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On the Inversion and Dimension Pairs of Row-Strict Tableaux

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Abstract: *In this article, we consider two algorithms, dimension and inversion pairs of rows-strict, used for the computation of Betti numbers of Springer varieties and then show that the sequences respectively generated by these algorithms are dual to each other, (except for $\lambda = 1^n$ where $I_k = D_k$) and that the sum $I_k + D_k$ gives another sequence which is palindromic. We also show that for each row-strict tableau τ of shape $\lambda = n - r, 1^r (0 \leq r \leq n-1)$, the dimension of the corresponding Springer varieties equals the cardinality of the union of the set of inversions and dimensions of τ.* This research contributes to a deeper understanding of the rich combinatorial landscape of tableaux, opening up new avenues for further research.

Keywords: Flag variety, Springer variety, Standard tableaux, Row-strict tableaux, Betti numbers

1 Introduction

The full flag variety $\mathcal{F}\ell_n(\mathbb{C})$ over the general linear group $GL_n(\mathbb{C})$, is the collection of sequences (V_i) of subspaces of $n-$dimensional complex vector space ordered by inclusions [8]. That is,

$$\mathcal{F}\ell_n(\mathbb{C}) = \{V_\bullet : V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n\} \tag{1.1}$$

such that $dimV_i = i$ for each i.

There is a family of sub-varieties of $\mathcal{F}\ell_n(\mathbb{C})$ parameterized by a linear operator X and a non-decreasing function $h : [n] \longrightarrow [n]$ (called Hessenberg function), such that $h(i) \geq i$ for all i. These subvarieties are called Hessenberg varieties $H(X, h)$ and defined as

$$H(X, h) = \{V_\bullet \in \mathcal{F}\ell_n(\mathbb{C}) : XV_i \subset V_{h(i)} \; \forall \, i\}. \tag{1.2}$$

For example, if $h(i) = n$ for all i and X is principal nilpotent, then $H(X, h)$ becomes the full flag varieties, if $h(i) = i + 1$, $(1 \leq i \leq n-1)$ and X is regular nilpotent, then $H(X, h)$ gives rise to Peterson varieties, if $h(i) = i$ for all i and X is a nilpotent matrix in Jordan form of type λ (Where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ is a partition of n determined by the Jordan blocks of X), $H(X, h)$ yields Springer varieties.
In the late 80s, Hessenberg varieties introduced by De Mari and Shayman, in order to efficiently compute the eigenvalues and eigenspace of a linear operator in question related to numerical analysis. Ever since the introduction, Hessenberg varieties have been objects of current research in combinatorics, geometry, representation theory and topology, see [1], [2], and [7]. In [11], Tymoczko showed that Hessenberg varieties are not always pure dimensional. Jordan [6] proved that the QR-algorithm restricted on the subset of Hessenberg varieties are generically controllable. In [10], Tymoczko used dimension pairs of row strict tableaux to compute the Betti numbers of cohomology of Springer varieties in type A, Fresse [5] use an algorithm called Inversion, and determined the Betti numbers of cohomology of Springer varieties in type A by constructing a cell decomposition of Springer varieties, the number of inversions $|inv|_{(\tau)}$ for each row strict tableau τ were computed and hence, the k^{th} Betti number $b_k = dimH^{2k}(Spr_\lambda, \mathbb{Z})$ which is the number of row-strict tableaux with $d - k$ inversions, (where $d = dim(Spr_\lambda)$).

By row strict tableaux τ of shape λ, we mean a filling of a Young diagram from $[n]$ such that, numbers in the rows increase from left to right. For example, say

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}.$$

However, it is referred to as a conventional Young tableaux and is occasionally denoted by T if the filling of such a τ grows from top to bottom and from left to right. We define $(rst)^{\lambda}$ to be the set of all row strict tableaux corresponding to λ.

Since it is possible to explore the topological properties of a geometrical object via two different algorithms on the same combinatorial, hence, deem it fit to investigate the two algorithms in [10] and [5] with the intention to see the connections between these two algorithms.

