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Blow up at Infinite Time of Solutions for a Plate Equation with Delay Term

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Abstract: *In this article, we investigate the viscoelastic plate equation with both delay and source terms. Initially, we give the local-global existence results. Later, we establish the blow-up results at infinite time by utilizing the energy method when $E(0) < 0$ under suitable conditions. Delays effect generally seems in many practical problems for instance medicine, biological, chemical, physical, thermal, economic phenomena, electrical engineering systems and mechanical applications.*

Keywords: Blow up, Delay, Viscoelastic plate equation

1 Introduction

This manuscript is devoted to the blow-up at infinite time of solutions for the viscoelastic plate equation with delay term as follows

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t \varpi(t-q)\, \Delta^2 u(q)\, dq + \mu_1 u_t & \\ + \int_{\tau_1}^{\tau_2} |\mu_2(q)|\, u_t(x, t-q)\, dq = b\, |u|^{p-2} u, & x \in \Omega,\ t > 0, \\ u(x,t) = \frac{\partial u(x,t)}{\partial \upsilon} = 0, & x \in \partial\Omega,\ t \in [0, \infty), \\ u_t(x, -t) = f_0(x,t), & (x,t) \in \Omega \times (0, \tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where b, μ_1 are positive constants, $p > 2$ and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$ and μ_2 is bounded-function, and ϖ is a differentiable function. The unit outward normal vector is υ.

For many researchers, problems involved the mathematical behavior results for PDE's with delay have become attractive because delays generally seem in many practical problems for instance, electrical engineering systems, mechanical applications, medicine, thermal, economic phenomena, physical, chemical, biological. Furthermore, delay effects may devastate the stabilizing properties of a system. There are some examples that show how delay effects destabilize the control systems in the literature [7, 8].

Datko et al. [5], in 1986, showed delay is source of in-stability. In [14], Nicaise and Pignotti studied the equation with time delay term as follows

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0. \tag{2}$$

In the case $0 < \mu_2 < \mu_1$, they proved some stability results. Zuazua [23] obtained exponentially stability for the equation (2) in the absence of delay.

Cavalcanti et al. [4], concerned the equation as follows

$$u_{tt} + \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\, \Delta^2 u(s)\, ds + a(t)\, u_t = 0 \tag{3}$$

in $\Omega \times (0, \infty)$. They proved the decay of solutions if $\gamma = 0$ for (3). Rivera et al. [22] showed that the energy decay of the (3). Moreover, see also Lagnese [9] for more detail on (3).

Mukiawa [11], handled the equation as follows

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\, \Delta^2 u(s)\, ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0 \tag{4}$$

with delay term. The author obtained decay of solutions for the equation (4).

In [12], Mustafa and Kafini studied the equation with delay as follows

$$u_{tt} + \Delta^2 u - \int_0^\infty g(s)\, \Delta^2 u(t-s)\, ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = u\,|u|^\gamma . \tag{5}$$

They proved general decay of solutions when $|\mu_2| \leq \mu_1$ for the equation (5).

In [2], the authors studied the equation as follows

$$u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\, \Delta u(s)\, ds + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, u_t(x, t-\rho)\, d\rho = b\,|u|^{p-2}\, u. \tag{6}$$

Choucha et al. [2] proved the blow up results under suitable conditions for the equation (6). In [3], the authors indicated the growth results for the equation (6). Recently, some other authors considered related equations (see [10, 16, 17, 18, 19, 20, 21]).

There is no research about the blow up at infinite time of solutions of the equation (1) with delay term to our best knowledge. Hence, the manuscript is generalization of the above studies. The main aim in this work is to establish the blow up results at infinite time of the equation (1) with delay. This paper is planned as following: In section 2, we give needed materials for the proof. In section 3, we establish the main result.

