Approximation by Some Stancu Type Linear Positive Operators

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Abstract: *Present paper is the study about Stancu type generalization of modified Beta-Szasz operators and their q-analogues. We obtain some approximation properties for these operators and estimate the rate of convergence by using the first and second order modulus of continuity. Author also investigates the statistical approximation properties of the q-Beta-Stancu operators using Korokvin theorem.*

Keywords: Linear positive operators, q-integral, Modulus of continuity, Weighted approximation, Rate of convergence, q-Beta-Stancu operators, Beta-Szasz operators

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1 Introduction

Stancu [28] introduced Beta operators L_n of the second kind in order to approximate the Lebesgue integrable functions on the interval $[0, \infty)$ as

$$L_n(f,x) = \frac{1}{B(nx, n+1)} \int_0^{\infty} \frac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)dt. \tag{1}$$

For the above operators (1), Abel and Gupta [1] and Gupta et al. [12] estimated the rate of convergence for functions and for the functions with derivatives of bounded of variation respectively. In the last two decades, many researchers focused their attention on the study of a generalized version in q-calculus of the well-known linear positive operators, see [2, 4, 5, 6, 21, 23, 27]. In 1987, Lupaş [19] initiated the convergence of Bernstein operators based on q-integers and in 1997 another generalization of these operators was introduced by Phillips [22]. More results on q-discrete operators were obtained by [13, 14, 15] etc.

Buyukyazici and Atakut [7, 8, 9] and Maheshwari et el. [20, 26] gave the Stancu variants of several well-known operators and estimated some direct results. Recently Sharma [24] and Sharma-Abid [25] have also discussed approximation properties of these summation-integral type using (p, q)-calculus.

Actually, the Stancu variant is based on two parameters α and β, which are two non negative real numbers and it generalizes the original operator. Motivated by the recent research on Stancu type operators [3] and [18], we introduced the Stancu type generalization of the Beta-Szasz operators. For $0 < q < 1$ and $0 \leq \alpha \leq \beta$, we propose the q-Beta-Szasz-Stancu operators as follows

$$B^q_{n,\alpha,\beta}(f,x) \;=\; \sum_{v=0}^{\infty} b^q_{n,v}(x) \int_0^{\frac{q}{1-q^n}} q^{-v-1} s^q_{n,v}(t) f\left(\frac{[n]_q t q^{-v-1} + \alpha}{[n]_q + \beta}\right) d_q t, \tag{2}$$

where $b^q_{n,v}(x)$ and $s^q_{n,v}(t)$ are Beta and Szasz basis functions respectively, defined as

$$b^q_{n,v}(x) = \frac{q^{\frac{v(v-1)}{2}}}{B_q(v+1,n)} \frac{x^v}{(1+x)_q^{n+v+1}} \quad \text{and} \quad s^q_{n,v}(t) = E_q(-[n]_q t) \frac{([n]_q t)^v}{[v]_q!}.$$

As a special case when $\alpha = \beta = 0$ and $q = 1$, the above operators (2) reduce to the Beta-Szasz operators introduced by Gupta and Srivastava [16].

For the brief explanation of the present work, some notations of q-calculus are given as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

$$[n]_q = \frac{1-q^n}{1-q},$$

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\ldots\left(1+q^{n-1}x\right), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is given as

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-qx)}, x \neq 0.$$

According to [17], there are two q-analogues of exponential function e^z

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]!} = \frac{1}{(1-(1-q)z)_q^{\infty}}, \qquad |z| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1+(1-q)z)_q^{\infty}, \quad |q| < 1,$$

where

$$(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x).$$

The q-integral by parts is given as

$$\int_0^{\infty} f(t) D_q(g(t)) d_q t = [f(t)(g(t))_a^b - \int_a^b g(q(t) D_q(f(t)) d_q t].$$

Following integrals are known as q-Jackson integrals and q-improper integrals and which are defined as

$$\int_0^a f(t) d_q t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad a > 0$$

$$\int_0^{\frac{\infty}{A}} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0.$$

The two q-Gamma functions are defined as

$$\Gamma q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q(-qt) d_q t,$$

$$\gamma_q^A(x) = \int_0^{\infty} A(1-q) t^{x-1} e_q(-t) d_q t.$$

For every $A, x > 0$, we can get

$$\gamma_q^{(x)} = K(A; x) \gamma_q^A(x),$$

where

$$K(A; x) = \frac{1}{(1+A)A^x(1+(\frac{1}{A})_q^x)(1+A)_q^{1-x}}.$$

In particular, for any $n > 0$,

$$K(A; n) = q^{\frac{n(n-1)}{2}}$$

and

$$\Gamma_q(n) = q^{n(n-1)} \gamma_q^A(n).$$

The present work deals with the q-analogue of the well-known Stancu-Beta operators. Here we obtain moments, the recurrence relation and some direct estimates in terms of higher order modulus of continuity of the studied q-Stancu-Beta operators.