Theorem 1.1. *The sequence $I_k(\lambda)$ is dual to the sequence $D_k(\lambda)$ for all partitions of n, except when $\lambda = 1^n$, where $I_k(\lambda) = D_k(\lambda)$ is the number of row strict tableaux of shape λ with k inversions and $D_k(\lambda)$ is the number of row strict of shape tableaux λ with k dimensions.*

1.1 Springer varieties Spr_{λ}

Of important interest to us is the set of flags $V_{\bullet} \in \mathcal{F}\ell_n(\mathbb{C})$ stabilized by a nilpotent operator X in Jordan form of type λ, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l) \vdash n$ is a partition of n, determined by the Jordan blocks of X, such that $\sum_{i=1}^{l} \lambda_i = n$.

Example 1.2. Let $n = 6$ and $\lambda = 3, 2, 1$, then $X = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 1 & \\ & & & 0 & 0 & \\ & & & & & 0 \end{bmatrix}$.

Definition 1.3. Let $X : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a Linear operator in Jordan form of Type λ. We define Springer variety $Spr_{\lambda} = \{V_{\bullet} \in \mathcal{F}\ell_n(\mathbb{C}) : XV_i \subseteq V_i,\ 1 \leq i \leq n\}$.

Spr_{λ} is a subvariety of $\mathcal{F}\ell_n(\mathbb{C})$ which is also called Springer fiber as it coincides with the fiber over X of the Springer resolution [9].

The structure of Spr_{λ} depends on the form of nilpotent operator X. For instance, if $\lambda = n$, X has only one Jordan block, then, Spr_{λ} consists the single flag $V_0 \subset V_2 \subset \cdots \subset V_n$, where $V_i = ker(X^i)$. At the other extreme, if $\lambda = 1^n$, $X = 0$, here, Spr_{λ} coincides with the whole full flag .

Remark 1.4. Springer varieties are irreducible in the two extremes described above, but are reducible to irreducible components for every other case. For the sake of clarity, we shall henceforth denote by X_{λ}, the nilpotent operator and the corresponding variety as Spr_{λ}, where λ is clear.

Theorem 1.5. *[5]*
For a given partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$, the dimension d of Spr_{λ} is combinatorially expressed as

$$d = \sum_{i=1}^{k} \frac{\lambda'_1(\lambda'_i - i)}{2}, \tag{1.3}$$

where $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_k)$ is the conjugate partition of λ. That is, λ'_i is the number of $\lambda_j \geq i$.

Example 1.6. Let $n = 6$ and $\lambda = (2, 2, 1, 1)$, $\lambda' = (4, 2)$, so, $d = 7$.

1.1.1 Cell decomposition of Springer varieties

It was shown in [5] that the variety Spr_{λ} admits a cell decomposition parameterized by row-strict tableaux of shape λ.

A more convenient way to construct a cell decomposition of Spr_{λ} as described in [10] is to take the intersection of Spr_{λ} with the Schubert cells of the flag variety with the condition that the Borel subgroup B is carefully chosen. The connection between the two descriptions above is revealed in theorem 3.2 of [12].

Remark 1.7. One of the most celebrated result on Young tableaux is the hook length formula, which we explain in the sequel.

The hook length h_{ij} of a given box (ij) in a frame of a young diagram of shape λ is the length of the right-angled path in the frame with that box as the upper left vertex (i labels rows j labels column).

For instance, the hook length of the asterisked box in Figure 1 is 9 (i.e. $h_{2,1} = 9$)

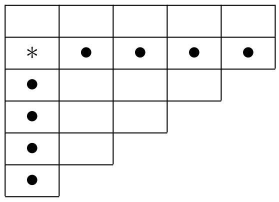

Figure 1: Hook length of $h_{2,1}$.

