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Numerical Solution of Fourth-order Initial Value Problems Using Novel Fourth-order Block Algorithm

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Abstract: *In this paper, a one-step fourth-order block scheme for solving fourth order Initial Value Problems (IVPs) of the Ordinary Differential Equations (ODE) is developed using interpolation and collocation techniques. The derived schemes contain two hybrid points which are chosen such that* $0 < w_1 < w_2 < 1$ *where* w_1 *and* w_2 *are defined as hybrid points. The characteristics of the developed schemes are analyzed. The obtained schemes are applied in block form to solve some fourth-order IVPs and the numerical results show the accuracy and effectiveness of the block scheme compared with some existing methods.*

Keywords: Block algorithm, Differential equations, Hybrid points, Numerical solution

1 Introduction

The pursuit of effective and accurate numerical methods for solving fourth-order IVP of the form

$$\mu^{(iv)} = g(x, \mu, \mu', \mu'', \mu'''), \quad x \in [x_0, x_N]$$

subject to the initial conditions

$$\mu(x_0) = \mu_0, \quad \mu'(x_0) = \mu'_0, \quad \mu''(x_0) = \mu''_0, \quad \mu'''(x_0) = \mu'''_0, \tag{1}$$

where x_0 and x_N represent the start and end points of the integration interval respectively, μ_0, μ'_0, μ''_0, μ'''_0 are real constants and $g(x, \mu, \mu', \mu'', \mu''')$ is a continuous real function, has been a fundamental attention of scientific research in the field of mathematical modeling and numerical analysis. The fourth-order IVPs have its application in electric circuits [8], neural networks [17], fluid dynamics [4], beam-theory [10] and some other areas of real life problems. The exact solution of some fourth-order IVPs do not exist hence the need for numerical methods to provide approximate solutions to (1). Some authors had to convert the fourth-order IVPs to system of first-order IVPs before applying numerical methods to solve the resulting system of first-order IVPs. The conversion of (1) into systems of first-order initial value problems (IVPs) has resulted in significant computational challenges and inefficient utilization of computer resources. These setbacks were addressed by Awoyemi [6, 7] which was also mentioned in Kayode [12]. While some other scholars solved the fourth-order IVPs directly without reducing to system of first-order IVPs and examples of such authors are Adesanya et al. [1], Adeyeye and Omar [2], Akinnukawe and Odekunle [3], Areo and Omole [5], Jator [10], Kayode [11, 12], Kuboye et al. [13, 14], Mohammed [18] to mention few but the accuracy of their schemes in terms of error can be improved upon.

Hence, this study considers the development of a new numerical method with optimal hybrid points to mark a significant step in providing efficient numerical solution to fourth-order IVPs. In this study, optimized hybrid points will be chosen such that $0 < w_1 < w_2 < 1$ holds and these hybrid points are introduced in the derivation of the block scheme and the developed block method is applied to solve fourth-order IVPs (1) directly. This approach aims to produce better accuracy compared with the existing methods.

2 Derivation of the One-step Block Algorithm

The solution of equation (1) is assumed on an interval $[x_n, x_{n+1}]$ which is locally approximated by a polynomial of the form

$$\mu(x) = \sum_{r=0}^{(p_1+p_2-1)} \phi_r x^r, \tag{2}$$

with corresponding derivatives as

$$\mu'(x) = \sum_{r=1}^{p_1+p_2-1} r\phi_r x^{r-1}, \tag{3a}$$

$$\mu''(x) = \sum_{r=2}^{p_1+p_2-1} r(r-1)\phi_r x^{r-2}, \tag{3b}$$

$$\mu'''(x) = \sum_{r=3}^{p_1+p_2-1} r(r-1)(r-2)\phi_r x^{r-3}, \tag{3c}$$

where $p_1 = 1$ and $p_2 = 7$ are the interpolation and collocation points respectively. The two hybrid points (w_1 and w_2) are considered in such a way that $0 < w_1 < w_2 < 1$ holds. Interpolating equation (2) and collocationg equations $(3a) - (3c)$ at given grid points, we have

