) If a given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is deferred Cesàro statistically convergent to a finite integrable random variable X_0 with $\mathbb{E}|X_0| < \infty$, then it is statistically deferred Cesàro summable to a finite integrable random variable X_0, but not conversely.

In view of the non-validity of the converse statement of Theorem 2.1, we establish here a counter example.

Example 2.1. Suppose that $u_n = 2n$ and $v_n = 4n$, and $(\mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ be a sequence (monotonically increasing) of 0-mean independent random variables of the σ-fields with $(X_n) \in \mathcal{F}_n$ such that $X_n = 1 \quad (n = 2m; m \in \mathbb{N})$ and $X_n = -1 \quad (n = 2m+1; m \in \mathbb{N})$. Here, the martingale $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ sequence does not ordinarily converge. Also, it does not statistically deferred Cesàro converge. But, it has deferred Cesàro sum $\frac{1}{2}$. Thus, it is statistically deferred Cesàro summable to $\frac{1}{2}$.

We now present an inclusion theorem for deferred Cesàro statistically convergent ($\mathrm{D}'\mathrm{MC}$) martingale sequence.

Theorem 2.2. Let (u_n), (v_n), (u'_n) and $(v'_n) \in \mathbb{Z}^{0+}$ with $u_n \leqq u'_n < v_n \leqq v'_n$ $(\forall\, n \in \{0\} \cup \mathbb{N})$. Then

$$\mathrm{stat_{D'MC}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0 \Longrightarrow \mathrm{DMC_{stat}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

Proof. Let $\epsilon > 0$, and suppose that $\text{stat}_{\text{D}'\text{MC}} \lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0$. Now consider the set in the following form:

$$\begin{aligned}\{i : u_n < i \leqq v_n \quad \text{and} \quad &|(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\} \\ &= \{i : u_n < i \leqq u_n' \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\} \\ &\quad \cup \{i : u_n' < i \leqq v_n' \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\} \\ &\quad \cup \{i : v_n' < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}.\end{aligned}$$

Following the deferred Cesàro mean, the statistical versions of the set can be written as

$$\begin{aligned}\frac{1}{v_n - u_n}&|\{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}| \\ &\leqq \frac{1}{v_n' - u_n'}|\{i : u_n < i \leqq u_n' \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}| \\ &\quad + \frac{1}{v_n' - u_n'}\{i : u_n' < i \leqq v_n' \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\} \\ &\quad + \frac{1}{v_n' - u_n'}\{i : v_n' < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}.\end{aligned}$$

Now, taking limit on both sides as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{1}{v_n - u_n}|\{i : u_n < i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}| = 0.$$

Hence,

$$\text{DMC}_{\text{stat}} \lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0.$$

□

Next, we see that under certain specific condition the converse of the Theorem 2.2 also holds.

Theorem 2.3. Let (u_n), (v_n), (u_n') and $(v_n') \in \mathbb{Z}^{0+}$ with $u_n \leqq u_n' < v_n \leqq v_n'$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to} \frac{v_n - u_n}{v_n' - u_n'} = \theta > 0.$$

Then,

$$\text{DMC}_{\text{stat}} \lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0 \Longrightarrow \text{stat}_{\text{D}'\text{MC}} \lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0.$$

Proof. Since, $\text{DMC}_{\text{stat}} \lim_{n\to\infty}(X_n, \mathcal{F}_n) = X_0$, then it is easy to see that the inclusion

$$\begin{aligned}\{i : u_n' + 1 \leqq i \leqq v_n' \quad \text{and} \quad &|(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\} \\ &\subseteq \{i : u_n + 1 \leqq i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}\end{aligned}$$

and the inequality

$$\begin{aligned}|\{i : u_n' + 1 \leqq i \leqq v_n' \quad \text{and} \quad &|(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}| \\ &\leqq |\{i : u_n + 1 \leqq i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|\end{aligned}$$

are fairly hold.

Thus, we have

$$\begin{aligned}\frac{1}{v_n' - u_n'}&|\{i : u_n' + 1 \leqq i \leqq v_n' \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}| \\ &\leqq \left(\frac{v_n - u_n}{v_n' - u_n'}\right) \frac{1}{v_n - u_n}|\{i : u_n + 1 \leqq i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|.\end{aligned}$$

Taking limit as $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{1}{v'_n - u'_n} |\{i : u'_n + 1 \leqq i \leqq v'_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|$$
$$\leqq \lim_{n\to\infty} \left(\frac{v_n - u_n}{v'_n - u'_n}\right) \frac{1}{v_n - u_n} |\{i : u_n + 1 \leqq i \leqq v_n \quad \text{and} \quad |(X_i, \mathcal{F}_i) - X_0| \geqq \epsilon\}|.$$

Thus, clearly we have

$$\text{stat}_{\text{D}'\text{MC}} \lim_{n\to\infty} (X_n, \mathcal{F}_n) = X_0.$$

□

3 Deferred Cesàro Statistical Uniform Integrability

A martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is uniformly integrable if, for each $\epsilon > 0$ there is a number $K_\epsilon > 0$ such that

$$\lim_{n\to\infty} \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP < \epsilon. \tag{3.1}$$

Next, to get the stronger statistical convergence of martingale sequence it is necessary to introduce the notion of statistical uniform integrability.