Hook length formula. If λ is a Young diagram with n boxes, then the number f^λ of standard Young tableaux of shape λ is $n!$ divided by the product of the hook lengths of the boxes, i.e.,

$$f^\lambda = \frac{n!}{\prod h_{(i,j)}}. \tag{1.4}$$

This article contains two parts. In section 2, we give a deep comparison of the two algorithms: inversion and dimension pairs, section 3 contains details of our main results.

2 Methods: Comparison of Inversion and Dimension Pairs of Row Strict Tableaux τ

In this section, we compare the dimension and inversion pairs of row-strict tableaux, for all partitions of $n \geq 3$.

Definition 2.1. [10]
Let $\tau \in (rst)^\lambda$, a pair (i, j) is said to be a dimension pair of τ if it satisfies:

1. $i < j$;
2. j is below i and in the same column, or located anywhere at the left of i;
3. If i immediately bordered on the right by c then $j \leq h(c)$, where $h(c)$ is the value of Hessenberg function at c.

We shall denote the set of all such pairs in τ by $(DP)^\tau$.

Example 2.2. Let $h = (1, 2, 3, 4, 5)$ and

$$\tau = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 3 & 6 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array}$$

$(DP)^\tau = \{(1, 2), (1, 3), (5, 6)\}$

Remark 2.3. $(DP)^\tau = \{\}$ if and only if entries of τ are such that, they decrease from top to bottom for each column. For instance

$$(DP)^{\begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array}} = \{\}.$$

There is only one of such $\tau \in (rst)^\lambda$, such fillings of τ is referred to as base filling by Tymoczko in [10]. It is equally noted that the dimension pairs of $\tau \in (rst)^\lambda$ is maximum if τ is a standard tableaux.

Theorem 2.4. *[10] The dimension of $H^{2k}(Spr_{X_\lambda})$ is the number of $\tau \in (rst)^\lambda$ such that τ has k dimension pair(s).*

Theorem 2.5. *[10] Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ and h_i be the length of i^{th} column of any tableau τ of shape λ. Then, the maximum number of dimension pairs for any $\tau \in (rst)^\lambda$ is*

$$\sum_i \frac{h_i(h_i - 1)}{2}.$$

Moreover, this maximum number of dimension pairs is realized by precisely all $\left(\frac{n!}{\prod h_{(ij)}}\right)$ standard Young tableaux of shape λ.

In a similar approach, Fresse [5] gave a notion of dimension of a multi-column tableaux as follows :

Definition 2.6. A pair (i, j) such that $i < j$ on the same column, is said to be an inversion pair of τ if it satisfies one of the following:

1. either i or j lacks entry to its right and i is below j,
2. i is bordered immediately on its right by r_i , j is bordered immediately on its right by r_j and $r_i > r_j$

We denote the set of all inversion pairs of τ by $(Inv)^\tau$

Example 2.7. Let

$$\tau = \begin{array}{|c|c|c|} \hline 3 & 5 & 6 \\ \hline 1 & 4 & \\ \cline{1-2} 2 & & \\ \cline{1-1} \end{array}, \ (Inv)^\tau = \{(2, 3), (4, 5)\}.$$

Lucas [5] established that $\tau \in (rst)^\tau$ is standard if $(Inv)^\tau = \{\}$ and that, the definition of inversion on row-strict tableaux specializes to the notion of permutation inversion if we represent $\pi \in S_n$ by single-column tableau whose entries appear in the order $\pi(1), \cdots, \pi(n)$.
It was argued in [5] that, given $\tau \in (rst)^\lambda$ it is possible to reorder entries in each column of τ to produce a unique standard Young tableau. This standard Young is referred to as the standardization of τ, which we denote by $stdz(\tau)$.