2 Preliminaries

In this section, we denote the necessary materials. Usually, the notation $\|.\|_p$ indicates L^p norm, and $(.,.)$ is the L^2 inner product. Particularly, we give $\|.\|$ instead $\|.\|_2$(For detailed, see [1, 15]).
We, now, yield the needed assumptions
(A1) $\varpi \in (R_+, R_+)$ is a non-increasing function, such that

$$\varpi(t) \geq 0,\ 1 - \int_0^\infty \varpi(q)\, dq = l > 0. \tag{7}$$

(A2) There exists the $\xi > 0$ constant, so that

$$\varpi'(t) \leq -\xi \varpi(t),\ t \geq 0. \tag{8}$$

(A3) $\mu_2 : [\tau_1, \tau_2] \to R$ is bounded function, such that

$$\left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq \leq \mu_1,\ \delta > \frac{1}{2}. \tag{9}$$

Assume $B_p > 0$ be a constant satisfies [1]

$$\|\nabla v\|_p \leq B_p \|\Delta v\|_p,\ \text{ for } v \in H_0^2(\Omega). \tag{10}$$

It holds

$$\begin{aligned} \int_0^t \varpi(t-q)\left(\Delta u(q), \Delta u_t(t)\right) dq &= -\frac{1}{2}\varpi(t)\, \|\Delta u(t)\|^2 + \frac{1}{2}\left(\varpi' \circ \Delta u\right)(t) \\ &\quad -\frac{1}{2}\frac{d}{dt}\left[(\varpi \circ \Delta u)(t) - \left(\int_0^t g(q)\, dq\right) \|\Delta u(t)\|^2\right], \end{aligned} \tag{11}$$

where

$$(\varpi \circ \Delta u)(t) = \int_\Omega \int_0^t \varpi(t-q)\, |\Delta u(t) - \Delta u(q)|^2\, dq. \tag{12}$$

Firstly, similar to [13], we give the new function

$$y(x,\rho,q,t) = u_t(x,t-q\rho).$$

Hence, we have

$$\begin{cases} qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0 \\ y(x,0,q,t) = u_t(x,t). \end{cases} \tag{13}$$

Therefore, the problem (1) becomes

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t \varpi(t-q)\Delta^2 u(q)\,dq + \mu_1 u_t \\ +\int_{\tau_1}^{\tau_2} |\mu_2(q)|\,|y(x,1,q,t)|\,dq = b|u|^{p-2}u, \ x\in\Omega,\ t>0, \\ qy_t(x,\rho,q,t) + y_\rho(x,\rho,q,t) = 0, \end{cases} \tag{14}$$

with initial-boundary conditions

$$\begin{cases} u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0, \qquad x\in\partial\Omega, \\ y(x,\rho,q,0) = f_0(x,q\rho), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \tag{15}$$

where

$$(x,\rho,q,t) \in \ \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

We, now, give without proof the theorem of local existence similar to [6].

Theorem 2.1. Assume that (7)-(9) hold. Suppose

$$\begin{cases} p\geq 2,\ n = 1,2,3,4, \\ 2 < p < \frac{2n-2}{n-4},\ n>4. \end{cases} \tag{16}$$

For any initial data with compact support

$$(u_0,\ u_1,\ f_0) \in H_0^2(\Omega)\times H_0^2(\Omega)\times L^2(\ \Omega\times(0,1)\times(\tau_1,\tau_2))$$

the problem (14)-(15) has a unique solution

$$u \in C\left([0,T]; H_0^2(\Omega)\times H_0^2(\Omega)\times L^2(\ \Omega\times(0,1)\times(\tau_1,\tau_2))\right)$$

for some $T>0$.

We, now, denote the global existence results similar to [24].

Theorem 2.2. Assume that (7)-(9) and (16) hold. If $u_0\in L^2$, $u_1\in H_0^2(\Omega)$ and

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{17}$$

where C_* is the constant of Poincare. Hence, the local solution is global.

Lemma 2.0.1. Suppose that (7-(9) and (16) satisfy and let $u(t)$ be a solution of (14), then $E(t)$ is non-increasing such that

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1-\int_0^t \varpi(q)\,dq\right)\|\Delta u\|^2 + \frac{1}{2}(\varpi\circ\Delta u)(t) \\ &\quad +\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} q|\mu_2(q)|\left|y^2(x,\rho,q,t)\right| dq d\rho dx - \frac{b}{p}\|u\|_p^p \end{aligned} \tag{18}$$

satisfies

$$E'(t) \leq -c_1\left(\|u_t\|^2 + \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(q)|\left|y^2(x,1,q,t)\right| dq dx\right). \tag{19}$$