$$\mu_{n+j} = \mu(x_{n+j}), \;\; j = 0, \tag{4a}$$

$$\mu'_{n+j} = \mu'(x_{n+j}), \;\; j = 0, \tag{4b}$$

$$\mu''_{n+j} = \mu''(x_{n+j}), \;\; j = 0, \tag{4c}$$

$$\mu'''_{n+j} = \mu'''(x_{n+j}), \;\; j = 0, \tag{4d}$$

$$\mu^{(iv)}_{n+j} = g(x_{n+j}), \;\; j = 0, w_1, w_2, 1, \tag{4e}$$

where μ_{n+j} and g_{n+j} are approximations for $\mu(x_{n+j})$ and $\mu^{(iv)}(x_{n+j})$ respectively. The system of eight equations from equations $(4a) - (4e)$ is written in compact form as

$$\mu A = G, \tag{5}$$

where

$$\mu = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+w_1} & 360x_{n+w_1}^2 & 840x_{n+w_1}^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+w_2} & 360x_{n+w_2}^2 & 840x_{n+w_2}^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 \end{bmatrix}$$

$$A = [\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7]^T$$

and

$$G = [\mu_n, \mu_n', \mu_n'', \mu_n''', g_n, g_{n+w_1}, g_{n+w_2}, g_{n+1}]^T$$

.

Solving equation (5) simultaneously gives the corresponding coefficients of $\phi_r, r = 0(1)7$. Substituting the resulting coefficients $\phi_r, r = 0(1)7$ into equation (2) and its derivatives yields a continuous implicit scheme of the form,

$$\alpha_z \mu_{n+z} = \alpha_0 \mu_n + h\beta_{10}\mu_n' + h^2\beta_{20}\mu_n'' + h^3\beta_{30}\mu_n''' + h^4 \sum_{j=0}^{1} \rho_j g_{n+j} + h^4 \sum_{j=1}^{2} \rho_{w_j} g_{n+w_j}, z = w_1, w_2, 1. \quad (6a)$$

and its first, second and third derivatives as

$$\alpha_z \mu_{n+z}^{(q)} = \alpha_0 \mu_n + h\beta_{10}\mu_n' + h^2\beta_{20}\mu_n'' + h^3\beta_{30}\mu_n''' + h^4 \sum_{j=0}^{1} \rho_j g_{n+j} + h^4 \sum_{j=1}^{2} \rho_{w_j} g_{n+w_j}, z = w_1, w_2, 1. \quad (6b)$$

where q signifies derivatives, that is, $q = 1(1)3$ and all the coefficients beta, alpha and rho values will change at equation $(6b)$. To obtain the approximate values of w_1 and w_2 hybrid points, optimize the local truncation errors of one of the schemes in equation (6) and ensure that the hybrid points satisfy the interval $0 < w_1 < w_2 < 1$, specifically we choose μ_{n+1}' [19]. To get the local truncation error, expand the Taylor series about the point x_n of the scheme to obtain

$$L[\mu'(x_{n+1}), h] = \frac{h^8 \mu^{(8)}(x_n) V_1}{20160} + O(h^9) \quad (7)$$

where

$$V_1 = -21w_1w_2 + 7w_1 + 7w_2 - 3 = 0, \quad (8a)$$

$$0 < w_1 < w_2 < 1. \quad (8b)$$

Imposing that the principal term (V_1) in the local truncation error (7) is zero. Solve $(8a)$ and its constraint

(8b) to obtain w_1 and w_2 and this yields the possible solution as $w_1 = \frac{1}{4}$ and $w_2 = \frac{5}{7}$.

