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A Comprehensive Study of Fractional-Order Derivative and Their Interplay with Basic Functions

Hem Raj Pandey[1,2,*], Ganga Ram Phaijoo[2] and Dil Bahadur Gurung[2]

[1]School of Engineering, Faculty of Science and Technology, Pokhara University, Nepal
[2]Department of Mathematics, School of Science, Kathmandu University, Nepal

*Correspondence to: Hem Raj Pandey, Email: hpandey@pu.edu.np

Abstract: *Fractional calculus from the nineteenth century to date has gained considerable attention due to its versatile applications in various scientific and engineering domains. This work examines the complex relationship between fractional-order derivative and basic functions, unraveling the profound interplay between mathematics and simulation. In this study, we illustrate the Mittag-Leffler function, Grunwald-Letnikov's, Riemann-Liouville's, and Caputo's fractional derivative and integral are presented with examples of basic functions and their graphical presentations. The purpose of this study is to examine the features of fractional derivatives from the perspective of researchers' motivations and interests.*

Keywords: Mittag-Leffler function, Grunwald-Letnikov's fractional derivative and integration, Riemann-Liouville's fractional derivative and integration, Caputo's fractional derivative, Graphical simulation

1 Introduction

Fractional calculus is a branch of calculus that deals with derivatives and integrals of non-integer order, i.e., fractional derivatives and fractional integrals. The idea of fractional calculus dates back to the mid-17^{th} century when Leibniz considered the possibility of using non-integer orders in his work on calculus [9, 23, 36]. However, it was not until the 19^{th} century that the first systematic study of fractional calculus was carried out by Liouville and Riemann [2, 28, 29]. For suitable functions, the n^{th} derivative of f, namely $D^n f(x) = \dfrac{d^n f(x)}{dx^n}$, is well defined when n is a positive integer. In 1695, L'Hopital asked Leibniz "What happens if $n = \dfrac{1}{2}$ in $D^n f$?", the insufficient but short answer of Leibniz was: *"It leads to a paradox, from which one day useful consequences will be drawn"*. In the same year, the derivative of general order was mentioned in the letter from Leibniz to J. Bernoulli [3, 10, 15, 29, 45].

Since that time, fractional calculus has drawn the attention of many famous mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. liouville, B. Riemann M. Riesz, and H. Weyl, who have high contributions to the elaboration of this field [11, 35, 40, 43]. But it was not until 1884 that the theory of generalized operators achieved such a level in its development so as to make it suitable as a point of departure for the modern mathematician [4, 8, 26]. The attraction behind the development of fractional calculus is rooted in the desire to generalize the classical calculus to a wider class of functions. Fractional calculus has applications in various fields, including physics, engineering, finance, and biology [22, 39]. In physics, fractional calculus is used to describe the behavior of complex systems, such as viscoelastic materials, fractals, and anomalous diffusion [13, 16]. In engineering, fractional calculus is used in the analysis and design of control systems, signal processing, and image processing [7]. In finance, fractional calculus is used to model stock price fluctuations and risk managements [11, 21]. In biology, fractional calculus is used to model the dynamics of biological systems, such as the spread of diseases and the growth of populations [30, 31, 32]. Fractional integrals and derivatives also appear in the theory of control of dynamical systems, when the controlled system and the controller is described by a fractional differential equation [7]. Fractional derivatives are used to describe the attributes of systems and processes leads to the mathematical modeling and simulation of fractional order differential equations [17, 24].

The fractional order derivative in differential equations have an ability to describe and capture the behavior of complex systems and phenomena that cannot be adequately represented by classical integer-order differential equations. By introducing non-locality and memory effects, fractional order derivatives provide a more versatile and accurate mathematical framework for modeling a wide range of natural and engineering systems [1, 5, 14, 20, 25, 27, 41]. Fractional order derivatives also describes like; anomalous diffusion where the spreading of particles or quantities deviates from classical diffusion behavior, model diffusion processes in fractal or heterogeneous media (including geophysics, hydrology and ecology), model wave propagation in media with power-law attenuation, where the wave amplitude decays in a non-exponential manner, viscoelastic system, time-fractional heat conduction and control theory [13, 16, 35, 36, 40, 41, 46].

In this study, we explore fractional order derivative together with basic functions, to uncovering the fundamental relationships and insights that appear from their interplay. Basic functions, often comprising elementary functions like polynomials, exponential functions, trigonometric functions, and power functions, form the building blocks of mathematical relations. We use the computation application software (MATLAB, Mathematica) for the graphical simulation [6, 37]. In summary, the history and motivation behind the development of fractional calculus are rooted to generalize the classical calculus to a wider class of functions and to describe phenomena that cannot be explained by classical calculus.

The present work contained following details: In Section 2, we present a brief summary of the fundamental concepts of fractional derivative and integration terms with Mittag-Leffler function, Grunwald-Letnikov's fractional derivative and integration, Riemann - Liouville's fractional derivative and integration, and Caputo's fractional derivative. In Section 3, we simulate the fractional term with basic functions. We discuses the perceptions between Riemann- Liouville's and Caputo fractional derivative in Section 4. The conclusion of the overall analysis is presented in the final section of the article.

2 Terminologies in Fractional Calculus

2.1 Gamma and Beta function

Definition 2.1.1 (Gamma function)**.** The gamma function $\Gamma(z)$ is defined on a complex plane as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad (\text{Re } z > 0), \tag{1}$$

with the relations; $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$.

