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On I-Convergence Difference Sequence Spaces Defined by Orlicz Function in 2−Normed Space

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Abstract: *The difference sequence spaces of type I-convergent, I-null, bounded I-convergent, and bounded I-null in 2-normed space are introduced and studied using the Orlicz function. Investigations into the pertinent characteristics of these spaces led us to the establishment of some inclusion relations.*

Keywords: Orlicz function, 2-Normed space, Difference sequence space, Ideal convergence

1 Introduction and Preliminaries

In the development of calculus and some other branches of mathematics, infinite sequences and series played an important role. Numerous disciplines of mathematics have been studied using infinite sequences and series. However, the mathematicians were facing the problem of calculating the limits of infinite series, in particular with those having divergent in behavior. Then mathematicians developed the various types of convergence to assign a limit, in some sense, to divergent series and sequences. In this work, we introduce and study the sequence spaces defined by Orlicz function and ideal convergence. We recall the definitions and notations used in this work before moving on to the main results.

Definition 1.1. [9] A function $M : [0, \infty) \longrightarrow [0, \infty)$ is said to be an Orlicz function if it is continuous, convex and non decreasing with the conditions $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M can be represented in the following integral form

$$M(x) = \int_0^x q(t)dt,$$

where q is known as the kernel of Orlicz function M which is right- differentiable for $t \geq 0$, $q(0) = 0, q(t) > 0$ for $t > 0$, q is non decreasing, and $q(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Definition 1.2. [9] An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$, for all $x \geq 0$. The Δ_2-condition is equivalent to the satisfaction of the inequality $M(Qx) \leq KQM(x)$ for all values of x for which $Q > 1$.

Orlicz sequence spaces are one of the natural generalizations of classical sequence spaces l_p. They were first considered by Orlicz in 1936. Afterwards, Lindenstrauss and Tzafriri [10] used Orlicz function in order to construct Orlicz sequence space l_M given by

$$l_M = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

of scalars (x_k). The first detailed study on Orlicz spaces was given by Krasnosel'skiî and Rutickiî [9]. The Orlicz sequence space l_M becomes a Banach space with the norm given by

$$||x||_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Moreover, l_M is closely related to the space l_p with $M(x) = x^p; 1 \leq p < \infty$.

Definition 1.3. [2] Let X be a vector space with $\dim(X) > 1$. A function $\|.,.\|$: $X \times X \longrightarrow \mathbb{R}$ satisfying

1. $\|x, y\| \geq 0$, $\|x, y\| = 0$ if and only if x and y are linearly dependent
2. $\|x, y\| = \|y, x\|$
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$ for any real α
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$

is called a $2-$norm on X. The pair $(X, \|.,.\|)$ is called a $2-$ normed space.

The 2-normed space is the generalization concept of normed space.
Geometrically, 2-norm measures the area of the parallelogram spanned by the two vectors.

Definition 1.4. [7] Kizmaz defined the difference sequence spaces by

$$c_o(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and showed that these spaces are Banach spaces with the norm given by $||x|| = |x_1| + ||(\Delta x)||_\infty$.

The notion of ideal convergence was introduced as a generalization of statistical convergence, first by Kostyrko et al. [8]. For more details about these types of sequence spaces, one may refer to Dutta [1], Ghimire [3], Ghimire and Pahari [4, 5], Hazarika et al. [6], Mursaleen and Alotaibi [11], Mursaleen and Mohiduddine [12], Mursaleen and Sharma [13], Sahiner et al. [15], Salat et al. [16], Savas and Gurdal [17], Savas [18, 19], Tripathy and Hazarika [20] and many others.

Definition 1.5. [8] A family of sets $I \subseteq 2^X$ is said to be an ideal on X if

1. $\phi \in I$
2. $A \in I$ and $B \subset A$ implies that $B \in I$ (Hereditary Property)
3. $A, B \in I$ implies $A \cup B \in I$ (Additive Property).

Definition 1.6. [8] Let $I \subset 2^{\mathbb{N}}$ be non trivial ideal in $\mathbb{N}$. The sequence (x_n) in a normed space $(X, \|.\|)$ is said to be ideal convergent ($I-$ convergent) to $x \in X$ if the set

$$A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\} \in I, \text{ for each } \epsilon > 0.$$

This can be written as

$$I - \lim_{n \longrightarrow \infty} x_n = x.$$

Definition 1.7. [20] A sequence space S is said to be sequence algebra if $(x_k).(y_k) = (x_k y_k) \in S$ whenever $(x_k), (y_k) \in S$.