Definition 3.1. A martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly integrable if, for each $\epsilon > 0$ there is a number $K_\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ i : i \leqq n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP \right| \geqq \epsilon \right\} \right| = 0.$$

We write

$$\text{stat}_{\text{UI}} \lim_{n\to\infty} \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP < \epsilon.$$

Theorem 3.1. If a martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly integrable, then for each $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that for any event B, if $\text{Prob}(B) < \delta_\epsilon$ then

$$\text{stat} \lim_{n\to\infty} \int_B | X_n | \, dP < \epsilon.$$

Proof. Let B be any event, and let for each $n \in \{0\} \cup \mathbb{N}$,

$$\text{stat} \lim_{n\to\infty} \int_B | X_n | \, dP = \text{stat} \lim_{n\to\infty} \int_{B \cap \{|X_n| \leqq K\}} | X_n | \, dP$$
$$+ \text{stat} \lim_{n\to\infty} \int_{B \cap \{|X_n| > K\}} | X_n | \, dP$$
$$\leqq K \ \text{Prob}(B) + \text{stat} \lim_{n\to\infty} \int_{\{|X_n| > K\}} | X_n | \, dP.$$

For every $\epsilon > 0$, select $K > 0$, so that

$$\text{stat} \lim_{n\to\infty} \int_{\{|X_n| > K\}} | X_n | \, dP < \frac{\epsilon}{2}$$

for all $n \in \{0\} \cup \mathbb{N}$.

For $K > 0$, we chose $\delta_\epsilon = \frac{\epsilon}{2K}$, therefore, $B \in \mathcal{F}$ for which $\text{Prob}(B) < \delta_\epsilon$. Thus, the inequality

$$\text{stat} \lim_{n\to\infty} \int_B | X_n | \, dP < \epsilon$$

holds for all $n \in \{0\} \cup \mathbb{N}$. □

Theorem 3.2. If a martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly integrable, and if it is statistically convergent to a random variable X_0 (a.s.), then

$$\text{stat} \lim_{n\to\infty} (X_n, \mathcal{F}_n) \to X_0.$$

in rth mean $(r = 1)$.

Proof. Since $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly integrable for each $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\text{stat} \lim_{n\to\infty} \int_B \mid X_n \mid dP < \epsilon$$

for all $n \in \{0\} \cup \mathbb{N}$, if Prob(B) $< \delta_\epsilon$. Also, since $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically convergent to a random variable X_0 (a.s.), then there is an event $\mathfrak{A}$ such that $\text{Prob}(\mathfrak{A}) < \delta_\epsilon$ and let $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly convergent over $\mathfrak{A}^c$.

Thus,

$$\begin{aligned} \text{stat} \int_\Omega \mid X_n - X_m \mid dP &\leqq \text{stat} \int_{\mathfrak{A}^c} \mid X_n - X_m \mid dP \\ &\quad + \text{stat} \int_{\mathfrak{A}} \mid X_n \mid dP + \text{stat} \int_{\mathfrak{A}} \mid X_m \mid dP \\ &\leqq \text{stat} \int_{\mathfrak{A}^c} \mid X_n - X_m \mid dP + 2\epsilon. \end{aligned}$$

As $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly convergent over $\mathfrak{A}^c$, we have

$$0 \leqq \text{stat} \int_{\mathfrak{A}^c} \mid X_n - X_m \mid dP < 3\epsilon \quad (\forall\ m, n \in \{0\} \cup \mathbb{N}).$$

Moreover, $L_1(\Omega, \mathcal{F}, P)$ being complete, there exists a random variable $X_0^{'} \in L_1(\Omega, \mathcal{F}, P)$ such that

$$\text{stat}(X_n, \mathcal{F}_n) \to X_0^{'}$$

in rth mean $(r = 1)$. Therefore, there exists a martingale subsequence $(X_{n,k}, \mathcal{F}_{n,k})$ of $(X_n, \mathcal{F}_n)$ such that

$$\text{stat}(X_{n,k}, \mathcal{F}_{n,k}) \to X_0^{'}\ (a.s.).$$

However,

$$\text{stat}(X_n, \mathcal{F}_n) \to X_0\ (a.s.)$$

implies that

$$\text{stat}(X_{n,k}, \mathcal{F}_{n,k}) \to X_0\ (a.s.).$$

Hence, $X_0 = X_0^{'}$ and

$$\text{stat} \lim_{n\to\infty} (X_n, \mathcal{F}_n) \to X_0$$

in rth mean $(r = 1)$. □

We now introduce and investigate the notions of deferred Cesàro statistically uniformly integrability and statistically deferred Cesàro uniformly summability of martingale sequences.

Definition 3.2. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$. A martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is deferred Cesàro statistically uniformly integrable if, for each $\epsilon > 0$ there is a number $K_\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{u_n - v_n} \left| \left\{ i : u_n < i \leqq v_n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} \mid X_n \mid dP \right| \geqq \epsilon \right\} \right| = 0.$$

We write

$$\text{DUI}_{\text{stat}} \lim_{n\to\infty} \int_{\{|X_n| \geqq K_\epsilon\}} \mid X_n \mid dP < \epsilon.$$

Definition 3.3. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$. A martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically deferred Cesàro uniformly summable if, for each $\epsilon > 0$ there is a number $K_\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ i : i \leqq n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | \mathcal{D}(X_n, \mathcal{F}_n) | \, dP \right| \geqq \epsilon \right\} \right| = 0.$$

We write

$$\text{stat}_{\text{DUI}} \lim_{n\to\infty} \int_{\{|\mathcal{D}(X_n,\mathcal{F}_n)| \geqq K'_{\epsilon_1}\}} | \mathcal{D}(X_n, \mathcal{F}_n) | \, dP < \epsilon.$$

Now we establish an inclusion theorem concerning these two new fascinating notions, that every deferred Cesàro statistically uniformly integrable martingale sequence is statistically deferred Cesàro uniformly summable but it is not generally true in the converse sense.