2 Auxiliary Results

In this section, we obtain the following lemmas for the proofs of main results.

Lemma 1. *For the above operators and for $\alpha = \beta = 0$ and $0 < q < 1$, following equalities hold*

(i) $B_n^q(1, x) = 1$

(ii) $B_n^q(t, x) = x(1 + \frac{1}{q[n]_q}) + \frac{1}{[n]_q}$

(iii) $B_n^q(t^2, x) = \frac{[n+1]_q[n+2]_q}{q^3[n]_q^2}x^2 + \frac{[n+1]_q}{q^2[n]_q^2}(1 + 2q + q^2)x + \frac{[2]_q}{[n]_q^2}$.

Proof of above lemma can be seen in [15].

Lemma 2. *For $0 < q < 1$ and $0 \leq \alpha \leq \beta$, we have*

$$\begin{aligned}
B_{n,\alpha,\beta}^q(1, x) &= 1 \\
B_{n,\alpha,\beta}^q(t, x) &= \frac{x(q[n]_q + 1) + q(1 + \alpha)}{q([n]_q + \beta)} \\
B_{n,\alpha,\beta}^q(t^2, x) &= \frac{x^2[n+2]_q[n+1]_q + x(q[n+1]_q + (1 + 2q + q^2) + 2\alpha([n]_q q^3 + q^2)) + ([2]_q + \alpha^2 + 1)q^3}{([n]_q + \beta)^2 q^3}.
\end{aligned}$$

Proof. From Lemma 1, it is obvious that

$$B_{n,\alpha,\beta}^q(1, x) = 1.$$

Further, we have

$$\begin{aligned}
B_{n,\alpha,\beta}^q(t, x) &= \sum_{v=0}^{\infty} b_{n,v}^q(x) \int_0^{\frac{q}{(1-q^n)}} q^{-v-1} s_{n,v}^q(t) \left(\frac{[n]_q t q^{-v-1} + \alpha}{[n]_q + \beta} \right) d_q t \\
&= \frac{[n]_q}{[n]_q + \beta} B_n^q(t, x) + \frac{\alpha}{[n]_q + \beta} B_n^q(1, x) \\
&= \frac{[n]_q}{[n]_q + \beta} \left(x(1 + \frac{1}{q[n]_q}) + \frac{1}{n_q} \right) + \frac{\alpha}{[n]_q + \beta} \\
&= \frac{x(q[n]_q + 1) + q(1 + \alpha)}{q([n]_q + \beta)}.
\end{aligned}$$

We have,

$$\begin{aligned}
B_{n,\alpha,\beta}^q(t^2, x) &= \sum_{v=0}^{\infty} b_{n,v}^q(x) \int_0^{\frac{q}{(1-q^n)}} q^{-v-1} s_{n,v}^q(t) \left(\frac{[n]_q t q^{-v-1} + \alpha}{[n]_q + \beta} \right)^2 d_q t \\
&= \left(\frac{[n]_q}{[n]_q + \beta} \right)^2 B_n^q(t^2, x) + \frac{2[n]_q \alpha}{([n]_q + \beta)^2} B_n^q(t, x) + \frac{\alpha^2}{([n]_q + \beta)^2} B_n^q(1, x) \\
&= \left(\frac{[n]_q}{[n]_q + \beta} \right)^2 \left(\frac{[n+1]_q[n+2]_q}{q^3[n]_q^2} x^2 + \frac{[n+1]_q}{q^2[n]_q^2}(1 + 2q + q^2)x + \frac{[2]_q}{[n]_q^2} \right) \\
&\quad + \frac{2[n]_q \alpha}{([n]_q + \beta)^2} + \left(x(1 + \frac{1}{q[n]_q}) + \frac{1}{[n]_q} \right) \left(\frac{\alpha}{[n]_q + \beta} \right)^2 \\
&= \frac{x^2[n+2]_q[n+1]_q + x(q[n+1]_q + (1 + 2q + q^2) + 2\alpha([n]_q q^3 + q^2)) + ([2]_q + \alpha^2 + 1)q^3}{([n]_q + \beta)^2 q^3}.
\end{aligned}$$