The discrete schemes and its derivatives derived by evaluating (6) at grid and non-grid points $\left(\frac{1}{4}, \frac{5}{7}, 1\right)$ are given in equations (9) − (12). These schemes are used to form a block of hybrid method and its derivative methods as

$$\begin{bmatrix} \mu_{n+\frac{1}{4}} \\ \mu_{n+\frac{5}{7}} \\ \mu_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \begin{bmatrix} h\mu'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{25}{98} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^2\mu''_n \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{1}{384} \\ \frac{125}{2058} \\ \frac{1}{6} \end{bmatrix} \begin{bmatrix} h^3\mu'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{217}{1843200} \\ 0 & 0 & \frac{129125}{29647548} \\ 0 & 0 & \frac{11}{900} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{1}{18432} & -\frac{49}{3686400} & \frac{1}{245760} \\ \frac{50000}{7411887} & -\frac{125}{302526} & \frac{3125}{19765032} \\ \frac{16}{585} & \frac{49}{23400} & 0 \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \tag{9}$$

The first, second and third derivative block methods are in equations (10) − (12) respectively.

$$\begin{bmatrix} h\mu'_{n+\frac{1}{4}} \\ h\mu'_{n+\frac{5}{7}} \\ h\mu'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h\mu'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \begin{bmatrix} h^2\mu''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{25}{98} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^3\mu'''_n \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & \frac{1063}{614400} \\ 0 & 0 & \frac{3875}{201684}/ \\ 0 & 0 & \frac{11}{300} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix} + \begin{bmatrix} \frac{311}{299520} & -\frac{3773}{15974400} & \frac{53}{737280} \\ \frac{80000}{1966419} & \frac{125}{214032} & \frac{625}{2420208} \\ \frac{64}{585} & \frac{343}{15600} & -\frac{1}{720} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \tag{10}$$

$$\begin{bmatrix} h^2\mu''_{n+\frac{1}{4}} \\ h^2\mu''_{n+\frac{5}{7}} \\ h^2\mu''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^2\mu''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{5}{7} \\ 1 \end{bmatrix} \begin{bmatrix} h^3\mu'''_n \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & \frac{229}{12800} \\ 0 & 0 & \frac{1525}{28812} \\ 0 & 0 & \frac{7}{100} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix} + \begin{bmatrix} \frac{581}{37440} & -\frac{1029}{332800} & \frac{43}{46080} \\ \frac{50000}{280917} & \frac{425}{15288} & \frac{625}{172872} \\ \frac{176}{585} & \frac{343}{2600} & -\frac{1}{360} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \tag{11}$$

$$\begin{bmatrix} h^3\mu'''_{n+\frac{1}{4}} \\ h^3\mu'''_{n+\frac{5}{7}} \\ h^3\mu'''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^3\mu'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{391}{3840} \\ 0 & 0 & \frac{55}{1029} \\ 0 & 0 & \frac{1}{15} \end{bmatrix} \begin{bmatrix} h^4 g_{n-\frac{5}{7}} \\ h^4 g_{n-\frac{1}{4}} \\ h^4 g_n \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{103}{624} & -\frac{2401}{99840} & \frac{11}{1536} \\ \frac{2000}{4459} & \frac{265}{1092} & -\frac{125}{4116} \\ \frac{16}{39} & \frac{343}{780} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} h^4 g_{n+\frac{1}{4}} \\ h^4 g_{n+\frac{5}{7}} \\ h^4 g_{n+1} \end{bmatrix}. \tag{12}$$

Equations $(9)-(12)$ form the Novel Fourth-order One-step Block Algorithm (NFOBA) with optimal hybrid points developed for the direct approximation of linear and non-linear fourth-order IVPs of ODEs (1).

3 Analysis of NFOBA

In this section, the properties of the derived method are examined.