Theorem 2.8. *[5] Let $\lambda \vdash n$ and consider Spr_{X_λ}. if $dim(Spr_{X_\lambda}) = d$, then the k^{th} Betti numbers $b_k = dim(H^{2k}(Spr_{X_\lambda}, \mathbb{Q}))$ equals the number of $\tau \in (rst)^\lambda$ with $|(Inv)^\tau| = d - k$.*

Remark 2.9. To each standard Young tableaux T, there are collection of row-strict tableaux τ obtained by permuting entries in each column of T such that the standardization of τ is T. We denote the collection of all such τ by $Inv(T) = \{\tau : stdz(\tau) = T\}$.

3 Results

Now, let $I_k(\lambda)$ be the number of $\tau \in (rst)^\lambda$ with k inversion(s) and $D_k(\lambda)$ the number of $\tau \in (rst)^\lambda$ with k dimension pair(s), $(0 \leq k \leq d)$.

Theorem 3.1. *The sequence $I_k(\lambda)$ is dual to the sequence $D_k(\lambda)$ for all partitions of n, except when $\lambda = 1^n$ where $I_k(\lambda) = D_k(\lambda)$.*

Proof. Let $I_{max}(\lambda)$ be the number of $\tau \in (rst)^\lambda$ with maximum inversions and $I_{min}(\lambda)$ be the number of $\tau \in (rst)^\lambda$ with minimum inversion(s). Doing the same for dimension pairs, we have $D_{max}(\lambda)$ and $D_{min}(\lambda)$. By computation $I_{max}(\lambda) = 1 = I_d(\lambda)$, where d is the dimension of Spr_λ.
Since $(Inv)^\tau = \{\}$ if τ is standard, then $I_{min}(\lambda) = \frac{n!}{\prod h_{(i,j)}}$. At the other end, $D_{max}(\lambda) = \frac{n!}{\prod h_{(i,j)}} = D_d$. From remark 2.3, we have that $D_{min}(\lambda) = 1$.

Following Theorem 2.4 and Theorem 2.5, we define a map

$$\phi : I_{max-k}(\lambda) \longrightarrow D_{min+k}(\lambda), \ 0 \leq k \leq d. \tag{3.1}$$

For $\lambda = (1^n), I_{max-k}(\lambda) = D_{min+k}(\lambda)$. ☐

We consider the cases of $\lambda \vdash n$ for $n \geq 3$, since the examples of $n = 1$ and $n = 2$ are trivial cases.

Example 3.2. (i) For $\lambda = (n)$, there is nothing to compute since there is only one row strict tableau of shape λ in this case.
(ii) For $n = 3$ and $\lambda = (2, 1)$ we have,

τ	Inversion Pairs $(Inv)^T$	Dimension Pairs $(DP)^T$
$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\{\}$	$\{(2,3)\}$
$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\{\}$	$\{(1,2)\}$
$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$	$\{(1,2)\}$	$\{\}$

Table 1: Table of inversion and dimension pairs for $\lambda = (2, 1)$

(iii) For $n = 3$ and $\lambda = (2, 1)$, we obtain for the corresponding I_k and D_k

k	$I_k(\lambda)$	$D_k(\lambda)$
0	2	1
1	1	2

Table 2: Table of values for I_k and D_k corresponding to $\lambda = (2, 1)$

τ	Inv Pairs $(Inv)^T$	Dim Pairs $(DP)^T$
$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	$\{\}$	$\{(1,2),\ (1,3),\ (2,3)\}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$	$\{(2,3)\}$	$\{(1,2),\ (1,3)\}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array}$	$\{(1,2)\}$	$\{(1,3),\ (2,3)\}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}$	$\{(1,2)\ (1,3)\}$	$\{(2,3)\}$
$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}$	$\{(1,3)\ (2,3)\}$	$\{(1,2)\}$
$\begin{array}{\|c\|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$	$\{(1,2)\ (1,3),\ (2,3)\}$	$\{\}$

Table 3: Table of inversion and dimension pairs $\lambda = (1, 1, 1)$

k	$I_k(\lambda)$	$D_k(\lambda)$
0	1	1
1	2	2
2	2	2
3	1	1

Table 4: Table of values for I_k and D_k corresponding to $\lambda = (1, 1, 1)$

Continuing this way, we have values for all partitions of 4 as follows:

	$\lambda = (3, 1)$	$\lambda = (2, 2)$	$\lambda = (2, 1, 1)$	$\lambda = (1, 1, 1, 1)$
k	0, 1	0,1,2	0,1,2,3	0,1,2,3,4,5,6
I_k	3,1	2,3,1	3,5,3,1	1,3,5,6,5,3,1
D_k	3,1	1,3,2	1,3,5,3	1,3,5,6,5,3,1

Table 5: For all partitions of 4 and the corresponding I_k and D_k

Also, for all partitions of 5, we have

	$\lambda = (4, 1)$	$\lambda = (3, 2)$	$\lambda = (3, 1, 1)$	$\lambda = (2, 2, 1)$	$\lambda = (2, 1, 1, 1)$	$\lambda = (1, 1, 1, 1)$
k	0,1	0,1,2	0,1,2,3	0,1,2,3,4	0,1,2,3,4,5,6	0,1,2,3,4,5,6,7,8,9,10
I_k	4,1	5,4,1	6,9,4,1	5,11,9,4,1	4,11,16,15,9,4,1	1,4,9,15,20,,15,9,4,1
D_k	1,4	1,4,5	1,4,9,6	1,4,9,11,5	1,4,9,15,16,11,4	1,4,9,15,20,15,9,4,1

Table 6: For all partitions of 5 and the corresponding I_k and D_k

Corollary 3.3. *The sum $I_k(\lambda) + D_k(\lambda)$ gives a palindromic sequence A_k.*

Proof. $I_{max-k}(\lambda) + D_{max-k}(\lambda) = I_{min+k}(\lambda) + D_{min+k}(\lambda)$. □

Example 3.4. Let $\lambda = 2, 1, 1, 1$. Then $I_k + D_k = 5, 15, 25, 30, 25, 15, 5$.

The next Theorem shows the equality between d and $|(inv)^\tau \cup (DP)^\tau|$.

Theorem 3.5. *Let X be a nilpotent operator whose Jordan type is given by the partition $\lambda = (n-r, 1^r), (1 \leq r \leq n-1)$. For any $\tau \in (rst)^\lambda$, the cardinality of the union of the set of inversion and dimension pairs of τ equals the dimension of the corresponding Springer variety, i.e.,*

$$d = |(inv)^\tau \cup (DP)^\tau|.$$

Proof. We prove by induction.
If $r = 0$, then $\lambda = (n)$. This is a trivial case as there exist only one row strict tableaux with zero inversion and dimension pair, and the Springer variety corresponding to this form of partition is of dimension zero.
Now let $r = 1$, then $\lambda = (n-1, 1)$. In this case, $d = 1$.
By computation, there are n row strict tableaux of shape $\lambda = (n-1, 1)$, $n-1$ of which are standard tableaux and are of the form: Since they are standard tableaux, then $(inv)^T = \{\}$ and

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & n \\ \hline 2 \\ \cline{1-1}\end{array}\;,\quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline 3 \\ \cline{1-1}\end{array}\quad \cdots \quad \begin{array}{|c|c|c|c|}\hline 1 & & \cdots & n \\ \hline n-1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|c|c|}\hline 1 & & \cdots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$$

$$(DP)^T = \{(1, 2)\}, \{(2, 3)\}, \{(3, 4)\}, \cdots \{(n-1, n)\}.$$

Hence $|(DP^T)| = 1$ for each $T \in (rst)^{(n-1,1)}$, for the only row-strict tableau τ which is not a standard Young tableaux is of the form $\begin{array}{|c|c|c|c|}\hline 2 & 3 & \cdots & n \\ \hline 1 \\ \cline{1-1}\end{array}$ has $(1, 2)$ as its only inversion pair and is of zero dimension pair. Therefore,

$$|(inv)^\tau \cup (DP)^\tau| = 1 = d,$$

where d is the dimension of Spr_λ.

