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Error Estimates in the Maximum Norm for the Solution of Poisson's Equation Approximated by the Five-Point Laplacian Using the Discrete Maximum Principle

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Abstract: *In this paper, we study error estimates in the maximum norm in the context of solving Poisson's equation numerically when approximated the Five-Point Laplacian method using the discrete maximum principle. The primary objective is to assess the accuracy of this numerical approach in solving Poisson's equation and to provide insights into the behavior of error estimates. We focus on the estimates of maximum norm of the discrete functions defined on a grid in a unit square as well as in a square of side* s*, and estimate errors measured in the maximum norm.*

Keywords: Maximum principle, Poisson's equation, Laplacian, Maximum norm, Error estimates

1 Introduction

1.1 Definitions

Poisson's equation: It is a second-order partial differential equation that defines a scalar field when there are localized sources. Poisson's equation commonly arises in electrostatics, where it describes the electric potential produced by a charge distribution. Mathematically, two dimensional Poisson's equation is expressed as [6]

$$\frac{\partial^2 \alpha(x,y)}{\partial x^2} + \frac{\partial^2 \alpha(x,y)}{\partial y^2} = f(x,y) \tag{1}$$

where the non-homogeneous term $f : \Omega \to \mathbb{R}$ is called a source term, depending on the application.

On the other hand, Laplace's equation that describes a steady-state or equilibrium situation in which a scalar field, does not vary with time. It represents situations where the field is in a state of equilibrium and has no sources or sinks. Mathematically, two dimensional Laplace's equation is expressed as [5],[6]

$$\frac{\partial^2 \alpha(x,y)}{\partial x^2} + \frac{\partial^2 \alpha(x,y)}{\partial y^2} = 0. \tag{2}$$

Harmonic function[5]: A function $\alpha \in C^2(\Omega) \cap C(\overline{\Omega})$ is said to be harmonic in Ω if it satisfies the Laplace's equation. If $\triangle \alpha \geq 0$ then α is called sub-harmonic in Ω. If $\triangle \alpha \leq 0$, then α is called super-harmonic in Ω.

1.2 Maximum principle

Maximum and minimum values are crucial concepts used in a wide range of fields, including mathematics, statistics, optimization, programming, and real-world applications, to determine the maximum and minimum values or to optimize various parameters.

The maximum principles have been applied in mathematical literature since the early 19th century for solutions of second-order elliptic partial differential equations (PDEs). The maximum principle of harmonic functions was introduced by Gauss in 1839 on the basis of the mean-value theorem. Over the years, it has been further developed and expanded by various authors. In 1927, H. Hopf extended strong maximum principle for harmonic function, stating that the function cannot have an interior maximum unless it is constant [7],[15]. It had an amazing impact in the theory and applications of PDEs which later became an essential tool for establishing important results of linear and non-linear PDEs.

Indeed, the maximum principle is intricately linked to the non-negativity property, a fundamental trait of classical solutions for second-order PDEs. The significant attributes of the maximum principles encompass solution uniqueness and stability, the comparison principle, and the non-negativity characteristic. These principles hold substantial importance in both theoretical and numerical aspects of solving such equations. They are critical for understanding the behavior of solutions and ensuring their reliability in various mathematical and computational contexts [1],[4].

The applicability of the maximum principle is primarily limited to second-order elliptic equations, notably Laplace's and Poisson's equations. This limitation stems from the fact that second-order derivatives of a function provide insights into the function's behavior at extrema, essentially stating that the gradient of a function at a maximum or minimum point is zero. The strong maximum principle asserts that if a function attains its maximum value within the domain's interior, the function must be uniformly constant throughout. On the other hand, the weak maximum principle indicates that the maximum value of the function is located on the boundary of the domain [2],[10],[12] as shown in Figure 1.

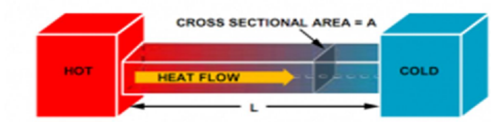

Figure 1: Extreme temperature at the boundaries.

1.3 Discrete maximum principle

The term 'discrete' implies that the numerical solution is available at finite number of specific points within the domain. The discrete maximum principles, appeared in early 1970s which are the modified form of the maximum principles. Over the past few decades, there has been extensive research conducted on them within the realm of linear PDEs and more recently also in non-linear PDEs. This is used to determine the highest and lowest values of grid function at grid points of the mesh [13],[15].