□

Lemma 3. *For $x \in [0, \infty)$ and $q \in (0, 1)$, we get the central moments given as below*

$$B_{n,\alpha,\beta}^q(t - x, x) = \frac{x(1 - q\beta) + q(1 + \alpha)}{q([n]_q + \beta)}.$$

$$B^q_{n,\alpha,\beta}((t-x)^2,x) = x^2\left[\frac{[n+1]_q[n+2]_q}{([n]_q+\beta)^2q^3} - \frac{x(q[n]_q+1)}{q([n]_q+\beta)} + 1\right] + x\left[\frac{[n+1]_q+(1+2q+q^2)}{([n]_q+\beta)^2q^2} + \frac{2\alpha(q[n]_q+1)}{q([n]_q+\beta)^2} - \frac{2(1+\alpha)}{[n]_q+\beta}\right] + \frac{[2]_q+\alpha^2+2\alpha}{([n]_q+\beta)^2}.$$

Operators $B^q_{n,\alpha,\beta}$ are linear in nature.

3 Main Results

Definition 1. *$C_B[0,\infty)$ denotes the space of real valued bounded and uniformly continuous functions f on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by*

$$||f|| = \sup_{0\leq x<\infty} |f(x)|.$$

Definition 2. *The Peetre's K-functional is defined by*

$$K_2(f,\delta) = \inf\{||f-g|| + \delta||g''|| : g \in W^2_\infty\},$$

where $W^2_\infty = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. Following [10], there exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}), \delta > 0$$

and the second order modulus of smoothness is given by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{0\leq x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Definition 3. *The usual modulus of continuity for $f \in C_B[0,\infty)$ is given by*

$$\omega_2(f,\delta) = \sup_{0<h\leq\delta} \sup_{0\leq x<\infty} |f(x+h) - f(x)|.$$

Definition 4. *Let $H_{x^2}[0,\infty)$, be the set of all functions f defined on the positive real axis and satisfying the condition $|f(x)| \leq K_f(1+x^2)$, where K_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on the class $C^*_{x^2}[0,\infty)$ is defined as*

$$||f||_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

We denote the modulus of continuity of the function f on the interval $[0,a]$, $a > 0$ as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta} \sup_{t\in[0,\infty)} |f(t) - f(x)|.$$

We observe that for the function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Definition 5. *(Rate of Convergence): By $DB_q(0,\infty)$(where q is some positive integer), we mean the class of absolutely continuous functions f defined on $(0,\infty)$ satisfying the following conditions*

(i) $f(t) = O(t^q), t \to \infty$

(ii) The function f has the first derivative on the interval $(0,\infty)$ which coincide a.e. with a bounded variation function, which is of on every finite sub-interval of $(0,\infty)$. It can be observed that for all functions $f \in DB_q(0,\infty)$, we can have the representation
$f(x) = f(c)\int_c^x \Psi(t)dt, x \geq c > 0.$

3.1 Theorem

Let $f \in C_B[0,\infty)$ and $0<q<1$, then for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, there exists an absolute constant $C>0$ such that

$$|B^q_{n,\alpha,\beta}(f,x)-f(x)| \leq C\omega_2\left(f,\delta_n(x)\right)+\omega\left(f,\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right),$$

where $\delta_n^2(x)=\left(B^q_{n,\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2\right)^{1/2}$.

Proof. Introducing the auxiliary operators $\overline{B}^q_{n,\alpha,\beta}$ as

$$\overline{B}^q_{n,\alpha,\beta}(f,x) \;=\; B^q_{n,\alpha,\beta}(f,x)-f\left(x+\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)+f(x), \tag{3}$$

where $x\in[0,\infty)$.
The operators $\overline{B}^q_{n,\alpha,\beta}(f,x)$ are linear and preserve the linearity conditions

$$\overline{B}^q_{n,\alpha,\beta}(t-x,x)=0. \tag{4}$$

Let $g \in W^2_\infty$, using Taylor's expansion

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)\; g''(u)\; du, \quad t\in[0,\infty)$$

and from (4), we get

$$\overline{B}^q_{n,\alpha,\beta}(g,x) \;=\; g(x)+\overline{B}^q_{n,\alpha,\beta}\left(\int_x^t (t-u)\; g''(u)\; du, x\right).$$