Proof. Multiply the first equation of (14) by u_t and integrate over Ω, we get

$$\begin{aligned}
& \frac{d}{dt}\left\{\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left(1-\int_0^t \varpi(q)\,dq\right)\|\Delta u\|^2+\frac{1}{2}(\varpi \circ \Delta u)(t)-\frac{b}{p}\|u\|_p^p\right\} \\
= & -\mu_1\left\|u_t\right\|^2-\int_{\Omega} u_t \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right||y(x, 1, q, t)|\, dq dx \\
& +\frac{1}{2}\left(\varpi^{\prime} \circ \Delta u\right)(t)-\frac{1}{2} \varpi(t)\|\Delta u\|^2 \qquad (20)
\end{aligned}$$

and, we get

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} q\left|\mu_2(q)\right|\left|y^2(x, \rho, q, t)\right| dq d\rho dx \\
= & -\frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} 2\left|\mu_2(q)\right| y y_\rho\, dq d\rho dx \\
= & \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right|\left|y^2(x, 0, q, t)\right| dq dx \\
& -\frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right|\left|y^2(x, 1, q, t)\right| dq dx \\
= & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right| dq\right)\left\|u_t\right\|^2 \\
& -\frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right|\left|y^2(x, 1, q, t)\right| dq dx . \qquad (21)
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{d}{dt} E(t) = & -\mu_1\left\|u_t\right\|^2-\int_{\Omega} \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right|\left|u_t y(x, 1, q, t)\right| dq dx+\frac{1}{2}\left(\varpi^{\prime} \circ \Delta u\right)(t) \\
& -\frac{1}{2} \varpi(t)\|\Delta u\|^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right| dq\right)\left\|u_t\right\|^2 \\
& -\frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2}\left|\mu_2(q)\right|\left|y^2(x, 1, q, t)\right| dq dx . \qquad (22)
\end{aligned}$$

By (20) and (21), we have (18). From Young's inequality, (7)-(9) in (22), we have (19). □

Now, we get the main result. Firstly, we define

$$\begin{aligned}
H(t) = & -E(t) \\
= & \frac{b}{p}\|u\|_p^p-\frac{1}{2}\left\|u_t\right\|^2-\frac{1}{2}\left(1-\int_0^t \varpi(q)\, dq\right)\|\Delta u\|^2 \\
& -\frac{1}{2}(\varpi \circ \Delta u)(t)-\frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} q\left|\mu_2(q)\right|\left|y^2(x, \rho, q, t)\right| dq d\rho dx . \qquad (23)
\end{aligned}$$

3 Blow up at infinite time

In this section, we get the blow-up at infinite time of solutions for the problem (14)-(15).

Theorem 3.1. Let (7)-(9) and (16) hold. Suppose that $E(0)<0$ holds. Hence, the local solution for the problem (14) grows exponentially.

Proof. By (18), we get

$$E(t) \leq E(0) \leq 0. \qquad (24)$$

Therefore,

$$\begin{aligned} H'(t) &= -E'(t) \geq c_1 \left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \right) \\ &\geq c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \geq 0 \end{aligned} \tag{25}$$

and

$$0 \leq H(0) \leq H(t) \leq \frac{b}{p} \|u\|_p^p. \tag{26}$$

We set

$$\mathcal{K}(t) = H(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon \mu_1}{2} \int_\Omega u^2 dx, \tag{27}$$

where $\varepsilon > 0$ to be specified later. We multiply the first equation (14) by u and take derivative of (27), we have

$$\begin{aligned} \mathcal{K}'(t) &= H'(t) + \varepsilon \|u_t\|^2 + \varepsilon \int_\Omega \Delta u \int_0^t \varpi(t-q) \Delta u(q) dqdx \\ &\quad -\varepsilon \|\Delta u\|^2 + \varepsilon b \int_\Omega |u|^p dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |uy(x,1,q,t)| dqdx. \end{aligned} \tag{28}$$

By using

$$\begin{aligned} &\varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| |uy(x,1,q,t)| dqdx \\ \leq\ & \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \|u\|^2 + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx \right\}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\varepsilon \int_0^t \varpi(t-q) dq \int_\Omega \Delta u \Delta u(q) dxdq \\ =\ & \varepsilon \int_0^t \varpi(t-q) dq \int_\Omega \Delta u (\Delta u(q) - \Delta u(t)) dxdq \\ &+\varepsilon \int_0^t \varpi(q) dq \|\Delta u\|^2 \\ \geq\ & \frac{\varepsilon}{2} \int_0^t \varpi(q) dq \|\Delta u\|^2 - \frac{\varepsilon}{2} (\varpi \circ \Delta u)(t). \end{aligned} \tag{30}$$