Definition 1.8. [14] A sequence space X is said to be solid (normal) if for any sequence (x_k) in a sequence space X and for all sequences (λ_k) of scalars with $|\lambda_k| \leq 1$ for all $k \in \mathbb{N}$ implies that $(\lambda_k x_k) \in X$.

Definition 1.9. [15] Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. The sequence (x_n) of 2-normed space $(X, \|.,.\|)$ is called I-convergent to x if the set $\{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon\} \in I$ for each $z \in X$ and for each $\epsilon > 0$.
This can be written as

$$I - \lim_{n \longrightarrow \infty} \|x_n - x, z\| = 0.$$

2 Main Results

In this section, we introduce and investigate the following classes of difference sequence spaces defined by ideal convergence and Orlicz function in 2-normed space by extending the work done in [3].
Let$(X, \|.,.\|)$ be a 2-normed space. Assign ω to represent the space containing all vector-valued sequences defined over $(X, \|.,.\|)$. We now define the following classes of sequences

$$\begin{aligned} c^I(M,\Delta,\|.,.\|) &= \left\{x=(x_k)\in\omega : I-\lim_{k\longrightarrow\infty} M\left(\frac{\|\Delta x_k - l, z\|}{\rho}\right)=0,\ \text{for some } \rho>0, l\in X, z\in X\right\} \\ c_0^I(M,\Delta,\|.,.\|) &= \left\{x=(x_k)\in\omega : I-\lim_{k\longrightarrow\infty} M\left(\frac{\|\Delta x_k, z\|}{\rho}\right)=0, \text{for some } \rho>0, z\in X\right\} \\ l_\infty(M,\Delta,\|.,.\|) &= \left\{x=(x_k)\in\omega : \sup_k M\left(\frac{\|\Delta x_k, z\|}{\rho}\right)<\infty, \text{for some } \rho>0, z\in X\right\} \end{aligned}$$

Also, we define

$$\begin{aligned} m^I(M,\Delta,\|.,.\|) &= c^I(M,\Delta,\|.,.\|)\cap l_\infty(M,\Delta,\|.,.\|) \\ m_0^I(M,\Delta,\|.,.\|) &= c_0^I(M,\Delta,\|.,.\|)\cap l_\infty(M,\Delta,\|.,.\|) \end{aligned}$$

The classes of sequences denoted by $c^I(M,\Delta,\|.,.\|)$, $c_0^I(M,\Delta,\|.,.\|)$, $m^I(M,\Delta,\|.,.\|)$ and $m_0^I(M,\Delta,\|.,.\|)$ are categorized as I-convergent, I-null, bounded I-convergent and bounded I-null respectively.

Theorem 2.1. *The classes $c^I(M,\Delta,\|.,.\|)$, $c_0^I(M,\Delta,\|.,.\|)$, $m^I(M,\Delta,\|.,.\|)$ and $m_0^I(M,\Delta,\|.,.\|)$ are linear spaces.*

Proof. We prove that the class $c^I(M,\|.,.\|)$ is a linear space. One can show that the remaining classes are linear spaces in the same way.
Let $x=(x_k)$ and $y=(y_k)$ be sequences in $c^I(M,\Delta,\|.,.\|)$. Then, there exist $\rho_1,\rho_2>0$ such that

$$I-\lim_{k\longrightarrow\infty} M\left(\frac{\|\Delta x_k - l_1, z\|}{\rho_1}\right)=0$$

and

$$I-\lim_{k\longrightarrow\infty} M\left(\frac{\|\Delta y_k - l_2, z\|}{\rho_2}\right)=0$$

for some l_1, $l_2\in X$ and $z\in X$.
Thus, for given positive ϵ , we have

$$N_1=\left\{k\in\mathbb{N} : M\left(\frac{\|\Delta x_k - l_1, z\|}{\rho_1}\right)>\frac{\epsilon}{2}\right\}\in I \tag{1}$$

$$N_2=\left\{k\in\mathbb{N} : M\left(\frac{\|\Delta y_k - l_2, z\|}{\rho_2}\right)>\frac{\epsilon}{2}\right\}\in I \tag{2}$$

