Decomposition of Riemannian Recurrent Curvature Tensor Manifolds of First Order

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Abstract: *Takano [6] premeditated decomposition of curvature tensor in a recurrent Riemannian space. After that, Negi and Bisht [3] defined and deliberated the decomposition of recurrent curvature tensor fields in a Kaehlerian manifolds of first order. We have calculated the decomposition of Riemannian recurrent curvature tensor manifolds of first order and some theorems established using the decomposition tensor field.*

Keywords: Manifolds, Symmetric, Recurrent and Riemannian curvature tensor

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1 Introduction

Walter [7] has given the following properties of the decomposition curvature tensor, namely recurrent, symmetric, skews symmetric, and Bianchi identity

$$A_{jkl} = -A_{jlk}, A_{kl} = -A_{lk}, \nabla_n A^i = 0, \tag{1}$$

$$\nabla_n A_{kl} + \nabla_k A_{nl} + \nabla_l A_{nk} = 0, \tag{2}$$

$$A^i_{jk} = A^i_{kj}, A^i_{jk} = -A^i_{kj}. \tag{3}$$

The covariant derivative of the tensor A^p are A_q and covariant differentiation of a mixed tensor A^p_q are defined as

$$\nabla_q A^p = \partial_q A^p + \Gamma^p_{qs} A^s; \nabla_r A_q = \partial_r A_q + \Gamma^s_{qs} A_s, \tag{4}$$

$$\nabla r A^p_q = \partial_r A^p_q + \Gamma^p_{rs} A^s_{qr} - \Gamma^s_{qr} A^p_s \tag{5}$$

also, in n-dimensional space, we describe the line element as ds through the quadratic form called the metric form as below

$$ds^2 = \sum_{p=1}^{N}\sum_{q=1}^{N} g_{pq} dx^p dx^q \,\text{or}\, ds^2 = g_{pq} dx^p dx^q \tag{6}$$

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$, then g_{pq} is defined as

$$g^{pq} = \frac{cofactor g_{pq}}{g}, \tag{7}$$

where g_{pq} is also a symmetric tensor known as conjugate tensor and some tensor manifold are represented by [6]

$$\{k, i, j\} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{jk} - \partial_k g_{ij}) \tag{8}$$

And

$$\{^l_{ij}\} = \Gamma^l_{ij} = g^{lk}[k, ij] \tag{9}$$

The Christoffel symbols $[k, ij]$ and τ^i_{ij} are symmetric in the indices j and k. The relation between is reciprocal through the following equation [4]

$$[j, ki] = g_{li}\Gamma^l_{ki} \,\text{and}\, g^{jm}[j, ki] = \Gamma^m_{ki}. \tag{10}$$

2 Riemannian Curvature Tensor Manifolds of First Order

Riemannian recurrent curvature correlates a tensor at every point of Riemannian manifold (M, g), and determines the degree to which the metric tensor in not narrowly equivalent to Euclidean space [1]. Riemannian recurrent curvature tensor with respect to Christoffel symbols has components R^i_{jkl} given by

$$R^i_{jkl} = \partial_j \Gamma^i_{kl} - \partial_k \Gamma^i_{jl} + \Gamma^i_{pj}\Gamma^p_{kl} - \Gamma^i_{pk}\Gamma^p_{jl}. \tag{11}$$

The curvature tensor is called Riemannian recurrent curvature tensor Manifolds or Riemannian – Christoffel curvature tensor of the second kind. The Riemannian recurrent curvature tensor Manifolds satisfy the following identities [4, 6]

$$R^i_{jkl} = -R^i_{kjl}, R^i_{jkl} = -R^i_{jlk}, \tag{12}$$

$$R^i_{jkl} + R^i_{ljk} + R^i_{klj} = 0, \tag{13}$$

$$\nabla_s R^i_{jkl} = \nabla_k R^i_{sjl} = \nabla_j R^i_{ksl} = 0. \tag{14}$$

Equations (13) and (14) respectively are called Bianchi's first and second identities.