Definition 2.1.2 (Beta function)**.** The beta function $B(z,\omega)$ is defined on a complex plane as

$$B(z,\omega) = \int_0^\infty t^{z-1}(1-t)^{\omega-1} dt, \quad (\text{Re } z > 0, \text{Re } \omega > 0), \tag{2}$$

with the relation, $B(z,\omega) = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}$.

2.2 Mittag-Leffler function

Definition 2.2.1. The Mittag-Leffler function appears in 1903 A.D., numerous field of mathematics and physics, including fractional calculus, probability theory, and the theory of fractional differential equation. It intimately connected to fractional calculus, which extends the traditional calculus to non-integer orders [17, 34, 36, 44]. This function is a special in mathematics that generalizes the exponential function to fractional and complex arguments. And two-parameter function is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta > 0 \tag{3}$$

When the parameter $\alpha = \beta = 1$ is equal to 1, the Mittag-Leffler function reduces to the exponential function: $E_1(z) = e^z$. Therefore, the Mittag-Leffler function generalizes the exponential function to fractional orders.

If $\beta = 1$, then we get

$$\begin{aligned} E_{\alpha,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} = E_\alpha(z) \\ &= 1 + \frac{z}{\Gamma(\alpha+1)} + \frac{z^2}{\Gamma(2\alpha+1)} + \frac{z^3}{\Gamma(3\alpha+1)} + \ldots, \end{aligned} \tag{4}$$

Where α is a complex parameter and z is a complex variable. This series converges for all complex z and is an entire function of z. The Mittag -Leffler function in different form as we change the value of parameter α and β as shown below;

1.	$E_{0,1}(z) = \frac{1}{1-z},\ \lvert z\rvert < 1$	5.	$E_{2,1}(-z^2) = \cos(z)$
2.	$E_{2,1}(z^2) = \cosh(z)$	6.	$E_{2,2}(-z^2) = \frac{1}{z}\sin(z)$
3.	$E_{1,3}(z) = \frac{e^z - 1 - z}{z^2}$		
4.	$E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right\}$		

Table 1: The different form of Mittag-Leffler function with the value of parameter α and β.

Relations:

1. For all $\alpha > 0$, $\beta > 0$, then $E_{\alpha,\beta} = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$.

   Proof. We know that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}.$$

   Put $k = n + 1$,

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{n=-1}^{\infty} \frac{z^{n+1}}{\Gamma(\alpha(n+1)+\beta)} \\ &= \sum_{n=-1}^{\infty} \frac{z^{n+1}}{\Gamma(\alpha n+\alpha+\beta)} \\ &= \frac{z^0}{\Gamma(\beta)} + \sum_{n=0}^{\infty} \frac{z^(n+1)}{\Gamma(\alpha n+\alpha+\beta)} \\ &= \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z). \end{aligned}$$

   □

2. The fractional derivative of $E_{\alpha,\beta}(z)$ is $\frac{1}{\alpha z}E_{\alpha,\beta-1}(z) - \frac{\beta-1}{\alpha z}E_{\alpha,\beta}(z)$.

2.3 Grunwald-Letnikov's (GL) fractional derivative and integration

Grunwald–Letnikov's derivative is a basic extension of the natural derivative to fractional one. It was introduced by Anton Karl Grunwald in 1867, and then by Aleksey Vasilievich Letnikov's in 1868 [9, 33, 36, 42].

2.3.1 Derivative

Consider the real valued continuous function $f(t)$. From the definition of derivative, we have

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t-h)-f(t)}{-h} = \lim_{h\to 0} \frac{f(t)-f(t-h)}{h} \tag{5}$$

Applying definition for the second, third and so on upto the n^{th} order, we get

$$f''(t) = \frac{d^2 f}{dt^2} = \lim_{h\to 0} \frac{f(t)-2f(t-h)+f(t-2h)}{h^2} \tag{6}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t)-3f(t-h)+3f(t-2h)-f(t-3h)}{h^3} \tag{7}$$

By induction we get

$$f^n(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(t-ih), \tag{8}$$

where, the binomial coefficient

$$\binom{n}{i} = \frac{n(n-1)(n-2),\ldots,(n-i+1)}{i!}$$

If we generalize as a fraction with $v, n, i \in \mathbb{N}$ and $h \in \mathbb{R}$, we get

$$f_h^v(t) = \lim_{h\to 0} \frac{1}{h^v} \sum_{i=0}^{n} (-1)^i \binom{v}{i} f(t-ih). \tag{9}$$

It follows for $v \leq n$,

$$\lim_{h\to 0} f_h^v(t) = f^v(t) = \frac{d^v f}{dt^v}.$$

By generalizing the order v in natural numbers to the fractional order α, the fractional derivative of a function $f(t)$ with order $\alpha > 0$ is obtained as

$${}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} f(t-ih), \tag{10}$$

where, $nh = t-a$ and a and t are called the terminals. The binomial term can be expressed in the form as $\binom{\alpha}{i} = \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)}$.