It is an important property of finite difference schemes of second-order PDEs that is the Poisson's equation is employed to derive various outcomes, including but not limited to uniqueness, boundedness, the comparison principle, and the non-negativity property. In brief, the discrete maximum principle mirrors the traditional maximum principle in a discrete version. It forms a fundamental attribute when dealing with solutions to elliptic PDEs, such as Poisson's equation, in the discrete domain. Essentially, it is a crucial concept for understanding and analyzing discrete solutions of these equations. When approximating Poisson's equation using the five-point Laplacian, the maximum principle ensures that the numerical solution satisfies certain qualitative properties, which can help us assess the correctness of the solution.

1.4 Error estimates

To estimate the error in the maximum norm when approximating solutions to Poisson's equation using the five-point Laplacian method, it is necessary to grasp various key concepts. These include comprehending the finite difference method, having a clear understanding of the maximum norm, and being skilled in error analysis techniques. These elements collectively contribute to the accurate assessment of error in the context of solving Poisson's equation.

It approximates derivatives by finite differences and discretizes the domain into a grid. In the context of solving Poisson's equation, we use a central difference scheme to approximate the second-order derivatives. The maximum norm, also known as the infinity norm or supremum norm, is a way to measure the maximum absolute value of a function over a given domain. The maximum norm of $\alpha(x, y)$ on D (domain) is defined as

$$\|\alpha\|_\infty = max\{|\alpha(x, y)| : (x, y) \in D.$$

Regarding solutions of Poisson's equation, we want to estimate the error in this maximum norm. In the process of assessing the error using the maximum norm in the finite difference method for approximating solutions, it is a standard procedure to compare the numerical solution to the exact solution. This comparative analysis serves as a common approach to ascertain the accuracy of the numerical approximation. This approach involves assessing the disparity between the computed result and the theoretically precise outcome. It is a standard procedure for evaluating the accuracy of the numerical approximation. The error E is defined as [12]

$$E = \|\alpha_{exact} - \alpha_{numerical}\|_\infty.$$

To estimate this error, we need to have the exact solution, which may not always be possible. If we have an analytical solution, we can compute the error directly.

However, in many practical cases, we might not have an exact solution, and we would rely on convergence analysis. This involves solving the problem on grids of increasing resolution (i.e., refining the mesh) and comparing the solutions. The error should decrease as we refine the mesh, and estimate the convergence rate. The convergence rate tells how fast the error decreases as we refine the grid. For many numerical methods, including finite differences, the error often scales with the grid spacing, i.e., h raised to a power, which can be determined through empirical observations or analysis. Hence, estimating the error in the maximum norm of Poisson's equation approximated by the five-point Laplacian involves comparing the numerical solution to the exact solution or using convergence analysis on successively refined grids to estimate the order of convergence. The precision of numerical solutions is assessed using the maximum norm as the measuring method. Regarding error estimates in the maximum norm, the maximum principle provides a useful qualitative check on the accuracy of the numerical solution.

1.5 Discrete form of Poisson's equation

The two dimensional Poission's equation is $\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} = f(x, y)$, where $\alpha(x, y)$ is the exact solution of Poisson's equation [3],[12]. Using the finite difference approximations and difference notations [14], we obtain the following finite difference schemes for Poission's equation.

The five point Laplacian scheme:
Let $\beta_{l,m}$ be the grid function at grid point (l, m). Using the central-difference operator for the second-order derivatives are [13]

$$\delta_x^2\beta_{l,m} = \frac{\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m}}{h^2}$$

and

$$\delta_y^2\beta_{l,m} = \frac{\beta_{l,m+1} - 2\beta_{(l,m)} + \beta_{l,m-1}}{k^2}.$$

Substituting in discrete version of Poisson's equation

$$\delta_x^2\beta_{l,m} + \delta_y^2\beta_{l,m} = f_{l,m}.$$

We obtain

$$\frac{\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m}}{h^2} + \frac{\beta_{l,m+1} - 2\beta_{l,m} + \beta_{l,m-1}}{k^2} = f_{l,m}. \quad (3)$$

For $h = k$,

$$\bigtriangledown_h^2\beta = \frac{1}{h^2}[\beta_{l+1,m} + \beta_{l-1,m} + \beta_{l,m+1} + \beta_{l,m-1} - 4\beta_{l,m}] \quad (4)$$

is the five-point Laplacian for Poisson's equation.