Let α and β be the scalars. Let us choose

$$\rho=\max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}.$$

Using non-decreasing and convexity properties of Orlicz function M, we have

$$\begin{aligned} M\left(\frac{\|(\alpha\Delta x_k+\beta\Delta y_k)-(\alpha l_1+\beta l_2), z\|}{\rho}\right) &\leq M\left(\frac{|\alpha|.\|\Delta x_k - l_1, z\|}{\rho}+\frac{|\beta|.\|\Delta y_k - l_2, z\|}{\rho}\right) \\ &\leq M\left(\frac{\|\Delta x_k - l_1, z\|}{\rho_1}\right)+M\left(\frac{\|\Delta y_k - l_2, z\|}{\rho_2}\right) \end{aligned}$$

From (1) and (2), we have

$$\left\{k\in\mathbb{N} : M\left(\frac{\|(\alpha\Delta x_k+\beta\Delta y_k)-(\alpha l_1+\beta l_2), z\|}{\rho}\right)>\epsilon\right\}\subset N_1\cup N_2.$$

Thus, $c^1(M,\Delta,\|.,.\|)$ is a linear space.

□

Theorem 2.2. *If the Orlicz functions M_1, M_2 satisfy the Δ_2-condition, then*

1. $S(M_1, \Delta, \|., .\|) \subseteq S(M_2 o M_1, \Delta, \|., .\|)$
2. $S(M_1, \Delta, \|., .\|) \cap S(M_2, \Delta, \|., .\|) \subseteq S(M_1 + M_2, \Delta, \|., .\|)$ *for* $S = c^I, c_0^I, m^I, m_0^I$.

Proof. We prove these results for the space $S = c_0^I$. The proofs for the remaining spaces can be done in the similar way.
1. Let $x = (x_k) \in c_0^I(M_1, \Delta, \|., .\|)$. Then there exists a positive number ρ such that

$$I - \lim_{k \longrightarrow \infty} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) = 0, \text{for } z \in X.$$

Let $\epsilon > 0$. Since M_1 is an Orlicz function, there exists $0 < \delta < 1$ such that $M_1(u) < \epsilon$, whenever $\delta \geq u \geq 0$. Now we define the following sets

$$D_1 = \left\{ k \in \mathbb{N} : M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \leq \delta \right\}$$

$$D_2 = \left\{ k \in \mathbb{N} : M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) > \delta \right\}$$

If $k \in D_2$ then

$$M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) < \frac{1}{\delta} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) < 1 + \left\{ \frac{1}{\delta} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\}$$

Since M_2 is convex and non- decreasing, we can write

$$\begin{aligned} M_2 \left\{ M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} &< M_2 \left\{ 1 + \left\{ \frac{1}{\delta} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} \right\} \\ &< \frac{1}{2} M_2(2) + \frac{1}{2} M_2 \left\{ 2 . \frac{1}{\delta} . M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} \end{aligned}$$

Since M_2 satisfies Δ_2-condition, we have

$$\begin{aligned} M_2 \left\{ M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} &< \frac{1}{2} K \left\{ \frac{1}{\delta} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} M_2(2) + \frac{1}{2} K \left\{ \frac{1}{\delta} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} M_2(2) \\ &= \frac{K}{\delta} M_2(2) M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \end{aligned}$$

For $k \in D_1$, we have

$$M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) < \delta \implies M_2 \left\{ M_1 \left(\frac{\|\Delta x_k, z\|}{\rho} \right) \right\} < \epsilon$$

Hence, we conclude that

$$c_0^I(M_1, \Delta, \|., .\|) \subset c_0^I(M_2 o M_1, \Delta, \|., .\|).$$

2. Let $x = (x_k) \in c_0^I(M_1, \Delta, \|., .\|) \cap c_0^I(M_2, \|., .\|)$.
For positive constants ρ_1 and ρ_2, we have

$$I - \lim_{k \longrightarrow \infty} M_1 \left(\frac{\|\Delta x_k, z\|}{\rho_1} \right) = 0, \text{for} z \in X.$$

and

$$I - \lim_{k \longrightarrow \infty} M_2 \left(\frac{\|\Delta x_k, z\|}{\rho_2} \right) = 0, \text{for} z \in X.$$