2.3.2 Integration

We will now show that for negative v for a regular integration. The binomial coefficient (8) can be expressed as

$$\binom{v}{i} = \frac{v(v-1)(v-2)\ldots(v-i+1)}{i!}.$$

Then, the value of v as negative, we get

$$\binom{-v}{i} = \frac{-v(-v-1)(-v-2)\ldots(-v-i+1)}{i!} = (-1)^i \binom{v}{i}. \tag{11}$$

Replacing v by $-v$ in the equation (9) we get,

$$f_h^{-v}(t) = \lim_{h\to 0} h^v \sum_{i=0}^{n} \binom{v}{i} f(t-ih). \tag{12}$$

Let n be defined by the relation $nh = t - a$, such that $n \to \infty$ as $h \to 0$, also a and t form the limits of integration. And fractional integration is defined as

$$ {}_aD_t^{-v} f(t) = \lim_{h\to 0} f_h^{-v}(t). $$

By generalizing the order v in natural numbers to the fractional order α, the fractional integration of a function $f(t)$ with order α is obtained as

$$ \begin{aligned} {}_aD_t^{-\alpha} f(t) &= \lim_{h\to 0} h^{\alpha} \sum_{i=0}^{n} \binom{\alpha}{i} f(t - ih) \\ &= \frac{1}{\Gamma(\alpha)} \lim_{h\to 0} h^{\alpha} \sum_{i=0}^{\frac{t-a}{h}} \frac{\Gamma(\alpha + i)}{\Gamma(i+1)} f(t - ih) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha - 1} f(s) ds. \end{aligned} \tag{13} $$

We can make similar discussion in Riemann-Liouville's fractional integral. The GL derivative with arbitrary order is defined as

$$ \begin{aligned} {}_aD_t^{\alpha} f(t) &= \sum_{i=0}^{n} \frac{(t-a)^{i-\alpha} f^i(t)}{\Gamma(i - \alpha + 1)} + \frac{1}{\Gamma(n - \alpha + 1)} \int_a^t (t-s)^{n-\alpha} f^{n+1}(s) ds, \\ &= \left(\frac{d}{dt}\right)^{n+1} \int_a^t (t-s)^{n-\alpha} f(s) ds. \end{aligned} \tag{14} $$

The formula (14) has been obtained under the assumption that the derivatives $f^i(t),\ i = 1, 2, \ldots, n+1$ are continuous in the closed interval $[a, t]$ and that n is an integer satisfying the condition $n > \alpha - 1$. The smallest possible value for n is determined by the inequality, $n < \alpha < n + 1$. This relation (14) is same in Riemann-Liouville's fractional derivative.

Example 2.3.1. The GL fractional derivative of ${}_aD_t^{\alpha} f(t)$ of power function $f(t) = (t-a)^u$ is $\dfrac{\Gamma(u+1)}{\Gamma(u - \alpha + 1)}(t-a)^{u-\alpha}$, where $0 \le \alpha < n + 1$.

Example 2.3.2. ${}_aD_t^{\alpha} t^k = 0$, for $k < \alpha$.

2.4 Riemann-Liouville's fractional derivative and integration

The Riemann-Liouville's integral operator is named after two scientists Bernard Riemann and Joseph Liouville's [35]. The Riemann-Liouville's (RL) differintegral is very similar to the Grunwald-Letnikov's (GL) differintegral [15, 36]. It is based on extending the Cauchy formula for repeated integration for non-integer values, in order to define fractional integration

$$ \int_a^t \int_a^{t_n} \int_a^{t_{n-1}} \cdots \int_a^{t_3} \int_a^{t_2} f(t_1) dt_1\, dt_2 \ldots dt_{n-1}\, dt_n = \frac{1}{(n-1)!} \int_a^t \frac{f(s)}{(t-s)^{1-n}} ds, $$

where fractional integral of order α for the function $f(t)$ can be written as

$$ {}_aD_t^{-\alpha} f(t) = {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds. \tag{15} $$

Fractional differentiation is defined by first applying fractional integration and afterwards performing regular differentiation, in a unique way. Positive values of α correspond to differentiation while negative values of α correspond to integration.

Definition 2.4.1 (Riemann-Liouville's fractional derivative)**.** Let α be a real non negative number. For a positive integer n ($n = \lceil \alpha \rceil$ smallest integer $> \alpha$) such that $n-1 < \alpha \leq n$, the Riemann-Liouville's fractional-order differential operator of a function $f(t)$ of order α is defined by

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t (t-s)^{n-\alpha-1} f(s) ds, & \text{if } n-1 < \alpha < n, \\ \dfrac{d^n f}{dt^n}, & \text{if } \alpha = n. \end{cases} \tag{16} $$

When $0 < \alpha \leq 1$, we get

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds. \tag{17} $$

Note:. The derivative of

(i) t^q is $\dfrac{\Gamma(q+1)}{\Gamma(q-\alpha+1)} t^{q-\alpha}$.

(ii) constant function C is $\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} C$.

Here, the derivative of constant function is not zero. But if $a = -\infty$, then derivative of constant is zero.

Definition 2.4.2 (Riemann-Liouville's fractional integral)**.** Let α be a real non negative number and $f(t)$ be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$. The Riemann-Liouville's fractional integral of $f(t)$ of order $\alpha > 0$, for $s > 0$ is defined as

$$ {}_aI_t^\alpha f(t) = {}_aD_t^{-\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \dfrac{f(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{18} $$

Note: The integral of t^q is $\dfrac{\Gamma(q+1)}{\Gamma(q+\alpha+1)} t^{q+\alpha}$.

Properties: Some of the properties of Riemann-Liouville's fractional derivative and integral are listed as

i. For a function $f(t)$ and $\alpha, \beta \geq 0$,
$$ {}_aI_t^\alpha \, {}_aI_t^\beta f(t) = {}_aI_t^{\alpha+\beta} f(t), $$
$$ \text{and } {}_aI_t^\alpha \, {}_aI_t^\beta f(t) = {}_aI_t^\beta \, {}_aI_t^\alpha f(t). $$

ii. For all scalar λ, μ is $D_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda \, D_t^\alpha f(t) + \mu \, D_t^\alpha g(t)$.

iii. ${}_aD_t^0 f(t) = f(t)$.

iv. ${}_aD_t^\alpha \, {}_aI_t^\alpha f(t) = f(t)$.

v. ${}_aD_t^\alpha (f(t)g(t)) = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} {}_aD_t^k f(t) {}_aD_t^{\alpha-k} g(t)$.