The covariant derivative of the Riemannian curvature tensor R^i_{jkl} is defined as

$$\nabla_m R^i_{jkl} = \partial_m R^i_{jkl} + R^s_{jkl}\Gamma^i_{ms} - R^i_{skl}\Gamma^s_{mj} - R^i_{jsl}\Gamma^s_{mk} - R^i_{jks}\Gamma^s_{ml}, \tag{15}$$

The commutation laws involving the curvature tensor field R^i_{jkl} are given by

$$\nabla_j \nabla_k \lambda^i - \nabla_k \nabla_j \lambda^i = \lambda^l R^i_{jkl}, \tag{16}$$

$$\nabla_j \nabla_k \lambda^i_t - \nabla_k \nabla_j \lambda^i_t = \lambda^l_t R^i_{jkl} - \lambda^i_l R^l_{jkt}, \tag{17}$$

$$2\nabla_{(j)}\nabla_{(k)}\lambda^i = \lambda^i R^i_{jkl}, \tag{18}$$

$$2\nabla_{(j)}\nabla_{(k)}A_i = -A_i R^i_{jkl}. \tag{19}$$

The equations (18) and (19) are known as Ricci laws for covariant differentiation. λ is the component of any vector tangential to the surface [6]. The Riemannian curvature tensor a non-zero vector λ_n, then satisfies the relation

$$\nabla_n R^i_{jkl} = -\lambda_n R^i_{jkl} \tag{20}$$

Riemannian recurrent curvature tensor field R^i_{jkl} and satisfies the following theorem.

Theorem 1. *If the associated curvature has components $R_{jklm} = R^n_{jkl} g_{nm.}$, then*

(i) R_{jklm} is skew-symmetric in the first two indices $R_{(jk)lm} = 0$,

(ii) skew-symmetric in the last two indices $R_{jk(lm)} = 0$,

(iii) satisfy Bianchi's identities $R_{[jkl]m} = 0$ and $\nabla_{[p}R_{jk]lm} = 0$,

(iv) are symmetric in two part of indices $R_{jklm} = R_{lmjk}$.

Proof. (i). Using (11) and (12), we obtain

$$R_{jklm} = 2g_{mn}\partial_{[j}\Gamma^n_{k]l} + 2g_{mn}\Gamma^m_{p[j}\Gamma^p_{k]l},$$

where $\partial_{[j}\Gamma^n_{k]l}$ is skew-symmetric with regard to indices j and k. It can be expressed as

$$R_{jklm} = 2g_{mn} \times \frac{1}{2}[\partial_j\Gamma^n_{kl} - \partial_k\Gamma^n_{jl}] + 2gmn \times \frac{1}{2}[\Gamma^m_{pj}\Gamma^p_{kl} - \Gamma^m_{pk}\Gamma^p_{jl}]. \tag{21}$$

On the other hand, we obtain

$$R_{jklm} = g_{mn}\partial_j\Gamma^n_{kl} - g_{mn}\partial_k\Gamma^n_{jl} + g_{mn}\Gamma^n_{pj}\Gamma^p_{kl} - g_{mn}\Gamma^n_{pk}\Gamma^p_{jl}$$

or,

$$R_{jklm}+R_{kjlm} = g_{mn}\partial_j\Gamma^n_{kl}-g_{mn}\partial_k\Gamma^n_{jl}+g_{mn}\Gamma^n_{pj}\Gamma^p_{kl}-g_{mn}\Gamma^n_{pk}\Gamma^p_{jl}+g_{mn}\partial_k\Gamma^n_{jl}-g_{mn}\partial_j\Gamma^n_{kl}+g_{mn}\Gamma^n_{pk}\Gamma^p_{jl}-g_{mn}\Gamma^n_{pj}\Gamma^p_{kl} = 0$$

or,

$$R_{jklm} + R_{kjlm} = 0.$$

This is equivalent to $R_{jk(lm)} = 0$.