2 Discrete Maximum Principle for Poisson's Equation

The maximum principle for Poisson's equation is fundamental property that governs the behavior of its solutions in a domain. The maximum principle of Poisson's equation states that if α is the solution of Poisson's equation $\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} = f(x, y)$ in a bounded domain D with certain regularity conditions, then maximum (or minimum) value of α occurs on the boundary ∂D, rather than in the interior of D, provided that f is continuous [5]. Likewise, the strong maximum principle states that the maximum (or minimum) value of α attain at the interior point of the domain D, then α must be a constant [5].

On the other hand, the discrete maximum principle adapted to the numerical solutions obtained through discretization methods, that is, the continuous domain is discretized Poisson's equation (3) on a grid in a domain D. Then the maximum (or minimum) value of $\beta_{l,m}$ over all grid points in the interior of D cannot be larger than the maximum value of $\beta_{l,m}$ on the boundary of D. It is important in ensuring the stability and accuracy of numerical solutions and preventing unphysical behaviors in the computed results. It is a discrete analog of the maximum principle [2].

Theorem 2.1. [13]
If condition $\triangledown_h^2\beta \geq 0$ is met within a specific region, then the highest value of β within that region is reached at its boundary. Likewise, if condition $\triangledown_h^2\beta \leq 0$ holds true, then the lowest value of β is achieved at the boundary of that region where $\triangledown_h^2 = \delta_x^2 + \delta_y^2$.

Proof. **(Using five-point Laplacian)**
From the equation (3), the discrete version of Poisson's equation is

$$\frac{\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m}}{h^2} + \frac{\beta_{l,m+1} - 2\beta_{l,m} + \beta_{l,m-1}}{k^2} = f_{l,m}$$

$$\frac{1}{h^2}[\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m} + \eta(\beta_{l,m+1} - 2\beta_{l,m} + \beta_{l,m-1})] = f_{l,m}, \qquad \text{where} \quad h \neq k, \quad \eta = \frac{h^2}{k^2}.$$

For $h \neq k$, the five-point Laplacian is

$$\triangledown_h^2\beta = \frac{1}{h^2}[\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m} + \eta(\beta_{l,m+1} - 2\beta_{l,m} + \beta_{l,m-1})].$$

Since $\triangledown_h^2\beta \geq 0$,

$$\frac{1}{h^2}[\beta_{l+1,m} - 2\beta_{l,m} + \beta_{l-1,m} + \eta(\beta_{l,m+1} - 2\beta_{l,m} + \beta_{l,m-1})] \geq 0$$

$$\beta_{l+1,m} + \beta_{l-1,m} + \eta\beta_{l,m+1} + \eta\beta_{l,m-1} \geq 2(1+\eta)\beta_{l,m}.$$

Let the maximum of β is obtained on an interior point $\beta_{l,m}$ and one of its neighborhood has value strictly less than $\beta_{l,m}$. Then,

$$2(1+\eta)\beta_{l,m} > \beta_{l+1,m} + \beta_{l-1,m} + \eta\beta_{l,m+1} + \eta\beta_{l,m-1} \geq 2(1+\eta)\beta_{l,m},$$

which is a contradiction. Hence, On the boundary, β reaches its maximum value for this region.
In the same manner, when $\triangledown_h^2\beta \leq 0$ by considering $\triangledown_h^2(-\beta) \geq 0$, we conclude that $\max(-\beta)$ is reached on the boundary and hence minimum (β) is reached on the boundary. □

3 Applications of Discrete Maximum Principle

The discrete maximum principle, which ensures that numerical solutions to partial differential equations maintain certain properties, is a crucial concept in various fields of science and engineering. Its applications are widespread and play a vital role in ensuring the accuracy, stability, and physical realism of numerical simulations. There are many more applications of discrete maximum principle, among them we focus on the error estimate of maximum norm of discrete version of Poisson's equation where discrete maximum principle provides a valuable tool for obtaining upper and lower bounds in the numerical solution [13].

3.1 Error in the maximum norm

Estimating the error in the maximum norm of discrete version Poisson's equation involves comparing numerical solutions to exact solutions. For a discrete function $\beta_{l,m}$ define on a domain Ω, the maximum norm is defined as [8],[12],[13]

$||\beta||_\infty = ||\beta||_{\infty,\Omega} = \max_{l,m} |\beta_{l,m}|$, where $\beta_{l,m}$ is discrete function.