2.5 Caputo's fractional derivative

The Caputo's fractional order derivative is a generalization of the classical derivative operator to non-integer orders [19, 35, 36]. It is a useful mathematical tool for describing and analyzing systems or phenomena that exhibits fractional dynamics. This makes it particularly suitable for fractional calculus problems and systems that exhibit memory or non-local behavior. The Caputo's derivative satisfies several properties, such as linearity, chain rule, and initial value conditions. It also has connections to other fractional calculus operators, such as the Riemann-Liouville fractional derivative and the Grunwald-Letnikov's fractional derivative [15, 35, 36]. The Caputo's fractional order derivative has found applications in various scientific and engineering fields, including physics, biology, signal processing, control systems, and finance. It allows more appropriate modeling and analysis of complex systems that explore fractional dynamics or anomalous behaviors [30, 36].

Definition 2.5.1 (Caputo's fractional derivative). Let α be a non negative integer and $n = \lceil \alpha \rceil$ such that $n-1 < \alpha \leq n$. Then the Caputo's fractional-order deivative operator of order α is defined by

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-s)^{n-1-\alpha} f^n(s)ds, & \text{if } n-1 < \alpha < n, \\ \dfrac{d^n f}{dt^n}, & \text{if } \alpha = n. \end{cases} \tag{19} $$

When $a = 0$, we get

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} f^n(s)ds, \ t > 0. \tag{20} $$

Notes: The derivative of

i. a constant C is zero.

ii. t^q is zero, $q = 0, 1 \ldots, \lceil \alpha \rceil - 1$.

iii. t^β does not exist, $\beta \in \mathbb{R}$, $\beta < \lceil \alpha \rceil - 1$

iv. t^q is $\dfrac{\Gamma(q+1)}{\Gamma(q+1-\alpha)} t^{q-\alpha}$, $q > \lceil \alpha \rceil - 1$.

Properties: Some of the properties of Caputo's fractional derivative are listed as

i. ${}_aD_t^\alpha \, {}_aI_t^\alpha f(t) = f(t)$.

ii. ${}_aD_t^k f(t) = f^k(t)$.

iii. ${}_aD_t^0 f(t) = f(t)$.

iv. ${}_aI_t^\alpha \, {}_aD_t^\alpha f(t) = f(t) - \displaystyle\sum_{k=0}^{m-1} \frac{{}_aD_t^k f(s)}{k!} (t-s)^k$.

v. For all scalar λ, μ is $D_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda \, D_t^\alpha f(t) + \mu \, D_t^\alpha g(t)$.

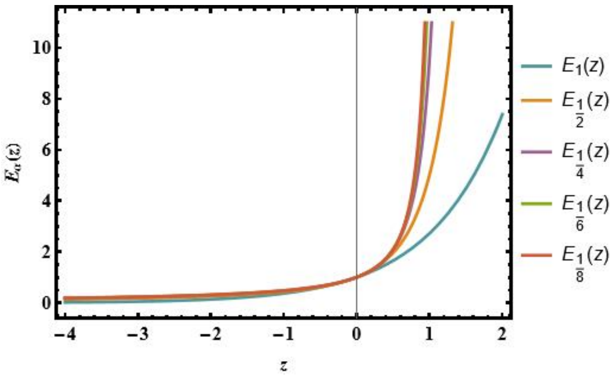
vi. ${}_aD_t^\alpha(f(t)g(t)) = \displaystyle\sum_{k=0}^{\infty} \binom{\alpha}{k} {}_aD_t^k f(t) {}_aD_t^{\alpha-k} g(t)$.

3 Simulations of Some Functions Using the Definition of Fractional Order Derivative

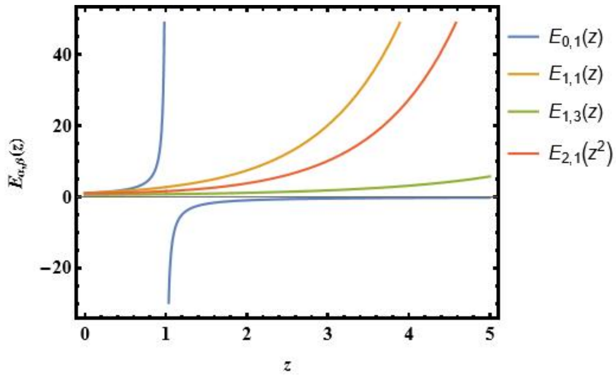
In this section, we simulate the fractional order derivative of a basic function using various values of α. The illustrated result demonstrates how the fractional derivative works with small changes, making the functions more descriptive and acceptable for analysis.

