Some Construction of Generalized Frames with Adjointable Operators in Hilbert C^*-modules

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Abstract: *Generalized frame called g-frame was first proposed using a sequence of adjointable operators to deal with all the existing frames as a united object. In fact, the g-frame is an extension of ordinary frames. Generalized frames with adjointable operators called K-g-frame is a generalization of a g-frame. It can be used to reconstruct elements from the range of a adjointable operator K. K-g-frames have a certain advantage compared with g-frames in practical applications. This paper is devoted to study some properties of K-g-frame in Hilbert C^*-module, we characterize the concept of K-g-frame by quotient maps. Also discus some result of the dual K-g-Bessel sequences of K-g-frame in Hilbert C^*-module. Our results are more general than those previously obtained. It is shown that the results we obtained can immediately lead to the existing corresponding results in Hilbert Spaces.*

Keywords: g-Frame, K-g-frame, C^*-algebra, Hilbert C^*-modules

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1 Introduction

Frames, introduced by Duffin and Schaefer [4] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [7] for signal processing. Today, frame theory is an exciting, dynamic and fast paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory, wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks, and more. The last decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence, g-frame was first proposed using a sequence of adjointable operators to deal with all the existing frames as a united object. In fact, the g-frame is an extension of ordinary frames. Generalized frames with adjointable operators called K-g-frame is a generalization of a g-frame. It can be used to reconstruct elements from the range of a adjointable operator K. K-g-frames have a certain advantage compared with g-frames in practical applications.

In 2000, Frank-Larson [6] extended the theory for the elements of C^*-algebra and Hilbert C^*-modules. Recentely, Khosravi and Khosravi [9] introduced the g-frame theory in Hilbert C^*-modules. Afterwards, Alijani and Dehghan [2] consider frames with C^*-valued bounds [2] in Hilbert C^*-modules. Bounader and Kabbaj [3] and Alijani [1] introduced the $*$-g-frames which are generalizations of g-frames in Hilbert C^*-modules. In 2016, Xiang and Li [12] gave a generalization of g-frames for operators in Hilbert C^*-modules. In this paper, we establish some new results for K-g-frames in Hilbert C^*-modules. Moreover, we investigate the duals of them. We also discuss the stability problem.

The paper is organized as follow, we continue this introductory section by briefly recalling the definitions and basic properties of Hilbert C^*-modules. In section 2, we construct some new K-g-frames and we characterize the concept of K-g-frames by quotient maps in Hilbert C^*-module. In section 3, we investigate the notion of dual K-g-Bessel sequence in Hilbert C^*-modules.

In the following, we briefly recall the definition and basic properties of Hilbert C^*-modules. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [8]. Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and U are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle \geq 0$ for all $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in H$.

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in H$.

For $x \in H$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|x| = \langle x, x \rangle^{\frac{1}{2}}$ for $x \in H$.

Throughout this paper, H is considered to be a countably generated Hilbert $\mathcal{A}-$module. Let $\{H_i\}_{i\in I}$ be a collection of Hilbert $\mathcal{A}$-modules, where I is a finite or countable index set. $End^*_{\mathcal{A}}(H, H_i)$ is the set of all adjointable operator from H to H_i. In particular, $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H. P_W denotes the orthogonal projection onto the closed submodule orthogonally complemented W of H, the range and null of K are denoted by $\mathcal{R}(K)$ and $\mathcal{N}(K)$, respectively. Define the module

$$l^2(\{H_i\}_{i\in I}) = \{\{x_i\}_{i\in I} : x_i \in H_i, \|\sum_{i\in I}\langle x_i, x_i \rangle\| < \infty\}$$

with $\mathcal{A}-$valued inner product $\langle x, y \rangle = \sum_{i\in I}\langle x_i, y_i \rangle$, where $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$, clearly $l^2(\{H_i\}_{i\in I})$ is a Hilbert $\mathcal{A}-$module.

We need the following lemmas to prove our results.

Lemma 1.2. [5] *Let E, H and L be Hilbert $\mathcal{A}$-modules, $T \in End^*_{\mathcal{A}}(E, L)$ and $T' \in End^*_{\mathcal{A}}(H, L)$. Then the following two statements are equivalent.*

(1) $T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$.

(2) There exists $\mu > 0$ such that $\|(T')^*z\| \leq \mu\|T^*z\|$ for all $z \in L$.

Lemma 1.3. [5] *Let E, H be Hilbert $\mathcal{A}$-modules and T be in $End^*_{\mathcal{A}}(E, H)$. Then the following statements are equivalent.*

(1) $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented.

(2) For any Hilbert $\mathcal{A}-$module L and any $T' \in End^*_{\mathcal{A}}(L, H)$, the equation $T' = TX$ for $X \in End^*_{\mathcal{A}}(L, E)$ is solvable whenever $\mathcal{R}(T') \subseteq \mathcal{R}(T)$.

(3) The equation $S = TX$ for $X \in End^*_{\mathcal{A}}(G, E)$ is solvable, where G and S are defined by $G = \overline{\mathcal{R}(T^*)}$ and S be the restriction of T on G.