Theorem 3.3. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$, and let a martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ is statistically uniformly integrable, then it is deferred Cesàro statistically uniformly integrable.

Proof. Let a martingale sequence $(X_n, \mathcal{F}_n, n \in \{0\} \cup \mathbb{N})$ be statistically uniformly integrable. Then, for each $\epsilon > 0$ there is a number $K_\epsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ i : i \leqq n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP \right| \geqq \epsilon \right\} \right| = 0.$$

As the given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is bounded with probability 1, then for every $\epsilon > 0$ and there exists $K_\epsilon > 0$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{v_n - u_n} \left| \left\{ i : u_n < i \leqq v_n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP \right| \geqq \epsilon \right\} \right| \\ &\subseteq \lim_{n\to\infty} \frac{1}{n} \left| \left\{ i : i \leqq n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP \right| \geqq \epsilon \right\} \right| = 0. \end{aligned}$$

Consequently, by Definition 3.2, we obtain

$$\lim_{n\to\infty} \frac{1}{v_n - u_n} \left| \left\{ i : u_n < i \leqq v_n \quad \text{and} \quad \left| \int_{\{|X_n| \geqq K_\epsilon\}} | X_n | \, dP \right| \geqq \epsilon \right\} \right| = 0.$$

□

Again, it is not hard to see that, every deferred Cesàro statistically uniformly integrable martingale sequence is statistically deferred Cesàro uniformly summable but it not true in the converse sense. Accordingly, we establish a theorem (below).

Theorem 3.4. If a given martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ is deferred Cesàro statistically uniformly integrable then it is statistically deferred Cesàro uniformly summable, but not conversely.

Proof. In the similar lines of proof of Theorem 3.3, the proof of the Theorem 3.4 can be established. We, thus, choose to skip the details involved.

For the non-validity of sufficient part of the theorem 3.4, here we consider an example (below).

Example 3.1. Let $u_n = 2n+1$, $v_n = 4n+1$, and let (X_n) be a martingale sequence relative to probability measurable functions $(\mathcal{F}_n)$ with mean 0 and $E(X_n)$ is finite, then for n is odd

$$X_n = \begin{cases} 1 & (n = 2^m; m \in \mathbb{N}) \\ \\ n & (\text{otherwise}). \end{cases}$$

and also for n is even

$$X_n = \begin{cases} \frac{1}{2} & (n = 2^m; m \in \mathbb{N}) \\ \\ n & (\text{otherwise}). \end{cases}$$

Here, the martingale sequence $(X_n, \mathcal{F}_n)$ is statistically deferred Cesàro uniformly summable, but not deferred Cesàro statistically uniformly integrable.

□

Corollary 3.1. For any martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$, statistical uniform integrability $\Longrightarrow$ deferred Cesàro statistical uniform integrability $\Longrightarrow$ statistical deferred Cesàro uniform summability. However, the converse is not true.

Proof. The proof of first implication (that is, statistical uniform integrability $\Longrightarrow$ deferred Cesàro statistical uniform integrability) is the consequence of Theorem 3.3, and the proof of second implication (that is, deferred Cesàro statistical uniform integrability $\Longrightarrow$ statistical deferred Cesàro uniform summability) is the direct consequence of Theorem 3.4. □

4 Distribution Convergence of Martingale Sequence

Let $(\Omega, \mathcal{F}, P)$ be a probability measurable space, and let $\mathfrak{F}_{(X_n, \mathcal{F}_n)}(x)$ and $\mathfrak{F}_{X_0}(x)$ be the real-valued continuous cumulative distribution functions of $(X_n, \mathcal{F}_n)$ and X_0, respectively. A given martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ converges in distribution (or converges weakly) to a finite integrable random variable X_0, if

$$\lim_{n\to\infty} \mathfrak{F}_{(X_n, \mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x) \quad (\forall\ x).$$

Next, for more details, it is necessary to introduce the notions of statistical distribution convergence (DC) and deferred Cesàro statistical distribution convergence (DDC) of martingale sequence.

Definition 4.1. Let $\mathfrak{F}_{(X_n, \mathcal{F}_n)}(x)$ be the cumulative distribution functions of a martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$, then it is statistically distribution convergent (or distribution convergent in statistics) to a distribution function $\mathfrak{F}_{X_0}(x)$ of a finite integrable random variable X_0 if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{i : i \leqq n \quad \text{and} \quad |\ \mathfrak{F}_{(X_n, \mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x)\ | \geqq \epsilon\right\}\right| = 0.$$

We write

$$\text{stat}_{\text{DC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n, \mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x).$$

Definition 4.2. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$, and let $\mathfrak{F}_{(X_n, \mathcal{F}_n)}(x)$ be the cumulative distribution functions of a martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$, then it is deferred Cesàro statistically distribution convergent (or deferred Cesàro distribution convergent in statistics) to a distribution function $\mathfrak{F}_{X_0}(x)$ of a finite integrable random variable X_0 if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{v_n - u_n} \left|\left\{i : u_n < i \leqq v_n \quad \text{and} \quad |\ \mathfrak{F}_{(X_n, \mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x)\ | \geqq \epsilon\right\}\right| = 0.$$

We write

$$\text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n, \mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x).$$

From the above definitions, it is necessary that every distribution convergence (DC) in statistic implies the deferred Cesàro distribution convergence (DDC) in statistics; however, the sufficient part is not true. In this context, here we present an example.