We get, by (28),

$$\begin{aligned} \mathcal{K}'(t) &\geq H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \left(1 - \frac{1}{2} \int_0^t \varpi(q) dq \right) \|\Delta u\|^2 \\ &\quad +\varepsilon b \|u\|_p^p - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \|u\|^2 \\ &\quad -\frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(q)| \left| y^2(x,1,q,t) \right| dqdx + \frac{\varepsilon}{2} (\varpi \circ \Delta u)(t). \end{aligned} \tag{31}$$

From (25) and set δ_1 such that, $\frac{1}{4\delta_1 c_1} = \kappa$, substitute in (31), we obtain

$$\begin{aligned} \mathcal{K}'(t) &\geq [1 - \varepsilon\kappa] H'(t) + \varepsilon \|u_t\|^2 \\ &\quad - \left[\left(1 - \frac{1}{2} \int_0^t \varpi(q) dq \right) \right] \|\Delta u\|^2 + \varepsilon b \|u\|_p^p \\ &\quad -\frac{\varepsilon}{4c_1\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(q)| dq \right) \|u\|^2 + \frac{\varepsilon}{2} (\varpi \circ \Delta u)(t). \end{aligned} \tag{32}$$

For $0 < a < 1$, by (23)

$$\begin{aligned}
\varepsilon b \|u\|_p^p &= \varepsilon p(1-a) H(t) + \frac{\varepsilon p(1-a)}{2} \|u_t\|^2 + \varepsilon b a \|u\|_p^p \\
&+ \frac{\varepsilon p(1-a)}{2} \left(1 - \int_0^t \varpi(q)\, dq\right) \|\Delta u\|^2 \\
&+ \frac{\varepsilon}{2} p(1-a)(\varpi \circ \Delta u)(t) \\
&+ \frac{\varepsilon p(1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q |\mu_2(q)| \left|y^2(x,\rho,q,t)\right| dq d\rho dx. \qquad (33)
\end{aligned}$$

Substitute in (32), we get

$$\begin{aligned}
\mathcal{K}'(t) &\geq [1 - \varepsilon\kappa] H'(t) + \varepsilon \left[\frac{p(1-a)}{2} + 1\right] \|u_t\|^2 \\
&+ \varepsilon \left[\left(\frac{p(1-a)}{2}\right)\left(1 - \int_0^t \varpi(q)\, dq\right) - \left(1 - \frac{1}{2}\int_0^t \varpi(q)\, dq\right)\right] \|\Delta u\|^2 \\
&- \frac{\varepsilon}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq\right) \|u\|^2 + \varepsilon p(1-a) H(t) + \varepsilon b a \|u\|_p^p \\
&+ \frac{\varepsilon p(1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q |\mu_2(q)| \left|y^2(x,\rho,q,t)\right| dq d\rho dx \\
&+ \frac{\varepsilon}{2}(p(1-a)+1)(\varpi \circ \Delta u)(t). \qquad (34)
\end{aligned}$$

Utilizing Poincare's inequality, we have

$$\begin{aligned}
\mathcal{K}'(t) &\geq [1 - \varepsilon\kappa] H'(t) + \varepsilon \left[\frac{p(1-a)}{2} + 1\right] \|u_t\|^2 \\
&+ \frac{\varepsilon}{2}(p(1-a)+1)(\varpi \circ \Delta u)(t) \\
&+ \varepsilon \left\{\left(\frac{p(1-a)}{2} - 1\right) - \int_0^t \varpi(q)\, dq \left(\frac{p(1-a)-1}{2}\right)\right. \\
&\left. - \frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq\right)\right\} \|\Delta u\|^2 \\
&+ \varepsilon b a \|u\|_p^p + \varepsilon p(1-a) H(t) \\
&+ \frac{\varepsilon p(1-a)}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q |\mu_2(q)| \left|y^2(x,\rho,q,t)\right| dq d\rho dx. \qquad (35)
\end{aligned}$$