The global error global E is defined by [9],[12]

$$E = ||\alpha - \beta||_\infty = sup|\alpha(x_l, y_m) - \beta_{l,m}|. \tag{5}$$

Local Truncation Error: Local truncation error is the discrepancy at grid points between the differential equation and its finite difference approximation. It assesses the degree to which a discretization with finite differences approximates the differential equation.

For Poisson's equation with Dirichlet's boundary value is $\dfrac{\partial^2\alpha}{\partial x^2} + \dfrac{\partial^2\alpha}{\partial y^2} = f(x,y),\ 0 < x, y < 1$ and $\alpha(x,0) = \alpha(x,1) = \alpha(0,y) = \alpha(1,y) = 0$,
where $\alpha(x,y)$ is the exact solution of Poisson's equation.

Using central finite difference schemes for second-order derivatives in the x- and y-directions respectively. Let $\beta_{l,m}$ be the grid function at grid point (l,m). Then the discrete version of Poisson's equation is

$$\frac{\beta_{l-1,m} - 2\beta_{l,m} + \beta_{l+1,m}}{h_2^2} + \frac{\beta_{l,m-1} - 2\beta_{l,m} + \beta_{l,m+1}}{h_1^2} = f_{lm} + T_{lm}, \tag{6}$$

where $l = 1, 2, ..., m-1, m = 1, 2, ..., n-1$ and T_{lm} is the local truncation error at the grid point [9], where the solution is unknown, that is

$$T_{lm} \approx \frac{h_1^2}{12}\frac{\partial^4}{\partial x^4}\alpha(x_l, y_m) + \frac{h_2^2}{12}\frac{\partial^4}{\partial y^4}\alpha(x_l, y_m) + \circ(h_1^4, h_2^4).$$

When $h = max\{h_1, h_2\}$, $T_{lm} \approx \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha(x_l, y_m) + \frac{\partial^4}{\partial y^4}\alpha(x_l, y_m)]$ (7)

If $\bigtriangledown\alpha(x,y) = \dfrac{\partial^2\alpha}{\partial x^2} + \dfrac{\partial^2\alpha}{\partial y^2} = f(x,y)$ and $\bigtriangledown_h\beta_{l,m} = \frac{\beta_{l-1,m}+\beta_{l+1,m}+\beta_{l,m-1}+\beta_{l,m+1}-4\beta_{l,m}}{h^2}$ (five point Laplacian)
Then, $T(x_l, y_m) = \bigtriangledown_h\beta_{l,m} - \bigtriangledown\alpha(x_l, y_m)$.

A finite difference scheme is consistent if $T(x_l, y_m) = lim_{h\to 0}T(x_l, y_m) = lim_{h\to 0}\bigtriangledown_h\beta_{l,m} - \bigtriangledown\alpha(x_l, y_m) = 0$

Consider $A\alpha = F + T$, $A\beta = F$ and $E = \beta - \alpha$,
where A is the coefficient matrix of the finite difference equations, F is the modified source term, T is the local truncation error at the grid point, E is the global error.
Then, $A(\alpha - \beta) = T \Rightarrow -AE = T$.
If A is non-singular then $\parallel E \parallel = \parallel A^{-1}T \parallel \leq \parallel A^{-1} \parallel\parallel T \parallel$ [9].
Hence, global error E depends on both $\parallel A^{-1} \parallel$ and local truncation error $\parallel T \parallel$.
Hence, finite difference method for BVP is stable if A is invertiable and $\parallel A^{-1} \parallel \leq C$ (bounded by a constant) for all $0 < h < h_0$, where constants C and h_0 are independent of h.
Therefore, a finite difference method is consistent and stable iff it is convergent.

For example, for Poisson's equation with Dirichlet's boundary values is $\dfrac{\partial^2\alpha}{\partial x^2} + \dfrac{\partial^2\alpha}{\partial y^2} = f(x,y), 0 < x, y < 1$ and $\alpha(x,0) = \alpha(x,1) = \alpha(0,y) = \alpha(1,y) = 0$,
where $\alpha(x,y) = cos\pi x cos\pi x$ is the exact solution of Poisson's equation. The plot of comparative solutions that is exact and numerical solutions with error as shown in the Figure 2.