Example 4.1. Let $(u_n) = 2n$ and $(v_n) = 4n$ and let $\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)$ be the cumulative distribution functions of a martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ of the form

$$\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \begin{cases} x - \frac{\sin 2n\pi x}{2n\pi} & (0 \leqq x \leqq 1; n = m^2, m \in \mathbb{N}) \\ \\ 0 & (\text{otherwise}). \end{cases}$$

Now by Definition 4.2, we have

$$\text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = 0.$$

Here we observe that, the cumulative distribution function $\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)$ is not distribution convergent (DDC) in statistic; however, it is deferred Cesàro distribution convergent (DDC) in statistics to 0 distribution function of a random variable. Thus, the example is valid for the Definition 4.2, but it is not valid for the Definition 4.1.

Now in view of our Definitions 4.1 and 4.2, we establish a beautiful Theorem concerning a relation between statistical distribution convergence and deferred Cesàro statistical distribution convergence of martingale sequence.

Theorem 4.1. If the sequence

$$\frac{v_n}{v_n - u_n} \leqq \mathcal{K},$$

then

$$\text{stat}_{\text{DC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x) \text{ implies } \text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x),$$

but not conversely.

Proof. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = +\infty$.

Now from the Definition 4.1, we immediately get

$$\frac{1}{n}|\{i : i \leqq n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\}| = 0$$

for every $\epsilon > 0$.

Again, since (u_n) and (v_n) are non-negative integers such that $u_n < v_n$, we have

$$\frac{1}{v_n}|\{i : i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\}| = 0 \quad (n \in \mathbb{N}).$$

Consequently, the inclusion

$$\{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\} \\ \subseteq \{i : i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\}$$

holds.

Thus, the inequality

$$|\{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\}| \\ \leqq |\{i : i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x) - \mathfrak{F}_{X_0}(x)| \geqq \epsilon\}|$$

also holds.

Now by definition of deferred Cesàro distribution convergence in statistics, we have

$$\left(1+\frac{u_n}{v_n-u_n}\right)|\{i : u_n < i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x)-\mathfrak{F}_{X_0}(x)| \geqq \epsilon\}|$$
$$\leqq \frac{1}{v_n-u_n}|\{i : i \leqq v_n \quad \text{and} \quad |\mathfrak{F}_{(X_i,\mathcal{F}_i)}(x)-\mathfrak{F}_{X_0}(x)| \geqq \epsilon\}|.$$

Taking limit on both sides, we get

$$\text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x).$$

□

Next, under suitable condition, we establish and prove the sufficient part of the Theorem 4.1.

Theorem 4.2. Let (u_n) and $(v_n) \in \mathbb{Z}^{0+}$, and let $v_n = n$ for all n. Then $\text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x)$ implies $\text{stat}_{\text{DC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x)$.

Proof. Suppose that $\text{stat}_{\text{DDC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x)$. Then, for each $n \in \mathbb{N}$, we have

$$u_n = n_1 > u_{n_1} = n_2 > u_{n_2} = \cdots,$$

and we may write the set

$$\left\{i : i \leqq n \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$= \left\{i : i \leqq n_1 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$\cup \left\{i : n_1 < i \leqq n \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}.$$

Again the set

$$\left\{i : 1 < i \leqq n_1 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$= \left\{i : i \leqq n_2 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$\cup \left\{i : n_2 < i \leqq n_1 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$

and the set

$$\left\{i : i \leqq n_2 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$= \left\{i : i \leqq n_3 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$\cup \left\{i : n_3 < i \leqq n_2 \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\},$$

and if this process is continued, we get

$$\left\{i : i \leqq n_{t-1} \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$= \left\{i : i \leqq n_t \quad \text{and} \, | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\}$$
$$\cup \left\{t : n_t < i \leqq n_{t-1} \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) |\geqq \epsilon\right\},$$

for a positive integer $t > 0$ depending on n such that $n_t \geqq 1$ and $n_{t+1} = 0$. From the above relations, we have for each n,

$$\begin{aligned}\frac{1}{n}\Big\{i : i \leqq n \quad \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) | \geqq \epsilon\Big\} \\ = \sum_{s=0}^{t} \frac{n_s - n_{s+1}}{n} \frac{1}{n_s - n_{s+1}} \Big| \Big\{ t : n_{s+1} < i \leqq n_s \\ \text{and} \quad | \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) - \mathfrak{F}_{X_0}(x) | \geqq \epsilon \Big\} \Big| .\end{aligned}$$

Consequently, it indicates that $\text{stat}_{\text{DC}} \lim_{n\to\infty} \mathfrak{F}_{(X_n,\mathcal{F}_n)}(x) = \mathfrak{F}_{X_0}(x)$. □

Next, we establish a theorem under a continuous map of distribution functions of a martingale sequence.