Choose $\rho = \max\{\rho_1, \rho_2\}$. Then,

$$\begin{aligned}(M_1+M_2)\left(\frac{\|\Delta x_k, z\|}{\rho}\right) =& M_1\left(\frac{\|\Delta x_k, z\|}{\rho}\right) + M_2\left(\frac{\|\Delta x_k, z\|}{\rho}\right)\\ \leq& M_1\left(\frac{\|\Delta x_k, z\|}{\rho_1}\right) + M_2\left(\frac{\|\Delta x_k, z\|}{\rho_2}\right)\end{aligned}$$

This yields

$$c_0^I(M_1, \Delta, \|.,.\|) \cap c_0^I(M_2, \Delta, \|.,.\|) \subseteq c_0^I(M_1+M_2, \Delta, \|.,.\|)$$

□

Theorem 2.3. *Let M be an Orlicz function. Then we have*

$$c_0^I(M, \Delta, ||.,.||) \subset c^I(M, \Delta, ||.,.||) \subset l_\infty^I(M, \Delta, ||.,.||).$$

Proof. The inclusion $c_0^I(M, \Delta, ||.,.||) \subset c^I(M, \Delta, ||.,.||)$ is obvious.
Let $x = (x_k) \in c^I(M, \Delta, ||.,.||)$. There exists $\rho > 0$ such that

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\Delta x_k - l, z||}{\rho}\right) = 0$$

for $z \in X$ and $l \in X$.
Now

$$M\left(\frac{||\Delta x_k, z||}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{||\Delta x_k - l, z||}{\rho}\right) + \frac{1}{2}M\left(\frac{||l, z||}{\rho}\right)$$

Taking supremum over k on each side, one can easily obtain that $x \in l_\infty^1(M, \Delta, ||.,.||)$.
Thus , $c_0^I(M, \Delta, ||.,.||) \subset c^I(M, \Delta, ||.,.||) \subset l_\infty^I(M, \Delta, ||.,.||)$.

□

Theorem 2.4. *The spaces $c^I(M, \Delta, ||.,.||)$ and $c_0^I(M, \Delta, ||.,.||)$ are sequence algebra.*

Proof. We prove that the space $c^I(M, \Delta, ||.,.||)$ is sequence algebra. The result for the space $c^I(M, \Delta, ||.,.||)$ can be proved similarly.
Let $(x_k), (y_k) \in c_0^I(M, \Delta, ||.,.||)$. Then there exists positive constants ρ_1 and ρ_2 and for $z \in X$ such that

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\Delta x_k, z||}{\rho_1}\right) = 0$$

and

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\Delta y_k, z||}{\rho_2}\right) = 0$$

Choose $\rho = \rho_1\rho_2$.
Then one can easily show that

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\Delta x_k.\Delta y_k, z||}{\rho}\right) = 0$$

It follows that $x_k y_k = (x_k)(y_k) \in c_0^I(M, \Delta, ||.,.||)$. This shows that $c_0^I(M, \Delta, ||.,.||)$ is a sequence algebra. □

Theorem 2.5. *The spaces $c_0^I(M, \Delta, ||.,.||)$ and $m_0^I(M, \Delta, ||.,.||)$ are solid.*

Proof. We prove that the space $c_0^I(M, \Delta, ||.,.||)$ is solid. The result for the space $m_0^I(M, \Delta, ||.,.||)$ can be proved similarly.
Let $x = (x_k) \in c_0^I(M, \Delta, ||.,.||)$. Then there exists a positive constant ρ such that

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\Delta x_k, z||}{\rho}\right) = 0 \text{ for } z \in X.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all natural numbers k. Then

$$M\left(\frac{||\alpha_k \Delta x_k, z||}{\rho}\right) \leq |\alpha_k| M\left(\frac{||\Delta x_k, z||}{\rho}\right) \leq M\left(\frac{||\Delta x_k, z||}{\rho}\right)$$

This follows that

$$I - \lim_{k \longrightarrow \infty} M\left(\frac{||\alpha_k \Delta x_k, z||}{\rho}\right) = 0.$$

This shows that $\alpha_k x_k \in c_0^I(M, \Delta, ||.,.||)$ and hence $c_0^I(M, ||.,.||)$ is solid.

□

3 Conclusion

In this work, some of the results that characterize the linear topological properties of the difference sequence spaces of type I-convergent, I-null, bounded I-convergent, and bounded I-null in 2-normed space defined by the Orlicz function have been established. Further extensions of these findings to n-normed space are possible. Additionally, these findings can be used to create sequence spaces of these or similar types that are richer in algebraic and geometrical properties.

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