Here, take $a > 0$ so small, hence

$$\alpha_1 = \frac{p(1-a)}{2} - 1 > 0$$

and suppose

$$\int_0^\infty \varpi(q)\, dq < \frac{\frac{p(1-a)}{2} - 1}{\left(\frac{p(1-a)}{2} - \frac{1}{2}\right)} = \frac{2\alpha_1}{2\alpha_1 + \frac{1}{2}} \qquad (36)$$

then, choosing κ so large, such that

$$\alpha_2 = \left(\frac{p(1-a)}{2} - 1\right) - \int_0^t \varpi(q)\, dq \left(\frac{p(1-a)-1}{2}\right) - \frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2} |\mu_2(q)|\, dq\right) > 0.$$

Once κ and a are fixed, picking ε so small enough, such that

$$\alpha_4 = 1 - \varepsilon\kappa > 0$$

and

$$\mathcal{K}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{37}$$

Therefore, for some $\beta > 0$, the estimate (35) becomes

$$\begin{aligned} \mathcal{K}'(t) &\geq \beta\Big\{H(t) + \|u_t\|^2 + \|\Delta u\|^2 + (\varpi \circ \Delta u)(t) + \|u\|_p^p \\ &\quad + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx\Big\} \end{aligned} \tag{38}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0,\ t > 0. \tag{39}$$

From Poincare's and Young's inequalities, using (27), we get

$$\begin{aligned} \mathcal{K}(t) &= \left(H(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega u^2 dx\right) \\ &\leq c\left[H(t) + \left|\int_\Omega u u_t dx\right| + \|u\|^2 + \|\Delta u\|^2\right] \\ &\leq c\left[H(t) + \|\Delta u\|^2 + \|u_t\|^2\right]. \end{aligned} \tag{40}$$

For $c > 0$, since $H(t) > 0$, by (14), we obtain

$$\begin{aligned} &-\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\left(1 - \int_0^t \varpi(q)\,dq\right)\|\Delta u\|^2 - \frac{1}{2}(\varpi \circ \Delta u)(t) \\ &-\frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx + \frac{b}{p}\|u\|_p^p \\ > &\ 0 \end{aligned} \tag{41}$$

then

$$\begin{aligned} \frac{1}{2}\left(1 - \int_0^t \varpi(q)\,dq\right)\|\Delta u\|^2 &< \frac{b}{p}\|u\|_p^p < \frac{b}{p}\|u\|_p^p + (\varpi \circ \Delta u)(t) \\ &\quad + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx. \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} \|\Delta u\|^2 &< \frac{2b}{p}\|u\|_p^p + 2(\varpi \circ \Delta u)(t) + \left(\int_0^t \varpi(q)\,dq\right)\|\Delta u\|^2 \\ &\quad + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx. \end{aligned} \tag{43}$$

On the other hand, by using (7), to obtain

$$\begin{aligned} \|\Delta u\|^2 &< \frac{2b}{p}\|u\|_p^p + 2(\varpi \circ \Delta u)(t) + (1-l)\|\Delta u\|^2 \\ &\quad + 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx. \end{aligned} \tag{44}$$

Consequently, inserting (44) into (40), there is positive constant k_1 such that, for $\forall\, t > 0$

$$\begin{aligned} \mathcal{K}(t) &\leq k_1\left[H(t) + \|\Delta u\|^2 + \|u_t\|^2 + \frac{b}{p}\|u\|_p^p + (\varpi \circ \Delta u)(t)\right. \\ &\quad \left. + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} q\,|\mu_2(q)|\,\left|y^2(x,\rho,q,t)\right| dq d\rho dx\right]. \end{aligned} \tag{45}$$

Utilizing the inequalities (38) and (45), we get the differential inequality

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{46}$$

where $\lambda > 0$, depending only on β and k_1.
Integrating (46), we have

$$\mathcal{K}(t) \geq \mathcal{K}(0) e^{\lambda t}, \forall t > 0. \tag{47}$$

By (26) and (37), we obtain

$$\mathcal{K}(t) \leq H(t) \leq \frac{b}{p} \|u\|_p^p. \tag{48}$$

From (47) and (48), we get

$$\|u\|_p^p \geq Ce^{\lambda t}, \forall t > 0.$$

Hence, we complete the proof. □

4 Conclusion

In recent times, there have been many published works regarding to hyperbolic-type equations with delay. There are no blow up results at infinite time of the viscoelastic-plate equation with distributed delay term and source term to the best of our knowledge. Initially, we have given the local-global existence of solutions. Then, we have established the blow up results at infinite time under sufficient conditions.

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