Theorem 4.3. Let $(\mathcal{X}, d)$ be a metric space. Suppose the distribution functions $\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)$ of a martingale sequence $(X_n, \mathcal{F}_n; n \in \{0\} \cup \mathbb{N})$ is statistically distribution convergent to a distribution function $\mathfrak{F}_{X_0}(x)$ of a finite integrable random variable $X_0 \in (\mathcal{X}, d)$. Let $(\mathcal{Y}, d)$ be also a metric space such that there is a function $h : (\mathcal{X}, d) \to (\mathcal{Y}, d)$. Define

$$\mathcal{C}_h = \{x : h(x) \text{ is continuous at } x\}.$$

Suppose that $Prob(X_0) = 1$. Then

$$\text{stat}_{\text{DC}} h\left(\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)\right) = h\left(\mathfrak{F}_{X_0}(x)\right).$$

Proof. Let (W_n) be the distribution of $h\left(\mathfrak{F}_{(X_n,\mathcal{F}_n)}\right)$ and let W be the distribution of $h\left(\mathfrak{F}_{X_0}(x)\right)$. Subsequently, (Z_n) be the distribution of $\mathfrak{F}_{(X_n,\mathcal{F}_n)}$ and let Z be the distribution of $\mathfrak{F}_{X_0}$.

Let $\mathcal{B} \subset (\mathcal{Y}, d)$ which is closed, then there exists a sequence

$$\{a_n : n \in \mathbb{N}\} \subseteq h^{-1}(\mathcal{B})$$

such that

$$\text{stat} h(a_n) \to h(a) \in \mathcal{B},$$

since $\mathcal{B}$ is closed and $h(a_n) \in \mathcal{B}$. It implies that

$$\overline{h^{-1}(\mathcal{B})} \subseteq h^{-1}(\mathcal{B}) \cup \mathcal{C}_h^c.$$

We write

$$\begin{aligned}\text{stat}_{\text{DC}} \limsup_{n\to\infty} W_n(\mathcal{B}) &= \text{stat}_{\text{DC}} \limsup_{n\to\infty} Z_n(h^{-1}(\mathcal{B})) \\ &\leqq \text{stat}_{\text{DC}} \limsup_{n\to\infty} Z_n(\overline{h^{-1}(\mathcal{B})}) \leqq Z_n(\overline{h^{-1}(\mathcal{B})}) \\ &\leqq Z(h^{-1}(\mathcal{B})) + Z(\mathcal{C}_h^c) \\ &\leqq Z(h^{-1}(\mathcal{B})) = W(\mathcal{B}).\end{aligned}$$

Thus, we have

$$\text{stat}_{\text{DC}} h\left(\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)\right) = h\left(\mathfrak{F}_{X_0}(x)\right).$$

□

5 Applications of Deferred Cesàro Martingale Sequence

As the application's perspective, here we establish some new approximation of the Korovkin-type results based on different convergence criteria of a martingale sequence of positive linear operators via deferred Cesàro summability mean. Recently, under different settings of positive linear operators, the approximation of Korovkin-type results have been very useful in various fields of mathematics, such as sequence space, Measurable space, Probability space and so on. Here, we like to draw the attention of interested readers to the current works [18], [19] and [20].

Let $C([0,1])$ be the space of all continuous functions (real valued) over $[0,1]$ under the norm $\|.\|_\infty$, and let $C[0,1]$ be a Banach space. Then for $f \in C[0,1]$, the norm of f is given by,

$$\|f\|_\infty = \sup_{x\in[0,1]} \{|f(x)|\}.$$

The operator $\mathfrak{L}$ is called a martingale positive linear operator such that

$$f \geqq 0 \Longrightarrow \mathfrak{L}(f;x) \geqq 0, \text{ with } \mathrm{Prob}(\mathfrak{L}(f;x)) = 1.$$

Here, in view our Definitions 2.1 and 2.2, we have the following results.

Result 1. Let

$$\mathfrak{L}_m : C[0,1] \to C[0,1]$$

be the sequences of martingale positive linear operators, and let $f \in C[0,1]$. Then

$$\mathrm{DMC_{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{5.1}$$

if and only if

$$\mathrm{DMC_{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{5.2}$$

$$\mathrm{DMC_{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{5.3}$$

and

$$\mathrm{DMC_{stat}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{5.4}$$

Result 2. Let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be the sequences of martingale positive linear operators, and let $f \in C[0,1]$. Then

$$\mathrm{stat_{DMC}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{5.5}$$

if and only if

$$\mathrm{stat_{DMC}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{5.6}$$

$$\mathrm{stat_{DMC}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{5.7}$$

and

$$\mathrm{stat_{DMC}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{5.8}$$

Now in view of our Results 1 and 2, we present an example for a martingale sequence of positive linear operators in conjunction with the Bernstein polynomial, that does not fulfill the statistical convergence versions of Korovkin-type approximation Result 1 and also the results of Srivastava *et al.* [21], and Paikray *et al.* [13], but it satisfies the statistical summability versions of our Korovkin-type approximation Result 2. Thus, our Result 2 is a quite stronger approach than the Result 1 and also, the Results of Srivastava *et al.* [21] and Paikray *et al.* [13].

Example 5.1. We consider *Bernstein polynomial* $\mathcal{B}_m(f;\zeta)$ on $C[0,1]$ given by

$$\mathcal{B}_m(f;\zeta)=\sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m}\zeta^m(1-\zeta)^{n-m} \quad (\zeta\in[0,1]).$$

Next, we define martingale positive linear operators by using $\mathcal{B}_m(f;\zeta)$ over $C[0,1]$as follows,

$$\mathfrak{L}_m(f;\zeta)=[1+(X_n,\mathcal{F}_n)]\mathcal{B}_m(f;\zeta) \quad (\forall\ f\in C[0,1]) \tag{5.9}$$

where $(X_n,\mathcal{F}_n)$ already mentioned in Example 2.1.