If $r = 2$ then $\lambda = n-2, 1^2$. Here, there are $n(n-1)$ row-strict tableaux of shape λ out of which $\frac{(n-1)(n-2)}{2}$ are standard. We recall that all standard tableaux of any shape μ have equal number of inversion and dimension pair(s) which are zero and

$$\sum_{i=i}^{r} \frac{\lambda'_r(\lambda'_r - 1)}{2} = d \tag{3.2}$$

respectively. Hence, for all standard tableaux of shape λ,

$$|(inv)^\tau \cup (DP)^\tau| = \sum_{i=i}^{r} \frac{\lambda'_r(\lambda'_r - 1)}{2} = d. \tag{3.3}$$

For the remaining row-strict tableaux τ of the same shape which are not standard, we focus only on the first column to compute their inversion pairs.Doing this, we follow the definition of standardization of row-strict tableaux in [4] and let

$$Inv(T) = \{\tau \in (rst)^\lambda : stdz(\tau) = T\}. \tag{3.4}$$

Now if a pair of entries of T are swapped to obtain τ, then

$$|(inv)^\tau| = 1. \tag{3.5}$$

and

$$(DP)^\tau = \sum_{i=i}^{r} \frac{\lambda'_r(\lambda'_r - 1)}{2} - 1 = d - 1. \tag{3.6}$$

Therefore, if i pairs of entries of T are swapped to obtained τ, then

$$|(inv)^\tau| = i \tag{3.7}$$

and

$$(DP)^\tau = \sum_{i=i}^{r} \frac{\lambda'_r(\lambda'_r - i)}{2} - i = d - i. \tag{3.8}$$

Hence,

$$|(inv)^\tau \cup (DP)^\tau| = \sum_{i=i}^{r} \frac{\lambda'_r(\lambda'_r - 1)}{2} = d. \tag{3.9}$$

For $r = k, \lambda = (n-k, 1^k)$, there are $n(n-1)(n-2)\cdots(n-k+1)$ row-strict tableaux of shape λ, out of which

$$\frac{n(n-1)(n-2)\cdots 1}{n(n-k-1)(n-k-2)\cdots(k)(k-1)\cdots 1}$$

are standard tableaux. Following equations (3.5) to (3.9) for the argument of $\lambda = (n-2, 1^2)$, it is true for $r = k$.

Now for $k = n-1$, then $\lambda = 1^n$, (here, the Springer variety corresponding to λ is the full flag variety). There exist $n!$ row-strict tableaux of shape λ out of which only one is the standard tableaux, with $|(inv)^T| = 0$ and

$$(DP)^T = \frac{n(n-1)}{2}. \tag{3.10}$$

As usual, the number of pairs of entries of T swapped to obtain τ determines the number $|(inv)^\tau|$ and $|(DP)^\tau|$. Therefore $|(inv)^\tau \cup (DP)^\tau| = d$.

□

Corollary 3.6. *For any* $\tau \in (rst)^{\lambda},\ \lambda = n-r, 1^r, (inv)^{\tau} \cap (DP)^{\tau} = \phi.$

It follows easily from the definition of inversion and dimension pairs of τ.

4 Discussion

Our findings demonstrate a strong relationship between dimension pairs in row-strict tableaux and inversion. This implies a close relationship between the algebraic characteristics of the relevant representations and the combinatorial structure of tableaux. The finding has broad ramifications as it has possible applications in computer science, algebraic geometry, representation theory, and geometry. Additional linkages between tableaux and representation theory may be investigated in future studies, which could yield new knowledge and resources in this area.

5 Conclusion

Finally, our study shows that dimension pairs of row-strict tableaux and inversion have a strong relationship. This work advances our knowledge of the algebraic and combinatorial structures that underlie representation theory. We hope that this research will stimulate more studies in this field and result in fresh insights into the complex connections between algebraic geometry, tableaux, and representations. .

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