3.1 The error constant and order of the method

The local truncation error associated with the derived methods can be defined as the linear difference operator (see Lambert [15, 16])

$$L[\mu(x_n); h] = \sum_{j=0}^{1} \alpha_j \mu(x_n + jh) - \sum_{i=1}^{3} h^{(i)} \beta_{i0} \mu^{(i)}(x_n) - h^4 \sum_{j=0}^{1} \rho_j \mu^{(iv)}(x_n + jh)$$
$$-h^4 \sum_{j=1}^{2} \rho_{w_j} \mu^{(iv)}(x_n + (w_j)h). \tag{13}$$

Assuming that $\mu(x_n)$ is sufficiently differentiable, then using Taylor series expansion on $\mu^{(i)}(x_n + jh), i = 0(1)4$ about x_n and substitute into equation (13) to obtain

$$L[\mu(x_n); h] = C_0\mu(x_n) + C_1 h\mu'(x_n) + C_2 h^2 \mu''(x_n) + C_3 h^3 \mu'''(x_n) + ... + C_{m+4} h^{m+4} \mu^{(m+4)}(x_n) + \dots, \tag{14}$$

where $C_m, m = 0, 1, 2, \dots$ are constants given as

$$C_0 = \sum_{j=0}^{1} \alpha_j + \sum_{j=1}^{2} \alpha_{w_j}$$

$$C_1 = \left[\sum_{j=0}^{1} j\alpha_j + \sum_{j=1}^{2} w_j \alpha_{w_j} \right] - \beta_{10}$$

$$\vdots$$

$$C_{m+4} = \frac{1}{(m+4)!} \left[\sum_{j=0}^{1} j^{m+4} \alpha_j + \sum_{j=1}^{2} (w_j)^{m+4} \alpha_{w_j} \right] - \frac{1}{(m+3)!} \left[\sum_{j=0}^{1} j^{m+3} \beta_{1j} + \sum_{j=1}^{2} (w_j)^{m+3} \beta_{1w_j} \right]$$

$$-\frac{1}{(m+2)!}\left[\sum_{j=0}^{1} j^{m+2}\beta_{2j}+\sum_{j=1}^{2}(w_j)^{m+2}\beta_{2w_j}\right]-\frac{1}{(m+1)!}\left[\sum_{j=0}^{1} j^{m+1}\beta_{3j}+\sum_{j=1}^{2}(w_j)^{m+1}\beta_{3w_j}\right]$$

$$-\frac{1}{m!}\left[\sum_{j=0}^{1} j^{m}\rho_{j}+\sum_{j=1}^{2}(w_j)^{m}\rho_{w_j}\right] \tag{15}$$

The error constants and order of NFOBA are shown in Table 1.

Table 1: Order and error constants of NFOBA.

S/N	Scheme	Order(m)	Error Constant (C_{m+4})
1	$\mu_{n+\frac{1}{4}}$	4	$-\frac{1877}{55490641920}$
2	$\mu_{n+\frac{5}{7}}$	4	$-\frac{53125}{34865516448}$
3	μ_{n+1}	4	$-\frac{1}{423360}$
4	$\mu'_{n+\frac{1}{4}}$	4	$-\frac{13}{22020096}$
5	$\mu'_{n+\frac{5}{7}}$	4	$-\frac{8125}{1660262688}$
6	μ'_{n+1}	5	$\frac{1}{705600}$
7	$\mu''_{n+\frac{1}{4}}$	4	$-\frac{311}{41287680}$
8	$\mu''_{n+\frac{5}{7}}$	4	$\frac{625}{135531648}$
9	μ''_{n+1}	4	$\frac{1}{40320}$
10	$\mu'''_{n+\frac{1}{4}}$	4	$-\frac{83}{1474560}$
11	$\mu'''_{n+\frac{5}{7}}$	4	$\frac{2125}{19361664}$
12	μ'''_{n+1}	4	$\frac{1}{40320}$