3.1 Mittag - Leffler function

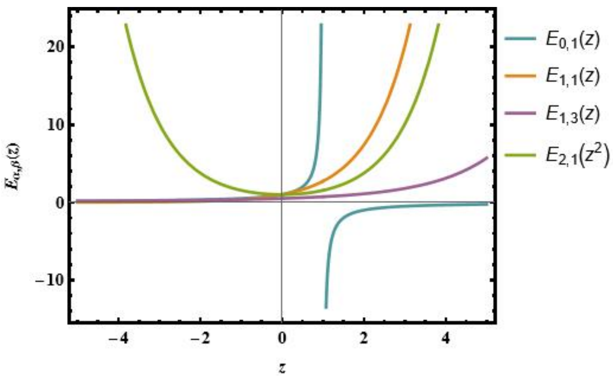
The different form of Mittag - Leffler function are presented in graphical form in the figures from $(a) - (d)$. The Mittag-Leffler function is computed with one and two variables at different intervals in Fig. $(1)(a-d)$. A single variable exhibits an exponential curve with a parameter $\alpha \in (0, 1]$ and $z \in [-4, 2]$. The curve is closer to the x-axis when a value is closer to zero. When we have a two-variable function, we get different types of curves, so converting Mittag-Leffler to a trigonometric function is beneficial while analyzing real-life situations.



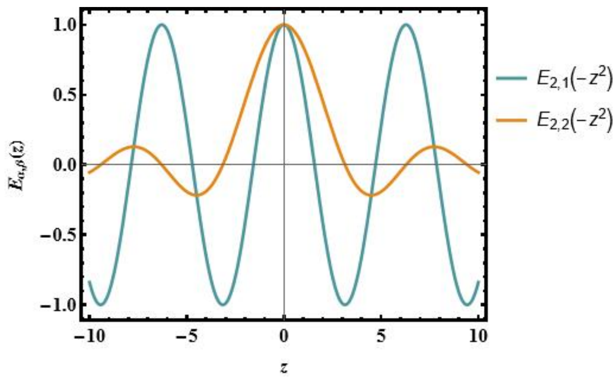
(a)



(b)



(c)



(d)

Figure 1: Graphical representation of Mittag- Leffler function of Table (1) (a) $E_\alpha(z)$ with $\alpha = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$ in interval $[-4, 2]$ (b) $E_{\alpha,\beta}$ with different value of (α, β) where $(0, 1)$, $(1, 1)$, $(1, 3)$, $(2, 1)$ in interval $[0, 5]$, (c) $E_{\alpha,\beta}$ with different value of (α, β) where, $(0, 1)$, $(1, 1)$, $(1, 3)$, $(2, 1)$ in interval $[-5, 5]$ and (d) $E_{\alpha,\beta}$ with different value of (α, β) where, $(2, 1)$, $(2, 2)$ in interval $[-10, 10]$.

3.2 Grunwald-Letnikov's (GL) fractional derivative

The basic function uses Grunwald-Letnikov's fractional derivative definition and is presented in graphical form in Fig. (2) for a function with $\alpha \in [0, 1]$ and in interval $t \in [0, 2\pi]$.

Relations:

1. The fractional derivative of $f(t) = e^{bt}$ is $e^{bt}b^\alpha$, where, $|e^{-bh}| \leq 1,\ Re(b) > 0$.

   Solution: Using the definition of GL fractional derivative (10), we get,

$$
\begin{aligned}
{}_aD_t^\alpha e^{bt} &= \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} e^{b(t-ih)}, \\
&= e^{bt} \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} e^{-bih}, \\
&= e^{bt} h \to 0 \frac{1}{h^\alpha} (1 - e^{-bh})^\alpha, \text{ where, } \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} = (1 - e^{-bh})^\alpha,
\end{aligned}
$$

$$
\begin{aligned}
{}_aD_t^\alpha e^{bt} &= e^{bt} \lim_{h\to 0} \frac{1}{h^\alpha}\left(bh - \frac{b^2h^2}{2} + \frac{a^3h^3}{3!} - \cdots\right)^\alpha, \\
&= e^{bt} b^\alpha, \text{ where, } |e^{-bh}| \le 1 \implies Re(b) > 0.
\end{aligned}
$$

2. The fractional derivative of $f(t) = \sin(t)$ and $\cos(t)$ is $\sin(t + \frac{\alpha\pi}{2})$ and $\cos(t + \frac{\alpha\pi}{2})$.

   Solution: We note that $e^{it} = \cos(t) + i\sin(t)$. Now,

$$
\begin{aligned}
{}_aD_t^\alpha e^{it} &= i^\alpha . e^{it} = e^{\alpha\frac{i\pi}{2}} . e^{it} = e^{i(t+\frac{\alpha\pi}{2})} \\
&= \cos(t + \frac{\alpha\pi}{2}) + i\sin(t + \frac{\alpha\pi}{2}) \\
{}_aD_t^\alpha(\cos(t) + i\sin(t)) &= \cos\left(t + \frac{\alpha\pi}{2}\right) + i\sin\left(t + \frac{\alpha\pi}{2}\right),
\end{aligned}
$$

   Equating real and imaginary part, we get

$$
{}_aD_t^\alpha \cos(t) = \cos\left(t + \frac{\alpha\pi}{2}\right), \text{ and } \quad {}_aD_t^\alpha \sin(t) = \sin\left(t + \frac{\alpha\pi}{2}\right).
$$