Now, we calculate the values of all the functions 1, 2ζ and $3\zeta^2$ by using our defined operators (5.9),

$$\mathfrak{L}_m(1;\zeta)=[1+(X_m,\mathcal{F}_m)]\cdot 1$$

$$\mathfrak{L}_m(2\zeta;\zeta)=[1+(X_m,\mathcal{F}_m)]\cdot 2\zeta$$

and

$$\mathfrak{L}_m(3\zeta^2;\zeta)=[1+(X_m,\mathcal{F}_m)]\cdot 3\left\{\zeta^2+\frac{\zeta(1-\zeta)}{m}\right\}$$

so that we have

$$\mathrm{stat}_{\mathrm{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(1;\zeta)-1\|_\infty=0,$$
$$\mathrm{stat}_{\mathrm{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(2\zeta;\zeta)-2\zeta\|_\infty=0$$

and

$$\mathrm{stat}_{\mathrm{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(3\zeta^2;\zeta)-3\zeta^2\|_\infty=0.$$

Consequently, the sequence $\mathfrak{L}_m(f;\nu)$ fulfills the conditions (5.6) to (5.8). Therefore, by Result 2, we have

$$\mathrm{stat}_{\mathrm{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(f;\zeta)-f\|_\infty=0.$$

Here, the given martingale sequence $(X_m,\mathcal{F}_m)$ in Example 2.1 is statistically deferred Cesàro summable, but not statistically deferred Cesàro convergent. Thus, our operators (5.9) satisfy the Result 2; however, it is not satisfying the Result 1.

Moreover, in view of our Definitions 3.2 and 3.3, we have the following Korovkin-type results via our proposed deferred Cesàro mean.

Result 3. Let $\mathfrak{L}_m : C[0,1]\to C[0,1]$ be the sequences of martingale positive linear operators, and let $f\in C[0,1]$. Then

$$\mathrm{DUI}_{\mathrm{stat}}\lim_{n\to\infty}\int_{[0,1]} \mid \mathfrak{L}_m(f;x)-f(x)\mid dP=0 \tag{5.10}$$

if and only if

$$\mathrm{DUI}_{\mathrm{stat}}\lim_{n\to\infty}\int_{[0,1]} \mid \mathfrak{L}_m(1;x)-2x\mid dP=0, \tag{5.11}$$

$$\mathrm{DUI}_{\mathrm{stat}}\lim_{n\to\infty}\int_{[0,1]} \mid \mathfrak{L}_m(2x;x)-2x\mid dP=0 \tag{5.12}$$

and

$$\mathrm{DUI}_{\mathrm{stat}}\lim_{n\to\infty}\int_{[0,1]} \mid \mathfrak{L}_m(3x^2;x)-3x^2\mid dP=0. \tag{5.13}$$

Result 4. Let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be the sequences of martingale positive linear operators, and let $f \in C[0,1]$. Then

$$\text{stat}_{\text{DUI}} \lim_{n\to\infty} \int_{[0,1]} \mid \mathfrak{L}_m(f;x) - f(x) \mid dP = 0 \tag{5.14}$$

if and only if

$$\text{stat}_{\text{DUI}} \lim_{n\to\infty} \int_{[0,1]} \mid \mathfrak{L}_m(1;x) - 2x \mid dP = 0, \tag{5.15}$$

$$\text{stat}_{\text{DUI}} \lim_{n\to\infty} \int_{[0,1]} \mid \mathfrak{L}_m(2x;x) - 2x \mid dP = 0 \tag{5.16}$$

and

$$\text{stat}_{\text{DUI}} \lim_{n\to\infty} \int_{[0,1]} \mid \mathfrak{L}_m(3x^2;x) - 3x^2 \mid dP = 0. \tag{5.17}$$

Now, in view of our Results 3 and 4, we present an example for the uniform integrability of a martingale sequence of positive linear operators in conjunction with the Bernstein polynomial, that does not satisfy the conditions of the deferred Cesàro uniform integrability in statistical versions of Result 3 and also the results of Srivastava *et al.* [21], and Paikray *et al.* [13], but it satisfies the deferred Cesàro uniform summability in statistical versions of our Result 4. Thus, our Result 4 is a quite stronger approach than the Result 3 and also, the Results of Srivastava *et al.* [21] and Paikray *et al.* [13].

Example 5.2. We consider *Bernstein polynomial* $\mathcal{B}_m(f;\xi)$ on $C[0,1]$ given by

$$\mathcal{B}_m(f;\xi) = \sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m}\xi^m(1-\xi)^{n-m} \quad (\xi \in [0,1]).$$

Next, we define martingale positive linear operators by using $\mathcal{B}_m(f;\xi)$ over $C[0,1]$ as follows,

$$\mathfrak{L}_m^{'}(f;\xi) = [1 + (X_n, \mathcal{F}_n)]\mathcal{B}_m(f;\xi) \quad (\forall\ f \in C[0,1]) \tag{5.18}$$

where $(X_n, \mathcal{F}_n)$ already mentioned in Example 3.1.