3.2 Zero-stability

The numerical method is said to be zero-stable if the roots of the first characteristic polynomial $\gamma(R)$ satisfy $|R_u| \leq 1, u = 1, \ldots, 12$ multiplicity not exceeding the order of the differential equation (3). The first characteristic polynomial $\gamma(R) = 0$ of the derived method is calculated as

$$\gamma(R) = \det(RA_{(1)} - A_{(0)})$$

where $A_{(1)}$ is a 12 by 12 identity matrix and

$$A_{(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{32} & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 1 & 0 & 0 & \frac{5}{7} & 0 & 0 & \frac{25}{98} & 0 & 0 & \frac{125}{2058} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{5}{7} & 0 & 0 & \frac{25}{32} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{5}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\gamma(R) = R^{u-e}(R-1)^e$, where e is the order of the differential equation and u is the order of the matrices $A_{(1)}$ and $A_{(0)}$. The NFOBA can be shown to be zero-stable since the first characteristic polynomial $\gamma(R) = R^8(R-1)^4$ satisfies $|R_u| \leq 1, u = 1(1)12$.

3.3 Consistency

The developed method is concluded to be consistent since according to Lambert [15], the necessary and sufficient condition for a numerical scheme to be consistent is for it to have order of at least one ($m \geq 1$). The derived method is of order 4 since the least order of the block is of order 4.

3.4 Convergence

A numerical method converges if it is consistent and zero-stable (Lambert [15]). This implies that NFOBA converges since the method is of order $m = 4 > 1$ and it satisfies the conditions for zero-stability.

4 Numerical Results

The following problems are considered in order to examine the accuracy and computational efficiency of the new block method (NFOBA). All computations are done using MATHEMATICA 13.0.

The notations used in representing the existing methods and the derived method in the result Tables are:

h	step size
$EIFBM$	Error of order 7 in First Block Method in [13]
EIM	Error in [18]
$EIFO$	Error in [9]
$EIOK$	Error in [14]
$EIAO$	Error in [5]
$EIBBCM$	Error in Block Bi-basis Collocation Method [3]
$EINFOBA$	Error in Novel Fourth order One-step Block Algorithm

where,

$$Error^{(i)} = |y^{(i)}_{exact} - y^{(i)}_{appro}|$$

These problems are chosen to aid comparison with other existing methods in literature.

Problem 1 [13]

$$\mu^{(iv)}(x) - (\mu')^2 + \mu\mu''' + 4x^2 - e^x(1-4x+x^2) = 0, \quad h = \frac{0.1}{32};$$

$$\mu(0) = 1, \quad \mu'(0) = 1, \quad \mu''(0) = 3,$$

$$\mu'''(0) = 1.$$

The exact solution is

$$\mu(x) = x^2 + e^x.$$

Table 2: Comparison of errors for problem 1.

t	EIFBM [13]	EIFO [9]	EIBBCM [3]	EINFOBA
0.103125	$1.8149238E-10$	$9.02145880E-10$	$4.44089E-16$	$2.22045E-16$
0.206250	$1.1543254E-08$	$1.21681228E-09$	$1.35003E-13$	$1.33227E-15$
0.309375	$1.2194148E-07$	$1.21681228E-09$	$1.13887E-12$	$1.11022E-15$
0.412500	$6.5296082E-07$	$1.71379609E-09$	$4.53237E-12$	$1.9984E-15$
0.515625	$2.3972196E-06$	$1.48197092E-08$	$1.24527E-12$	$3.33067E-15$
0.618750	$6.7092614E-06$	$3.05833850E-08$	$2.77791E-11$	$2.66454E-15$
0.721875	$1.6438756E-05$	$4.94185815E-08$	$5.39191E-11$	$8.88178E-16$
0.825000	$3.5549856E-05$	$7.12867908E-08$	$9.49667E-11$	$7.10543E-15$
0.928125	$6.9845227E-05$	$1.05877308E-07$	$1.55466E-10$	$1.82077E-14$
1.031250	$1.2716790E-04$	$1.44552007E-07$	$2.40636E-10$	$2.53131E-14$

.