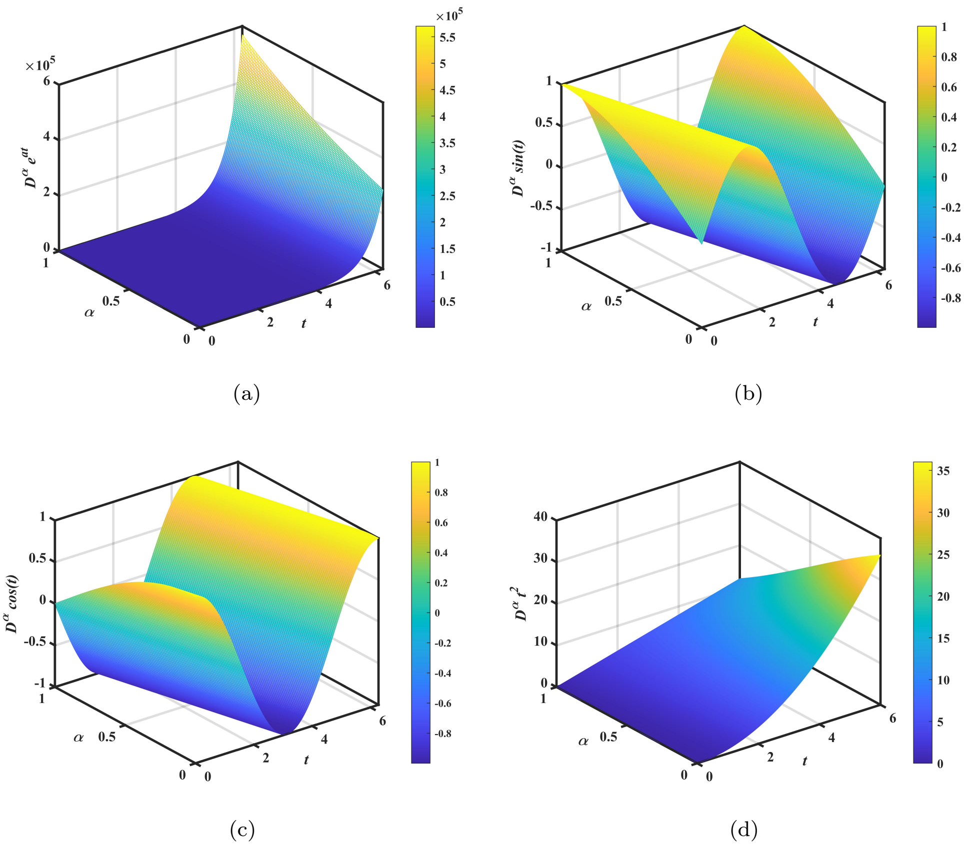
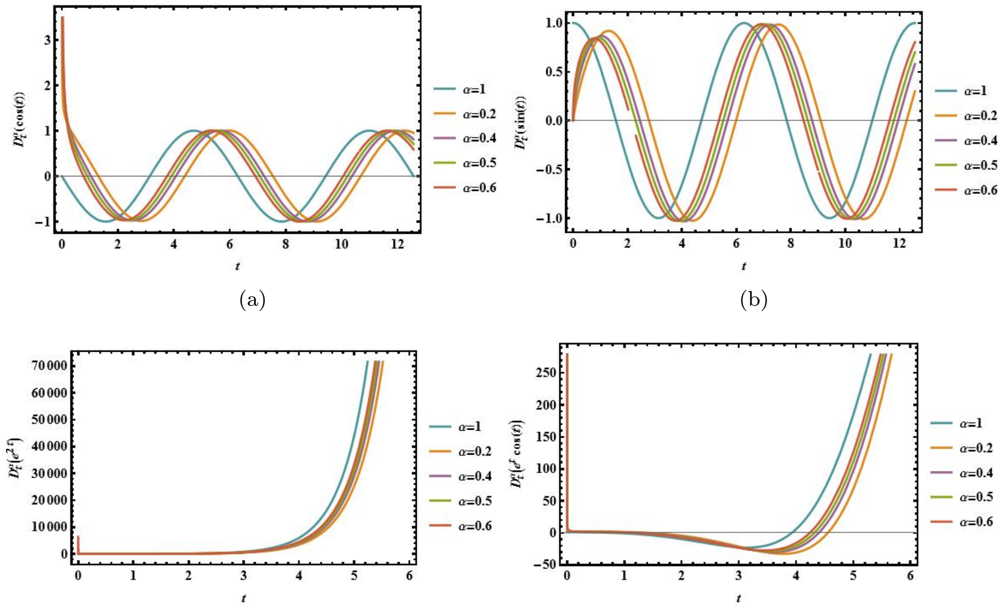


Figure 2: Graphical representation of Grunwald-Letnikov's (GL) fractional derivative of the function (a) e^{at}, (b) $\sin(t)$, (c) $\cos(t)$ and (d) t^2 for all $t \in [0, 2\pi]$.

All the panels of Fig. (2) are in a 3D mesh plot with $\alpha \in (0, 1]$, which integrates the figure dimension between 0 and 1 perfectly. Graphs of fractional derivatives of special functions such as algebraic functions, exponential functions, and trigonometric functions are noticeable. This indicates that the fractional order of the derivatives is an effective solution to many problems. It is additionally helpful in obtaining novel solutions to many existing problems and in analyzing and designing systems. They also provide a powerful tool to analyze and control chaotic systems.

3.3 Riemann-Liouville's (RL) fractional derivative

The basic function of exponential and trigonometric are used Riemann-Liouville's fractional derivative definition and is presented in graphical form as a function with $\alpha \in (0, 1]$ and in interval $t \in [0, 4\pi]$ and $t \in [0, 6]$.



(a) (b) (c) (d)

Figure 3: Graphical representation of Riemann-Liouville's (RL) fractional derivative of the function (a) $\cos(t)$, (b) $\sin(t)$, (c) e^{2t}, and (d) $e^t \cos(t)$ for all $t \in [0, 4\pi]$ and $t \in [0, 6]$.