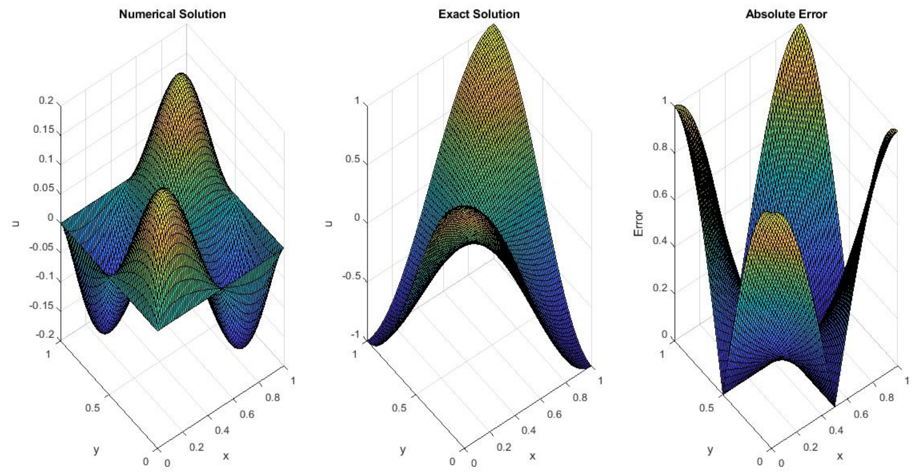

Figure 2: Numerical and exact solutions of Poission's equation with error.

Theorem 3.1. [8],[13] If a discrete function $\beta_{l,m}$ is defined on a grid in a square of side s with $\beta_{l,m} = 0$ on the boundary, then

$$\parallel \beta \parallel_{\infty} \leq \frac{s^2}{8} \parallel \bigtriangledown_h^2 \parallel_{\infty} .$$

Proof. Define the function $f_{l,m}$ in the interior of the square of side s by

$$\bigtriangledown_h^2 \beta_{l,m} = f_{l,m} \quad \text{and} \quad \beta_{l,m}|_{\partial\Omega} = 0,$$

where Ω is an open, connected and bounded open set in $\mathbb{R}^2$.
We have

$$||\beta||_{\infty} = \max_{(l,m)\in I} |\beta_{l,m}|, \quad \text{where} \quad I = \{(l,m) : (x_l, y_m) \in (0,s) \times (0,s)\}.$$

Since $\bigtriangledown_h^2 \beta_{l,m} = f_{l,m}$,

$$-max|f_{l,m}| \leq \bigtriangledown_h^2 \beta_{l,m} \leq max|f_{l,m}| \quad \Rightarrow \quad -||f||_{\infty} \leq \bigtriangledown_h^2 \beta_{l,m} \leq ||f||_{\infty} \qquad (i)$$

Let us construct a grid function,

$$\gamma_{l,m} = \frac{1}{4}[(x_l - \frac{s}{2})^2 + (y_m - \frac{s}{2})^2],$$

on all grid-points including those on the boundary. Note that γ is non-negative.
Claim-1:

$$\bigtriangledown_h^2 \gamma_{l,m} = 1 \quad and \quad \gamma_{l,m} \geq 0.$$

Clearly, $\gamma_{l,m} \geq 0$.
We have,

$$\bigtriangledown_h^2 \gamma_{l,m} = \delta_x^2 \gamma_{l,m} + \delta_y^2 \gamma_{l,m} \qquad (ii)$$

From the second-order symmetric central difference approximation to the second derivative of the grid function $\gamma_{l,m}$, we have

$$\begin{aligned}\delta_x^2 \gamma_{l,m} &= \frac{\gamma_{l+1,m} - 2\gamma_{l,m} + \gamma_{l-1,m}}{h^2} \\ &= \frac{1}{4h^2}\{[(x_{l+1} - \frac{s}{2})^2 + (y_m - \frac{s}{2})^2] - 2[(x_l - \frac{s}{2})^2 + (y_m - \frac{s}{2})^2] + [(x_{l-1} - \frac{s}{2})^2 + (y_m - \frac{s}{2})^2]\}\end{aligned}$$

$$= \frac{1}{4h^2}[(x_l + h - \frac{s}{2})^2 - 2(x_l - \frac{s}{2})^2 + [(x_l - h - \frac{s}{2})^2] = \frac{1}{2}.$$

Similarly, $\delta_y^2 \gamma_{l,m} = \frac{1}{2}$.
Hence, from (ii), we obtain $\bigtriangledown_h^2 \gamma_{l,m} = 1$.