Problem 2 [18]

$$\mu^{(iv)}(x) - x = 0, \quad h = 0.1;$$

$$\mu(0) = 0, \quad \mu'(0) = 1, \quad \mu''(0) = \mu'''(0) = 0.$$

The exact solution is

$$\mu(x) = \frac{x^5}{120} + x.$$

.

Problem 3 [5]

$$\mu^{(iv)}(x) - \mu(x) = 0, \quad h = \frac{1}{320};$$

$$\mu(0) = 1, \quad \mu'(0) = 0, \quad \mu''(0) = -2,$$

$$\mu'''(0) = 0.$$

The exact solution is

$$\mu(x) = -\frac{1}{4}exp(x) - \frac{1}{4}exp(-x) + \frac{3}{2}cos(x).$$

Table 3: Comparison of errors for problem 2.

t	EIM [18]	EIOK [14]	EINFOBA
0.1	$7.000024E-10$	$1.002087E-12$	$1.38778E-17$
0.2	$8.9999912E-10$	$0.000000E+00$	$2.77556E-17$
0.3	$2.599993E-09$	$0.00000E+00$	$5.55112E-17$
0.4	$5.100033E-09$	$0.00000E+00$	$5.55112E-17$
0.5	$7.799979E-09$	$1.002087E-12$	0
0.6	$1.180009E-08$	$2.755907E-12$	$1.11022E-16$
0.7	$1.180009E-08$	$3.597306E-12$	$1.11022E-16$
0.8	$1.410006E-08$	$3.597306E-12$	$2.22045E-16$
0.9	$1.880000E-08$	$4.175549E-12$	$2.22045E-16$
1.0	$1.008335E-08$	$4.759970E-12$	$2.22045E-16$

Table 4: Comparison of errors for problem 3.

t	EIAO [5]	EIBBCM [3]	EINFOBA
0.003125	$4.440892E-16$	$1.11022E-16$	$1.11022E-16$
0.006250	$2.176037E-14$	$1.11022E-16$	$2.22045E-16$
0.009375	$7.771916E-13$	$1.11022E-16$	$2.22045E-16$
0.012500	$7.666090E-13$	$1.11022E-16$	$1.11022E-16$
0.015625	$2.367773E-12$	$1.11022E-16$	$3.33067E-16$
0.018750	$5.932477E-12$	$1.11022E-16$	$1.11022E-16$
0.021875	$1.287681E-11$	$2.22045E-16$	$3.33067E-16$
0.025000	$2.517841E-11$	$1.11022E-16$	$4.44089E-16$
0.028125	$4.546752E-11$	$2.22045E-16$	$5.55112E-16$
0.031250	$7.712331E-11$	$0.00000E+00$	$4.44089E-16$

. **Problem 4 [3]**
Consider the ship dynamic problem

$$\mu^{(iv)}(x) + 3\mu'' + \mu(2 + \rho cos(wx)) = 0, \quad x > 0,$$
$$\mu(0) = \mu'(0) = \mu''(0) = \mu'''(0) = 0.$$

When $\rho = 0$, the exact solution is

$$\mu(x) = 2cos(x) - cos(\sqrt{2}x).$$

Table 5: Comparison of errors for problem 4.

t	Exact	NFOBA	EIBBCM [3]	EINFOBA ($h = 0.01$)
3	-1.527323135908539	-1.5273230429245035‘	$9.29840E-08$	$1.20615E-12$
6	2.510535059206013	2.5105347506092786	$3.08597E-07$	$7.22977E-13$
9	-2.8092394453688807	-2.8092390262763756	$4.19093E-07$	$1.15525E-11$
12	1.9910488550789986	1.9910487618540555	$9.32249E-08$	$2.27867E-11$
15	-0.8070186485444338	-0.8070191456233216	$4.97079E-07$	$1.75494E-11$

.