Now, we calculate the values of all the functions 1, 2ξ and $3\xi^2$ by using our defined operators (5.9),

$$\mathfrak{L}'_m(1;\xi) = [1 + (X_m, \mathcal{F}_m)] \cdot 1$$

$$\mathfrak{L}'_m(2\xi;\xi) = [1 + (X_m, \mathcal{F}_m)] \cdot 2\xi$$

and

$$\mathfrak{L}'_m(3\xi^2;\xi) = [1 + (X_m, \mathcal{F}_m)] \cdot 3\left\{\xi^2 + \frac{\xi(1-\xi)}{m}\right\}$$

so that we have

$$\text{stat}_{\text{DUI}} \lim_{m\to\infty} \int_{[0,1]} \mid \mathfrak{L}'_m(1;x) - 2x \mid dP = 0,$$

$$\text{stat}_{\text{DUI}} \lim_{m\to\infty} \int_{[0,1]} \mid \mathfrak{L}'_m(2x;x) - 2x \mid dP = 0$$

and

$$\text{stat}_{\text{DUI}} \lim_{m\to\infty} \int_{[0,1]} \mid \mathfrak{L}'_m(3x^2;x) - 3x^2 \mid dP = 0.$$

Consequently, the sequence $\mathfrak{L}'_m(f;\xi)$ fulfills the conditions (5.15) to (5.17). Therefore, by Result 4, we have

$$\text{stat}_{\text{DUI}} \lim_{m\to\infty} \int_{[0,1]} \mid \mathfrak{L}'_m(f;x) - f(x) \mid dP = 0$$

Here, the given martingale sequence $(X_m, \mathcal{F}_m)$ in Example 3.1 is statistically deferred Cesàro uniformly summable, but not statistically deferred Cesàro statistically uniformly integrable. Thus, our operators (5.18) satisfy the Result 4; however, it is not satisfying the Result 3.

Likewise, in view of the Definitions 4.1 and 4.2, we present the following results.

Result 5. Let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be the sequences of martingale positive linear operators, and let $f \in C[0,1]$. Then

$$\text{stat}_{\text{DC}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{5.19}$$

if and only if

$$\text{stat}_{\text{DC}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{5.20}$$

$$\text{stat}_{\text{DC}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{5.21}$$

and

$$\text{stat}_{\text{DC}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{5.22}$$

Result 6. Let $\mathfrak{L}_m : C[0,1] \to C[0,1]$ be the sequences of martingale positive linear operators, and let $f \in C[0,1]$. Then

$$\text{stat}_{\text{DDC}} \lim_{m\to\infty} \|\mathfrak{L}_m(f;x) - f(x)\|_\infty = 0 \tag{5.23}$$

if and only if

$$\text{stat}_{\text{DDC}} \lim_{m\to\infty} \|\mathfrak{L}_m(1;x) - 1\|_\infty = 0, \tag{5.24}$$

$$\text{stat}_{\text{DDC}} \lim_{m\to\infty} \|\mathfrak{L}_m(2x;x) - 2x\|_\infty = 0 \tag{5.25}$$

and

$$\text{stat}_{\text{DDC}} \lim_{m\to\infty} \|\mathfrak{L}_m(3x^2;x) - 3x^2\|_\infty = 0. \tag{5.26}$$

Next for our Results 5 and 6, we present an example for the distribution functions with a martingale sequence in conjunction with the Bernstein polynomial, that does not satisfy the conditions of the distribution convergence (DC) in statistical versions of Result 5 and also the results of Srivastava *et al.* [21], and Paikray *et al.* [13], but it satisfies the deferred Cesàro distribution convergence (DDC) in statistical versions of our Result 6. Thus, our Result 6 is quite a stronger approach than the Result 5 and also, the Results of Srivastava *et al.* [21] and Paikray *et al.* [13].

Next in view of our Example 5.3, consider the operator introduced by Al-Salam [1] (also, see [23])

$$\nu(1+\nu D) \quad \left(D = \frac{d}{d\nu}\right).$$

Example 5.3. We consider *Bernstein polynomial* $\mathcal{B}_m(f;\nu)$ on $C[0,1]$ given by

$$\mathcal{B}_m(f;\nu) = \sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m}\nu^m(1-\nu)^{n-m} \quad (\nu \in [0,1]).$$

Next, we define the martingale sequences by using $\mathcal{B}_m(f;\nu)$ and $\nu(1+\nu D)$ operators over $C[0,1]$ as follows,

$$\mathfrak{L}_m^*(f;\nu) = [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu(1+\nu D)\mathcal{B}_m(f;\nu) \quad (\forall\ f \in C[0,1]) \tag{5.27}$$

where $\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)$ mentioned in Example 4.1.

Now, we calculate the values of the each functions 1, 2ν and $3\nu^2$ by using our proposed operators (5.27),

$$\mathfrak{L}_m^*(1;\nu) = [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu(1+\nu D)1 = [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu,$$

$$\mathfrak{L}_m^*(2\nu;\nu) = [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu(1+\nu D)2\nu = [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu(1+2\nu),$$

and

$$\begin{aligned}\mathfrak{L}_m^*(3\nu^2;\nu) &= [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\nu(1+\nu D)3\left\{\nu^2 + \frac{\nu(1-\nu)}{m}\right\}\\ &= [1+\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)]\left\{\nu^2\left(6-\frac{9\nu}{m}\right)\right\},\end{aligned}$$

so that we have

$$\begin{aligned}&\mathrm{stat_{DDC}}\lim_{m\to\infty}\|\mathfrak{L}_m^*(1;\nu)-1\|_\infty = 0,\\ &\mathrm{stat_{DDC}}\lim_{m\to\infty}\|\mathfrak{L}_m^*(2\nu;\nu)-2\nu\|_\infty = 0\end{aligned}$$

and

$$\mathrm{stat_{DDC}}\lim_{m\to\infty}\|\mathfrak{L}_m^*(3\nu^2;\nu)-3\nu^2\|_\infty = 0.$$