Claim-2: $\gamma_{l,m} \leq \frac{s^2}{8}$ on $\partial\Omega$
We have,

$$\gamma_{l,m} = \frac{1}{4}[(x_l - \frac{s}{2})^2 + (y_m - \frac{s}{2})^2] \leq \frac{1}{4}[(s - \frac{s}{2})^2 + (s - \frac{s}{2})^2] = \frac{1}{4}(\frac{s^2}{4} + \frac{s^2}{4}) = \frac{s^2}{8}.$$

By using inequality (i) and linearity, we obtain

$$\bigtriangledown_h^2(\beta_{l,m} + ||f||_\infty \gamma_{l,m}) = \bigtriangledown_h^2 \beta_{l,m} + ||f||_\infty \bigtriangledown_h^2 \gamma_{l,m}) = \bigtriangledown_h^2 \beta_{l,m} + ||f||_\infty \geq 0.$$

By the discrete maximum principle,

$$\beta_{l,m} \leq \beta_{l,m} + ||f||_\infty \gamma_{l,m} \leq \beta_{l,m}|_{\partial\Omega} + ||f||_\infty \gamma_{l,m}|_{\partial\Omega} = 0 + ||f||_\infty \gamma_{l,m}|_{\partial\Omega} \leq \frac{s^2}{8}||f||_\infty.$$

Similarly, using inequality (i), we obtain $\beta_{l,m} \geq -\frac{s^2}{8}||f||_\infty$.
Combining them, we obtain

$$-\frac{s^2}{8}||f||_\infty \leq \beta_{l,m} \leq \frac{s^2}{8}||f||_\infty \Rightarrow |\beta_{l,m}| \leq \frac{s^2}{8}||f||_\infty \Rightarrow max|\beta_{l,m}| \leq \frac{s^2}{8}||f||_\infty$$

$$||\beta||_\infty \leq \frac{s^2}{8}||f||_\infty.$$

Since

$$|| \bigtriangledown_h^2 \beta||_\infty = \max_{(l,m)\in I} | \bigtriangledown_h^2 \beta| = \max_{(l,m)\in I} |f_{l,m}| = ||f||_\infty.$$

Therefore,

$$||\beta||_\infty \leq \frac{s^2}{8}|| \bigtriangledown_h^2 \beta_{l,m}||_\infty.$$

□

Corollary 3.1. [13] If a discrete function $\beta_{l,m}$ is defined on a grid on the unit square with $\beta_{l,m} = 0$ on the boundary, then

$$\| \beta \|_\infty \leq \frac{1}{8} \| \bigtriangledown_h^2 \|_\infty .$$

Theorem 3.2. [8],[13] If $\beta_{l,m}$ is a discrete function defined on a grid on the domain $-1 < x < 1, \quad -1 < y < 1$, except for $0 < x < 1, \quad -1 < y < 0$, i.e., the points in quadrants first, second and third with $|x| \quad and \quad |y|$ less than 1 and $\beta_{l,m} = 0$ on the boundary. For this domain prove the estimate

$$\| \beta \|_\infty \leq \frac{5}{2} \| \bigtriangledown_h^2 \beta \|_\infty .$$

Proof. Let given domain be $\Omega = \{(x, y) \in$ first, second and third quadrants with $|x|, \quad |y| < 1\}$.
Define the function $f_{l,m}$ in the interior of the Ω by

$$\bigtriangledown_h^2 \beta_{l,m} = f_{l,m} \quad and \quad \beta_{l,m}|_{\partial\Omega} = 0.$$

Since $\bigtriangledown_h^2 \beta_{l,m} = f_{l,m}$, $\quad -max|f_{l,m}| \leq \bigtriangledown_h^2 \beta_{l,m} \leq max|f_{l,m}|$

$$-||f||_\infty \leq \bigtriangledown_h^2 \beta_{l,m} \leq ||f||_\infty \qquad (i)$$

Let us construct a grid function,
$\gamma_{l,m} = (x_l + \frac{1}{2})^2 + (y_m - \frac{1}{2})^2$, on all grid-points including those on the boundary. Note that γ is non-negative.