Hence, by (3), we have
$|\overline{B}^q_{n,\alpha,\beta}(g,x)-g(x)|$

$$\begin{aligned}
&\leq \left| B^q_{n,\alpha,\beta}\left(\int_x^t (t-u)\; g''(u)\; du, x\right)\right| + \left|\int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left(\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}-u\right)\; g''(u)\; du\right| \\
&\leq B^q_{n,\alpha,\beta}\left(\left|\int_x^t |t-u|\; |g''(u)|\; du\right|, x\right) + \int_x^{\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}} \left|\frac{x(q[n]_q+1)+q(1+\alpha)}{q([n]_q+\beta)}-u\right|\; |g''(u)| du
\end{aligned}$$

$$\leq \left[B^q_{n,\alpha,\beta}\left((t-x)^2,x\right)+\left(\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)^2\right]\; \|g''\| = \delta_n^2(x)||g''||. \tag{5}$$

According to (3), we have

$$|\overline{B}^q_{n,\alpha,\beta}(f,x)| \;\leq\; |B^q_{n,\alpha,\beta}(f,x)|+2||f|| \leq 3||f|| \tag{6}$$

Now from (3), (5) and (6), we have

$$\begin{aligned}
|B^q_{n,\alpha,\beta}(f,x)-f(x)| \;&\leq\; |\overline{B}^q_{n,\alpha,\beta}(f-g,x)-(f-g)(x)| \\
&\quad + |\overline{B}^q_{n,\alpha,\beta}(g,x)-g(x)| + \left| f\left(x+\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)-f(x)\right| \\
&\leq\; 4\; \|f-g\| + \delta_n^2(x)\|g''\| \\
&\quad + \left| f\left(x+\frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right)-f(x)\right|.
\end{aligned}$$

Hence, taking the infimum on the right hand side over all $g \in W^2$, we get

$$|B^q_{n,\alpha,\beta}(f,x) - f(x)| \leq CK_2\left(f, \delta_n^2(x)\right) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

Using the property of Peetre's K-functional, we have

$$|B^q_{n,\alpha,\beta}(f,x) - f(x)| \leq C\omega_2\left(f, \delta_n(x)\right) + \omega\left(f, \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)}\right).$$

This completes the proof of the theorem. □

3.2 Theorem

Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$ then for each $f \in C^*_{x^2}[0,\infty)$, we have

$$\lim_{n\to\infty} ||B^{q_n}_{n,\alpha,\beta}(f,x) - f(x)||_{x^2} = 0$$

Proof. Following [11], we observe that it is sufficient to verify the following conditions

$$\lim_{n\to\infty} ||B^{q_n}_{n,\alpha,\beta}(t^k,x) - x^k||_{x^2} = 0, \ \ k = 0,1,2. \tag{7}$$

Since $B^{q_n}_{n,\alpha,\beta}(1,x) = 1$, hence (7) holds for $k = 0$.

$$||B^{q_n}_{n,\alpha,\beta}(t,x) - x||_{x^2} \quad = \quad \sup_{x\in[0,\infty)} \frac{x(1-q\beta)+q(1+\alpha)}{q([n]_q+\beta)} . \frac{1}{1+x^2}$$

Hence

$$\lim_{n\to\infty} ||B^{q_n}_{n,\alpha,\beta}(t,x) - x||_{x^2} = 0.$$

Now

$$\begin{aligned}
&||B^{q_n}_{n,\alpha,\beta}(t^2,x) - x^2||_{x^2} \\
&\leq \left(\frac{[n+1]_q[n+2]_q}{([n]_q+\beta)^2 q_n^3} + 1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\
&+ \left(\frac{q[n+1]_q(1+2q+q^2)+2\alpha(q^3[n]_q+q^2)}{([n]_q+\beta)^2 q_n^3}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\
&+ \left(\frac{[2]_q+\alpha^2+q^3}{([n]_q+\beta)^2 q_n^3}\right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} ||B^{q_n}_{n,\alpha,\beta}(t^2,x) - x^2||_{x^2} = 0.$$

Hence theorem is proved. □

4 Conclusions

The present paper deals with some specific generalisation (Stancu type) of the linear positive operators using their q-analogue defined on bounded or unbounded interval in terms of higher order modulus of continuity. There are several integral and other modifications, variations and basic extensions of the Beta type operators. Here we have chosen Beta and Szasz basis functions, others can choose other basis functions comparably and can apply (p,q)-analogue rather than q-analogue.

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