(ii). In view of Ricci identities, we find $2\nabla_{[j}\nabla_{k]}g_{mn} = -g_{mn}R^p_{jkm} - g_{mn}R^p_{jkn}$. Using $\nabla_k g_{mn} = 0$ and equation (12), we obtain $0 = -R_{jkmn} + R_{jknm}$ or equivalently in symmetric brackets $0 = R_{jk(mn)}$.

(iii). If we multiply equation (12) and (13) by g_{mn} and sum with respect to m, we obtain the results.

(iv). The equation (13) is equivalent to

$$R_{jklm} + R_{kljm} + R_{ljkm} = 0. \tag{22}$$

Therefore, we have the three similar equations are expressed in the form

$$R_{jklm} + R_{lmkj} + R_{mklj} = 0,$$
$$R_{lmjk} + R_{mjlk} + R_{jlmk} = 0,$$
$$R_{mjkl} + R_{jkml} + R_{kmjl} = 0.$$

Adding above results, we obtain

$$R_{jklm}+R_{kljm}+R_{ljkm}+R_{jklm}+R_{kljm}+R_{ljkm}+R_{lmjk}+R_{mjlk}+R_{jlmk}+R_{mjkl}+R_{jkml}+R_{kmjl} = 0. \tag{23}$$

Then using (12), we get

$$R_{ljkm} - R_{kmlj} - R_{jlkm} - R_{kmlj} = 0$$

or,

$$R_{ljkm} + R_{ljkm} - R_{kmlj} - R_{kmlj} = 0$$

or,

$$2R_{ljkm} - 2R_{kmlj} = 0$$

This proved

$$R_{ljkm} = R_{kmlj}.$$

□

Theorem 2. *Prove that the Christoffel symbol*

$$\Gamma^i_{jk} = \frac{1}{2}\partial_j logg = \partial_j log\sqrt{g}, \tag{24}$$

where $g = |g_{ij}|$

Proof. we know that

$$g^{jk} = \frac{\text{Co-factor of}\, g_{jk}}{g}$$

or,

$$g^{jk} = \frac{G_{(jk)}}{g}$$

or,

$$gg^{jk} = G_{(j,k)}$$

Multiplying the above equation with g_{jr}, we obtain

$$gg^{jk}g_{jr} = G_{(j,k)}g_{jr},\ \text{or}\, g\partial^k_r = G_{(j,k)}g_{jr}\ (\text{for}\, k = r),$$

then we have $g = G_(j,k)g_{jr}$.

Its differentiation with respect to X^m gives

$$\frac{\partial g}{\partial x^m} = G_{(j,k)}\frac{\partial g_{jr}}{\partial x^m} \tag{25}$$

The above result can be expanded as

$$\frac{\partial g}{\partial x^m} = gg^{jk}([j, rm] + [r, jm]). \tag{26}$$

Considering the effect of conjugate metric tensor, we find

$$\frac{\partial g}{\partial x^m} = g\{^{\,r}_{rm}\} + g\{^{\,r}_{jm}\} \tag{27}$$

The above result is equivalent to $\partial_m g = 2g\Gamma^r_{rm}$ (since $g\{^{\,r}_{rm}\} = \Gamma^r_{rm}$).
This is same as $\frac{1}{2}g\partial_m g = \Gamma^r_{rm}$ or $\partial_m Log\sqrt{g} = \Gamma^r_{rm}$.