Claim-1: $\nabla_h^2 \gamma_{l,m} = 4 \quad and \quad \gamma_{l,m} \geq 0,$
We have,

$$\nabla_h^2 \gamma_{l,m} = \delta_x^2 \gamma_{l,m} + \delta_y^2 \gamma_{l,m}. \qquad (ii)$$

From the second-order symmetric central difference approximation to the second derivative of the grid function $\gamma_{l,m}$, we obtain

$$\begin{aligned}\delta_x^2 \gamma_{l,m} &= \frac{\gamma_{l+1,m} - 2\gamma_{l,m} + \gamma_{l-1,m}}{h^2} \\ &= \frac{1}{h^2}\{[(x_{l+1}+\frac{1}{2})^2+(y_m-\frac{1}{2})^2]-2[(x_l+\frac{1}{2})^2+(y_m-\frac{1}{2})^2]+[(x_{l-1}+\frac{1}{2})^2+(y_m-\frac{1}{2})^2]\} \\ &= \frac{1}{h^2}[(x_l+h+\frac{1}{2})^2 - 2(x_l-\frac{1}{2})^2 + [(x_l-h+\frac{1}{2})^2] = 2.\end{aligned}$$

Similarly, $\delta_y^2 \gamma_{l,m} = 2$.
Hence, from (ii), we obtain $\nabla_h^2 \gamma_{l,m} = 4$.

Claim-2: $\gamma_{l,m} \leq \frac{5}{2}$ on $\partial\Omega$
We have, $\gamma_{l,m} = [(x_l + \frac{1}{2})^2 + (y_m - \frac{1}{2})^2] \leq [(1 + \frac{1}{2})^2 + (1 - \frac{1}{2})^2] \leq \frac{5}{2}$.

By using linearity and inequality (i), we obtain

$$\nabla_h^2(\beta_{l,m} + ||f||_\infty \gamma_{l,m}) = \nabla_h^2 \beta_{l,m} + ||f||_\infty \nabla_h^2 \gamma_{l,m}) = \nabla_h^2 \beta_{l,m} + 4||f||_\infty \geq 0.$$

By the discrete maximum principle,

$$\beta_{l,m} \leq \beta_{l,m} + ||f||_\infty \gamma_{l,m} \leq \beta_{l,m}|_{\partial\Omega} + ||f||_\infty \gamma_{l,m}|_{\partial\Omega} = 0 + ||f||_\infty \gamma_{l,m}|_{\partial\Omega} \leq \frac{5}{2}||f||_\infty.$$

$\therefore \ \beta_{l,m} \leq \frac{5}{2}||f||_\infty$.
Similarly, by using inequality (i), we obtain, $\beta_{l,m} \geq -\frac{5}{2}||f||_\infty$.
Combining them, we obtain

$$-\frac{5}{2}||f||_\infty \leq \beta_{l,m} \leq \frac{5}{2}||f||_\infty \Rightarrow |\beta_{l,m}| \leq \frac{5}{2}||f||_\infty \Rightarrow max|\beta_{l,m}| \leq \frac{5}{2}||f||_\infty \Rightarrow ||\beta||_\infty \leq \frac{5}{2}||f||_\infty.$$

Since

$$|| \nabla_h^2 \beta||_\infty = \max_{(l,m)\in I} | \nabla_h^2 \beta| = \max_{(l,m)\in I} |f_{l,m}| = ||f||_\infty.$$

Therefore, $||\beta||_\infty \leq \frac{5}{2}|| \nabla_h^2 \beta_{l,m}||_\infty$. □

Theorem 3.3. [8],[13] Let $\alpha(x, y)$ be the solution to $\nabla^2\alpha = f$ on the unit-square having Dirichlet boundary conditions and let $\beta_{l,m}$ be the solution to $\nabla_h^2 \beta = f$ with $\beta_{l,m} = \alpha(x_l, y_m)$ on the boundary. Then,

$$\| E \|_\infty = \| \alpha - \beta \|_\infty \leq ch^2 \| \partial^4 \alpha \|_\infty,$$

where $\| \partial^4 \beta \|_\infty$ represents the greatest absolute value among all the fourth derivatives of β within the interior of the square.