Fig. (3)(a and b) illustrate the trigonometric functions. When $t = 0$, the derivative of $\cos(t)$ captures the special nature of the curve, as $\alpha \to 0$ the curves display its entire property with a different value of α as the value of t increases. Similarly, the nature of the derivative for the $\sin(t)$ demonstrated with a unique nature of the curve. In Fig. (3)(c and d), the dynamics of the curve changes as the values of t increase. As α fluctuates between 0 and 1, the characteristics of the curve changes over the time interval. For illustrative purposes, some examples can be found on the RL-Fractional integral.

Example 3.3.1. RL fractional integral of the function $f(t) = \ln 2$ of order $\alpha = \frac{1}{2}$.

Solution: Using the definition (18), we get

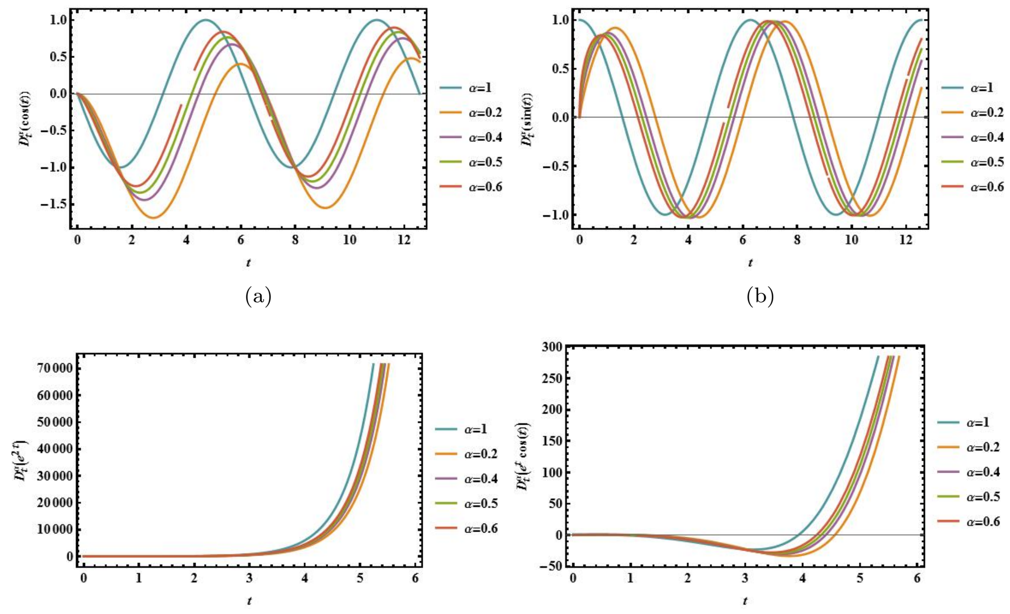
$$\begin{aligned} I^{\frac{1}{2}} \ln 2 &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{(\frac{1}{2}-1)} \ln 2 \, ds \\ &= \frac{\ln 2}{\Gamma(\frac{1}{2})} \int_0^t \frac{\ln 2}{\sqrt{t-s}} \, ds \\ &= \frac{2 \ln 2}{\sqrt{\pi}} \sqrt{t} \\ &= \ln 4 \sqrt{\frac{\pi}{t}}. \end{aligned}$$

Example 3.3.2. The RL fractional integral of the function $f(t) = e^t$ of order $\alpha = \frac{1}{2}$ is $e^t erf\ (\sqrt{t})$.

Example 3.3.3. The RL fractional integral of the function $f(t) = \sin(t),\ \cos(t),\ e^{bt}$ of order $\alpha > 0$ is $\sin\left(t - \frac{\alpha\pi}{2}\right),\ \cos\left(t + \frac{\alpha\pi}{2}\right),\ \frac{e^{bt}}{b^{\alpha}}$, which is true on the interval $(-\infty, t)$.

3.4 Caputo's fractional derivative

Using Caputo's definition of the fractional derivative, the basic functions of the exponential and trigonometric functions are presented graphically with $\alpha \in (0, 1]$ and in interval $t \in [0, 4\pi]$.



(a) (b) (c) (d)

Figure 4: Caputo's fractional derivative of the function $\cos(t)$, $\sin(t)$, e^{2t}, and $e^t \cos(t)$ for $t \in [0, 4\pi]$.

In comparison to the Riemann-Liouville's fractional derivative, Caputo's derivative is more effective in curve nature of trigonometric and exponential functions, which is graphically presented in Fig. (4)[$(a-d)$].

4 Perceptions on Riemann-Liouville's and Caputo's Fractional Derivative

In this section, we discussed about the Riemann-Liouville and Caputo fractional derivative relations from the literature [12, 18, 36, 38].

- The derivative of the Riemann-Liouville constant is non-zero, but Caputo's fractional derivative of constant function is zero.
- If $f(t) \in C^n[0, \infty)$ and $p - 1 < \alpha < p \in \mathbb{Z}^+$, then

$$ {}_0^C D_t^\alpha f(t) = {}_0^{RL} D_t^\alpha \left(f(t) - \sum_{k=0}^{p-1} \frac{t^k}{k!} f^k(t) \right). $$

- The Riemann-Liouville fractional derivative of an arbitrary function that is not continuous at the origin. One of the most important advantages of the Caputo fractional derivative is that it allows the inclusion of traditional boundary and starting conditions into the solution of the problem. The illustrate form of fractional differential equation for Riemann-Liouville's and Caputo's sense are presented.