□

Theorem 3. *Riemannian recurrent curvature tensor of second kind can be contracted in two modes. One yielding a zero and the other as a system tensor.*

Proof. Contracting indices n and l in the equation (11), we find

$$C^l_\epsilon R^n_{jkl} = C^l_\epsilon[2\partial_{[j}\Gamma^n_{k]l} + 2\Gamma^n_{pj[j}\Gamma^p_{k]l}$$

or,

$$R^l_{jkl} = 2\partial_{[j}\Gamma^l_{k]l} + 2\Gamma^n_{pj[j}\Gamma^p_{k]l} \tag{28}$$

We know that

$$\Gamma^l_{kl} = \frac{1}{2}\partial_{[j}\partial_{k]}logg = 0 \tag{29}$$

Also,

$$2\Gamma^l_{p[j}\Gamma^p_{k]l} = 2\Gamma^p_{l[j}\Gamma^l_{k]p}$$

or,

$$2 \times \frac{1}{2}[\Gamma^l_{pj}\Gamma^p_{kl} - \Gamma^l_{pk}\Gamma^p_{jl}] = 2 \times \frac{1}{2}[\Gamma^p_{lj}\Gamma^l_{kp} - \Gamma^p_{lk}\Gamma^l_{jp}] = 0 \tag{30}$$

Using (12) and (13) in (11) gives

$$R^l_{jkl} = 0 \tag{31}$$

This proves the first part.

Again, the equation (13) which Bianchi identify is equivalent to

$$R^m_{jkl} + R^m_{klj} + R^m_{ljk}$$

Setting $m = j$ in the above equation gives

$$R^j_{jkl} + R^j_{klj} + R^j_{ljk} = 0. \tag{32}$$

But $R^m_{klj} = 0$ in view of (12), thus above equation reduces to $R^j_{jkl} + R^j_{ljk} = 0$.

In view of skew symmetry property of R^j_{jkl}, we get

$$R^l_{jkl} - R^l_{ljk} = 0.$$

Contracting the above equation, it becomes

$$R_{kl} + R_{lk} = 0 \,\text{or}\, R^j_{[kl]} = 0.$$

Where $[\cdot]$ is skew symmetric brackets which proves the last part.

□

3 Decomposition of Riemannian Recurrent Curvature Tensor Manifolds of First Order

Decomposition is a mode of breaking up of the Riemannian curvature tensor into pieces with useful individual algebraic properties; it is the decomposition of the space of all tensors having the symmetries of the Riemannian tensor into its irreducible representation for the orthogonal group [4, 5]. We consider the decomposition of the Riemannian recurrent curvature tensor R^i_{jkl} in the following structure

$$R^i_{jkl} = X^i Y_{jkl}, \tag{33}$$

where Y_{jkl} is the decomposition tensor field and X^i is a vector field such that

$$X^i \lambda_i = 1. \tag{34}$$

Theorem 4. *In a Riemannian recurrent curvature tensor manifolds the decomposition tensor symmetric in the indices k and l, that is, $Y_{jkl} = -Y_{jkl}$.*

Proof. We have sited properties of decomposition tensor field Y_{jkl}. If we multiple (33) by λ_i, we obtain

$$\lambda_i R^i_{jkl} = \lambda_i X^i Y_{Jkl},$$

Since $X^i \lambda_i = 1$, the above equation becomes

$$\lambda_i R^i_{jkl} = Y_{jkl} \tag{35}$$

By interchanging the two indices k and l and adding in the above equation, we get

$$\lambda_i R^i_{jkl} + \lambda_i R^i_{jkl} = Y_{jkl} + Y_{jkl}$$

or,

$$\lambda_i (R^i_{jkl} + R^i_{jkl}) = Y_{jkl} + Y_{jkl}. \tag{36}$$

Since R^i_{jkl} is skew-symmetric in the indices k and l, in view of (20), that is,

$$R^i_{jkl} = -R^i_{jkl}$$

using the equation (20) in the equation (36), we have

$$\lambda_i(R^i_{jkl} - R^i_{jkl}) = Y_{jkl} + Y_{jkl}$$

This reduces to

$$0 = Y_{jkl} + Y_{jkl}.$$

This gives the following identity

$$Y_{jkl} = -Y_{jkl}. \tag{37}$$

□

Theorem 5. *The decomposition tensor Y_{kl} is skew-symmetric concerning its with two indices k and l, that is, $Y_{kl} = -Y_{kl}$.*