Proof. We have,

$$\| \partial^4 \alpha \|_\infty = \max_{(x,y)\in\Omega} \{|\frac{\partial^4 u}{\partial x^4}|, |\frac{\partial^4 \alpha}{\partial y^4}|\}.$$

Now,

$$\begin{aligned}\nabla_h^2 \alpha_{l,m} &= \nabla_h^2 \alpha(x_l, y_m) = \delta_x^2 \alpha_{l,m} + \delta_y^2 \alpha_{l,m} \\ &= \frac{\alpha_{l+1,m} - 2\alpha_{l,m} + \alpha_{l-1,m}}{h^2} + \frac{\alpha_{l,m+1} - 2\alpha_{l,m} + \alpha_{l,m-1}}{h^2}. \qquad (i)\end{aligned}$$

By Taylor series expansion, we obtain

$$\alpha_{l\pm1,m} = \alpha(x_l \pm h, y_m) = \alpha_{l,m} \pm h\frac{\partial}{\partial x}\alpha(x_l, y_m) + \frac{h^2}{2}\frac{\partial^2}{\partial x^2}\alpha(x_l, y_m) \pm \frac{h^3}{6}\frac{\partial^3}{\partial x^3}\alpha(x_l, y_m) + \frac{h^4}{24}\frac{\partial^4}{\partial x^4}\alpha(x_l + \theta_1 h, y_m).$$

and

$$\alpha_{l,m\pm1} = \alpha(x_l, y_m \pm h) = \alpha_{l,m} \pm h\frac{\partial}{\partial y}\alpha(x_l, y_m) + \frac{h^2}{2}h\frac{\partial^2}{\partial y^2}\alpha(x_l, y_m) \pm \frac{h^3}{6}\frac{\partial^3}{\partial y^3}\alpha(x_l, y_m) + \frac{h^4}{24}\frac{\partial^4}{\partial y^4}\alpha(x_l, y_m + \theta_2 h).$$

Putting these values in (i), we obtain

$$\begin{aligned}
\triangledown_h^2 \alpha_{l,m} &= [\frac{\partial^2}{\partial x^2}\alpha(x_l, y_m) + \frac{\partial^2}{\partial y^2}\alpha(x_l, y_m)] + \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}u(x_l + \theta_1 h, y_m) + \frac{\partial^4}{\partial y^4}u(x_l + y_m\theta_2 h)] \\
&= \triangledown_h^2 \alpha(x_l, y_m) + \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}u] \\
&= f(x_l, y_m) + \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}\alpha] \quad [since \triangledown^2 \alpha = f] \\
&= \triangledown_h^2 \beta_{l,m} + \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}\alpha].
\end{aligned}$$

Thus, we have

$$\begin{cases} \triangledown_h^2(\alpha_{l,m} - \beta_{l,m}) = \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}\alpha] \\ \alpha_{l,m} - \beta_{l,m} = 0 \quad on \quad \partial\Omega. \end{cases}$$

Then, by above mentioned corollary

$$\| E \|_\infty = \| \alpha_{l,m} - \beta_{l,m} \|_\infty \leq \frac{1}{8} \| \frac{h^2}{12}[\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}\alpha] \|_\infty \leq \frac{h^2}{96} \| [\frac{\partial^4}{\partial x^4}\alpha + \frac{\partial^4}{\partial y^4}\alpha] \|_\infty = ch^2 \| \partial^4\alpha \|_\infty .$$

Therefore,

$$\| \alpha - \beta \|_\infty \leq ch^2 \| \partial^4\alpha \|_\infty, \quad \text{where} \quad c = \frac{1}{96}.$$

□

4 Conclusion

In conclusion, this study was addressed the critical issue of error estimation in the maximum norm for numerical solution of Poisson's equation when employing the five-points Laplacian and the discrete maximum principle. Our finding have demonstrated the importance of accurate error assessment in ensuring the reliability and precision. We have observed that maximum norm of discrete function $\beta_{l,m}$ with $\beta_{l,m} = 0$ on the boundary is estimated on grid points in a square of side s as well as grid points in first, second and third quadrants on the unit square with $|x| < 1, |y| < 1$. We, also, observed that error estimates of analytic and numerical solutions of Poisson's equation on the unit square with Dirichlet's boundary condition. These insights have implications not only for the field of numerical method but also for practical applications in real-world phenomena like fluid dynamics as well as structural analysis.

Looking ahead, there are several promising direction for future research such as the application of error estimates to non-linear Poisson-like equation as well as extend to three-dimensional Poisson-like problem and time dependent equations. As we continue to refine error estimation techniques and explore new avenues, we can contribute to the ongoing advancement of scientific computing and mathematical modeling.

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