Consequently, the sequence $\mathfrak{L}_m^*(f;\nu)$ satisfies the conditions (5.24) to (5.26). Therefore, by Result 6, we have

$$\mathrm{stat_{DDC}}\lim_{m\to\infty}\|\mathfrak{L}_m^*(f;\nu)-f\|_\infty = 0.$$

Here, the given continuous distribution functions of martingale sequence $\mathfrak{F}_{(X_n,\mathcal{F}_n)}(x)$ in Example 4.1 is deferred Cesàro statistically distribution summable but not deferred Cesàro distribution convergent in statistic. Thus, our operators defined by (5.27) satisfies the Result 6; however, it is not satisfying the Result 5.

6 Remarkable Conclusion and Observations

In this remarkable segment of our investigation, we establish a few further remarks and observations with reference to several results which we have illustrated here.

Remark 6.1. Suppose $(X_n,\mathcal{F}_n;n\in\mathbb{N})$ be a martingale sequence as already mentioned in Example 2.1. Then, since

$$\mathrm{stat_{DMC}}\lim_{m\to\infty}(X_n,\mathcal{F}_n;n\in\mathbb{N}) = \frac{1}{2} \quad \text{on} \quad [0,1],$$

we have

$$\mathrm{stat_{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(f_j;x)-f_j(x)\|_\infty = 0 \quad (j=0,1,2).$$

So, by Result 2, we write

$$\mathrm{stat_{DMC}}\lim_{m\to\infty}\|\mathfrak{L}_m(f;x)-f(x)\|_\infty = 0,$$

where

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2.$$

Here, the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is neither convergent in statistic nor ordinary convergent; thus, the usual and statistical versions of approximation of the Korovkin-type results do not work for the already mentioned operators (5.9). Hence, this observation indicates that the Result 2 is a non-trivial extension (or generalization) of the usual and statistical versions of results (see [5] and [12]). Moreover, the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is not deferred Cesàro statistically convergent but it is statistically deferred Cesàro summable. Thus, Result 2 is certainly a non-trivial extension of Result 1. Therefore, Result 2 is quite stronger than Result 1.

Remark 6.2. Suppose $(X_n, \mathcal{F}_n)$ be a martingale sequence already mentioned in Example 3.1. Then, since

$$\text{stat}_{\text{DUI}} \lim_{m \to \infty} (X_n, \mathcal{F}_n) < \epsilon \;\text{ on }\; [0, 1],$$

so we have

$$\text{stat}_{\text{DUI}} \lim_{m \to \infty} \int_{[0,1]} \mid \mathfrak{L}'_m(f_j; x) - f(x) \mid dP = 0 \quad (j = 0, 1, 2).$$

So, by Result 4, we write

$$\text{stat}_{\text{DUI}} \lim_{m \to \infty} \int_{[0,1]} \mid \mathfrak{L}'_m(f; x) - f(x) \mid dP = 0$$

where

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2.$$

Here, the martingale sequence $(X_n, \mathcal{F}_n)$ is neither uniformly integrable in statistic nor simply uniformly integrable; thus, the martingale sequence (that is, usual and statistical versions) of approximation of the Korovkin-type results do not work for the already mentioned operators (5.18). Hence, this observation indicates that the Result 4 is a non-trivial generalization of the usual and statistical versions of uniformly integrable Korovkin-type results (see [12] and [22]). Moreover, the martingale sequence $(X_n, \mathcal{F}_n)$ is deferred Cesàro statistically uniformly integrable but not statistically deferred Cesàro uniformly summable. Therefore, the Result 4 is a non-trivial generalization of Result 3.

Remark 6.3. Suppose $(\mathfrak{F}_{(X_n, \mathcal{F}_n)}(x))$ be the distribution functions of martingale sequence mentioned in Example 4.1. Then, since

$$\text{stat}_{\text{DDC}} \lim_{m \to \infty} (\mathfrak{F}_{(X_n, \mathcal{F}_n)}(x)) = 0 \;\text{ on }\; [0, 1],$$

so we have

$$\text{stat}_{\text{DDC}} \lim_{m \to \infty} \|\mathfrak{L}^*_m(f_j; x) - f_j(x)\|_\infty = 0 \quad (j = 0, 1, 2).$$

So, by Result 6, we write

$$\text{stat}_{\text{DDC}} \lim_{m \to \infty} \|\mathfrak{L}^*_m(f; x) - f(x)\|_\infty = 0,$$

where

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2.$$

Here $(X_n, \mathcal{F}_n; n \in \mathbb{N})$, the distribution functions is neither distribution convergent in statistic nor usual distribution convergent; thus, the distribution convergence (usual and statistical versions) of approximation of the Korovkin-type results do not work for the already mentioned operators (5.27). Hence, the observation indicates that the Result 6 is a non-trivial generalization of the usual and statistical versions distribution convergence of Korovkin-type results (see [3] and [12]). Furthermore, the distribution $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ functions is not statistical distribution convergent, but it is deferred Cesàro statistical distribution convergent. Therefore, the Result 6 is a non-trivial generalization of our Result 5.

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