Problem 5 [3]

$$\mu^{(iv)}(x) + \mu''(x) = 0, \quad 0 \le x \le \frac{\pi}{2};$$

$$\mu(0) = 0,$$

$$\mu'(0) = -\frac{1.1}{72 - 50\pi},$$

$$\mu''(0) = \frac{1}{144 - 100\pi},$$

$$\mu'''(0) = \frac{1.2}{144 - 100\pi},$$

The exact solution is

$$\mu(x) = \frac{1 - x - cos(x) - 1.2sin(x)}{144 - 100\pi}.$$

Table 6: Comparison of errors for problem 5.

x	Exact	NFOBA	EIBBCM [3]	EINFOBA
0.01	0.00012899562284403668	0.0001289956228439637	$7.29668E-17$	$3.80555E-17$
0.02	0.0002573965432101358	0.0002573965432102199	$8.40799E-17$	$2.69424E-17$
0.03	0.00038519579791147405	0.00038519579791140846	$6.55942E-17$	$6.55942E-17$
0.04	0.000512386483927295	0.000512386483927374	$7.89299E-17$	$3.20924E-17$
0.05	0.0006389617590932021	0.0006389617590932817	$7.95804E-17$	$3.14419E-17$
0.06	0.0007649148427853702	0.0007649148427855135	$1.43223E-16$	$3.22008E-17$
0.07	0.0008902390165986053	0.0008902390165986818	$7.64363E-16$	$3.45860E-17$
0.08	0.0010149276250181782	0.0010149276250184247	$2.46548E-16$	$2.51535E-17$
0.09	0.0011389740760853638	0.0011389740760856526	$2.88831E-16$	$6.87384E-17$
0.10	0.0012623718420566414	0.0012623718420570196	$3.78170E-16$	$1.56125E-16$

.

Problem 6 [3]

$$\mu^{(iv)}(x) - \mu''' - \mu'' - \mu' - 2\mu = 0, \quad x \in [0, 2],$$

$$\mu(0) = \mu'(0) = \mu''(0) = 0,$$

$$\mu'''(0) = 30,$$

The exact solution is

$$\mu(x) = 2exp(2x) - 5exp(-x) + 3cos(x) - 9sin(x).$$

Table 7: Numerical results for problem 6.

t	Exact	NFOBA	EINFOBA ($h = 0.01$)
0.2	0.04217138626080574	0.04217138626080914	$1.01516E-14$
0.4	0.35789952803753966	0.3578995280376137	$1.82077E-13$
0.6	1.2904002491766824	1.2904002491771174	$1.04916E-12$
0.8	3.2933353381499177	3.2933353381515276	$3.80496E-12$
1.0	6.98638304633745	6.986383046342062	$1.07914E-11$
1.2	13.239103191447201	13.23910319145849	$2.62492E-11$
1.4	23.2971625812907	23.297162581315575	$5.76215E-11$
1.6	38.971816809968104	38.971816810019014	$1.17545E-10$
1.8	62.92373948426520	62.92373948436379	$2.27139E-10$
2.0	99.08750629903315	99.08750629921636	$4.21323E-10$

.

5 Discussion

The developed block scheme (NFOBA) is efficient and effective in the numerical integration of the fourth order linear and nonlinear initial value problems of ordinary differential equations as seen in Tables 2-7. NFOBA was compared with existing methods used to solve same problems and it shows superiority to these methods.

6 Conclusion

A one-step fourth-order block hybrid scheme is derived via interpolation and collocation techniques for the numerical solution of linear and non-linear fourth-order initial value problems of ordinary differential equations. The introduced hybrid points w_1 and w_2 in the derived scheme are computed in such a way that the points lies in the interval $0 < w_1 < w_2 < 1$. The approximate values of the hybrid points were obtained by optimizing the local truncation error of the derived scheme and its derivatives. Application of NFOBA on six problems shows the efficiency of the new scheme as seen in Tables 2-7 and the numerical results are compared with some existing methods. The analysis of the method were shown to be consistent, zero-stable and convergent. NFOBA has proven effective in the direct integration of fourth-order initial value problems of ordinary differential equations.

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