Proof. We have more decomposed the tensor field Y_{jkl} as

$$Y_{jkl} = \lambda_j Y_{kl} \tag{38}$$

Multiplying (38) by X^i, we obtain

$$X^j Y_{jkl} = X^j \lambda_j Y_{kl} \tag{39}$$

In view of (34), the above equation gives

$$X^j Y_{jkl} = Y_{kl}$$

By interchanging the two indices k and l in (39) and adding the respective results, we get

$$X^j(Y_{jkl} + Y_{jkl}) = Y_{kl} + Y_{kl} \tag{40}$$

Using (37), we get

$$X^j(Y_{jkl} - Y_{jkl}) = Y_{kl} + Y_{kl}$$

This reduces

$$0 = Y_{kl} + Y_{kl}$$

Therefore, we get

$$Y_{kl} = -Y_{kl}.$$

□

Theorem 6. *The decomposition of Riemannian recurrent curvature tensor field Y_{jkl} and Y_{kl} to be recurrent is that the vector field X^i is covariant. Also, the decomposition tensor field satisfies the Bianchi identity*

$$Y_{jkl} + Y_{klj} + Y_{ljk} = 0 \text{ and } \nabla_n Y_{kl} + \nabla_k Y_{nl} + \nabla_l Y_{nk}.$$

Proof. We solve equations (12) and (33), we get the following equation

$$X^J(Y_{jkl} + Y_{klj} + Y_{ljk}) \tag{41}$$

Transvacting (41) by λ^i, we get

$$X^j \lambda_i(Y_{jkl} + Y_{klj} + Y_{ljk}) = 0 \tag{42}$$

In view of (34) the following is obtained

$$Y_{jkl} + Y_{klj} + Y_{ljk} = 0, \tag{43}$$

as the first result.
Again, finding covariant differentiation of (33) about X^n and using (21), we obtain

$$\nabla_n R^i_{jkl} = \nabla_n X^i Y_{jkl} + X^i \nabla_n Y_{jkl}. \tag{44}$$

Consider X^i to be a covariant constant and using (33), the equation (44) gives

$$\nabla_n R^i_{jkl} = \nabla_n Y_{jkl} \tag{45}$$

By virtue of (33), the equation (44) gives

$$Y_{jkl} \nabla_n X^i = 0 \tag{46}$$

Since $Y_{jkl} \neq 0$, we have

$$\nabla_n X^i = 0. \tag{47}$$

Thus X^i is a covariant constant.

Now, we have in analysis of (22), (33) and $R^i_{jkl} = X^i Y_{jkl}$, the Bianchi identity of the form $\nabla_n R^i_{jkl} + \nabla_k R^i_{jnl} + \nabla_n R^i_{jnk} = 0$ is converted into

$$X^i [\nabla_n Y_{jkl} + \nabla_k Y_{jnl} + \nabla_l Y_{jnk}] = 0. \tag{48}$$

Transvecting (48) by h^j it gives

$$X^i [\nabla_n Y_{kl} + \nabla_k Y_{nl} + \nabla_l Y_{nk}] = 0. \tag{49}$$

Currently under the statement that X^i is covariant constant, the equation (49) reduces to

$$\nabla_n Y_{kl} + \nabla_k Y_{nl} + \nabla_l Y_{nk} = 0. \tag{50}$$

Which is the Bianchi identity for the decomposition tensor field is of $\nabla_k Y_{nl}$. □

4 Conclusions

Using the tensor field decomposition and the Bianchi identity, we developed some results on the Riemannian recursive curvature tensor manifold. We also proved that the Riemannian recursive curvature tensor of the second kind can be contracted in two ways. One produces zeros and the other as a system tensor.

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