  For Riemann-Liouville's inital value problem

$$ \begin{aligned} {}_0D_t^\alpha g(t) &= F(g(t), t),\ m - 1 \leq \alpha < m \in \mathbb{Z}^+,\ t > 0 \\ {}_0D_t^{\alpha-k} g(t)|_{t=0} &= g_0^k,\ k = 1, 2, \ldots, m \end{aligned} \tag{21} $$

and it's Laplace transform

$$\mathcal{L}[{}_0D_t^\alpha g(t)] = s^\alpha \mathcal{L}[{}_0D_t^\alpha g(t)] - \sum_0^{m-1} s^k {}_0D_t^{\alpha-k-1} g(t)|_{t=0},\ m-1 \le \alpha < m \in \mathbb{Z}^+,\ t>0 \tag{22}$$

For Caputo's sense initial value problem

$$\begin{aligned} {}_0D_t^\alpha g(t) &= F(g(t),t),\ m-1<\alpha\le m\in\mathbb{Z}^+,\ t>0 \\ g^k(0) &= g_0^k,\ k=0,1,2,\ldots,m \end{aligned} \tag{23}$$

and its Laplace transform

$$\mathcal{L}[{}_0D_t^\alpha g(t)] = s^\alpha \mathcal{L}[{}_0D_t^\alpha g(t)] - \sum_0^{m-1} s^{\alpha-k-1} g^k(0),\ m-1<\alpha\le m\in\mathbb{Z}^+,\ t>0 \tag{24}$$

For example, if $g(t)$ is a displacement, then $g'(t)$ will be the velocity, and $g''(t)$ the acceleration. Then, the initial value problem is difficult and complex, which do not have clear physical background and meaning in Riemann-Liouville's equation (21). But in Caputo's equation (23) has a clear physical meaning. So, Caputo's fractional order system is often used more for modeling and analysis of different model equations.

- For the Riemann-Liouville's fractional derivative, let $\alpha_1, \alpha_2 \in \mathbb{R}^+$. Then

$$ {}_0D_t^{\alpha_1} {}_0D_t^{\alpha_2} g(t) \ne {}_0D_t^{\alpha_1+\alpha_2} g(t) \ne {}_0D_t^{\alpha_2} {}_0D_t^{\alpha_1} g(t).$$

  For example, ${}_0D_t^{\frac{1}{2}} {}_0D_t^1 C = 0$, ${}_0D_t^1 {}_0D_t^{\frac{1}{2}} C = \frac{C}{\Gamma(\frac{1}{2})} t^{-\frac{1}{2}}$, ${}_0D_t^{\frac{3}{2}} C = \frac{C}{\Gamma(\frac{1}{2})} t^{-\frac{3}{2}}$.

  And for the Caputo's fractional derivative, let $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1+\alpha_2 \le 1$. Then

$$ {}_0D_t^{\alpha_1} {}_0D_t^{\alpha_2} g(t) = {}_0D_t^{\alpha_1+\alpha_2} g(t) = {}_0D_t^{\alpha_2} {}_0D_t^{\alpha_1} g(t).$$

  For example, as $\alpha_1+\alpha_2 \le 1$, ${}_0D_t^{0.4} {}_0D_t^{0.3}(t) = 1.1142 t^{0.3} = {}_0D_t^{0.3} {}_0D_t^{0.4}(t) = {}_0D_t^{0.7}(t)$. And $\alpha_1+\alpha_2 > 1$, we get ${}_0D_t^{0.6} {}_0D_t^{0.5}(t) = \frac{1}{\Gamma(0.9)} t^{-0.1}$, ${}_0D_t^{1.1}(t) = 0$.

- The fractional derivative of the instantaneous exponential and the Mittag-Leffler function has a singularity at the origin of an arbitrary function and is constant at the origin. The range of applications of the Riemann-Liouville fractional derivative is restricted by these limitations. Higher regularity requirements are needed for the differentiability to compute the fractional derivative of a function in the Caputo sense. To do this, we must first determine the derivative of the function. Caputo derivatives are defined only for differentiable functions, although fractional Riemann-Liouville derivatives of all degrees less than one can exist for functions without first-order derivatives.

- Riemann-Liouville's derivative have a memory effect, that they depend on the entire past history of the function. In Caputo's derivatives, along with the memory effect, they incorporate initial conditions that account for the history of the function making them more suitable for modeling real-world phenomena.

- Numerical approximations of Riemann-Liouville's derivative can be more challenging due to the non-causal nature of the derivative. Caputo's derivative are often easier to implement and more commonly used in practical applications.

5 Conclusion

In this study, the graphical simulation of basic functions applying Mittag-Leffler functions, Grunwald-Letnikov's fractional derivative, Riemann-Liouville's fractional derivative, and Caputo's fractional derivative yielded a noticeable impact in the generalization of classical functions. The solution of basic functions using fractional integrals inspires the solution of other analogous functions. So, we can conclude by discussing potential options for further exploration and research in the future direction of fractional-order derivatives with basic functions. Comparison between Riemann-Liouville's and Caputo's fractional system, Caputo's fractional order sense are appropriate for modeling and analysis of different model equations. As this field continues to progress, there are promising opportunities to delve deeper into its theoretical underpinnings and extend its practical applications. By unraveling the connections between fractional-order derivatives and fundamental mathematical constructs, researchers can expand their mathematical toolbox and pave the way for enhanced problem-solving capabilities across a spectrum of scientific and engineering domains.

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