Now, we recall the definitions of g-frames and K-g-frames in Hilbert C^*-modules.

Definition 1.4. [9] A sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(H, H_i) : i \in I\}$ is called a g-frame for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$, such that

$$A\langle x, x \rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x \rangle \leq B\langle x, x \rangle, \quad \forall x \in H. \tag{1.1}$$

the constants A and B are called the lower and upper bounds of g-frames, respectively. If $A = B$, we call $\{\Lambda_i\}_{i\in I}$ a tight g-frame; in particular if $A = B = 1$, $\{\Lambda_i\}_{i\in I}$ is called a Parseval g-frame. If only the right hand inequality of (1.1) holds, $\{\Lambda_i\}_{i\in I}$ is called a g-Bessel sequence for H.

Then we can define the adjointable operator $T : H \to l^2(\{H_i\}_{i\in I})$ by

$$T(x) = \{\Lambda_i x\}_{i\in I}, \quad \forall x \in H.$$

The adjoint operator $T^* : l^2(\{H_i\}_{i\in I}) \to H$ is given by

$$T^*(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* x_i, \quad \forall \{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I}).$$

By composing T^* with T, we can obtain the adjointable operator $S : H \to H$ defined by

$$Sx = T^*Tx = \sum_{i\in I} \Lambda_i^* \Lambda_i x, \quad \forall x \in H.$$

T, T^* and S are called the analysis operator, synthesis operator and g-frame operator, respectively.

Definition 1.5. [12] Let $K \in End_{\mathcal{A}}^*(H)$. A sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(H, H_i) : i \in I\}$ is called a K-g-frame for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle, \quad \forall x \in H. \tag{1.2}$$

The constants A and B are called the lower and upper bounds of K-g-frames, respectively.

It should be noted that the K-g-frame operator S is not invertible in general, however If K has closed range, then

$$S_\Lambda : \mathcal{R}(K) \to S(\mathcal{R}(K)) \tag{1.3}$$

is invertible and self-adjoint.

2 Construction of Some New K-g-Frames in Hilbert C^*-Modules

In inequality (1.2) we are comparing the positive elements in $\mathcal{A}$. The following theorem which characterize K-g-frames, we show that one can replace (1.2) with two inequalities in terms of the norm of elements. That we will used in the proof of the next results.

Theorem 2.1. *Let $K \in End_{\mathcal{A}}^*(H)$ and $\{\Lambda_i \in End_{\mathcal{A}}^*(H, H_i) : i \in I\}$ be a sequence. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H if and only if there exist two constants $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \|\sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle\| \leq B\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Proof. Assume that $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for H, then it is clear that we have (2.1).
Conversely, suppose that (2.1) holds.
We define the operator $T : H \to l^2(\{H_i\}_{i\in I})$ by $Tx = \{\Lambda_i x\}_{i\in I}$.
Let $\{x_i\}_{i\in H} \in l^2(\{H_i\}_{i\in I})$

$$\begin{aligned}
\|\sum_{i\in I} \Lambda_i^* x_i\| &= \sup_{\|y\|=1} \|\langle \Lambda_i^* x_i, y\rangle\| \\
&= \sup_{\|y\|=1} \|\sum_{i\in I} \langle x_i, \Lambda_i y\rangle\| \\
&\leq \sup_{\|y\|=1} \|\sum_{i\in I} \langle x_i, x_i\rangle\|^{\frac{1}{2}} \|\sum_{i\in I} \langle \Lambda_i y, \Lambda_i y\rangle\|^{\frac{1}{2}} \\
&\leq \sqrt{B}\|\{x_i\}_{i\in I}\|.
\end{aligned}$$

Then, $\sum_{i\in I} \Lambda_i^* x_i$ converge unconditionally in H, and we have $\forall x \in H$, $\forall \{x_i\}_{i\in I} \in l^2(\{H_i\}_{i\in I})$,

$$\langle Tx, \{x_i\}_{i\in I}\rangle = \sum_{i\in I} \langle \Lambda_i x, x_i\rangle = \langle x, \sum_{i\in I} \Lambda_i^* x_i\rangle,$$

so T is adjointable and $T^*(\{x_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* x_i$, also we have

$$A\|K^*x\|^2 \le \|Tx\|^2, \quad \forall x \in H.$$

Hence, by lemma 1.2, there exists a constant $\mu > 0$ such that

$$KK^* \le \mu T^*T,$$

therefore,

$$\frac{1}{\mu}\langle K^*x, K^*x\rangle \le \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle, \quad \forall x \in H.$$

And we have for each $x \in H$,

$$\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle = \langle Tx, Tx\rangle \le \|T\|^2\langle x, x\rangle, \quad \forall x \in H.$$

The proof is completed. □

The question of stability plays an important role in various fields of applied mathematics. The classical theorem of the stability of a base is due to Paley and Wiener [11]. It is based on the fact that a bounded operator T on a Banach space is invertible if $\|I - T\| < 1$.

Theorem 2.2. [11] *Let $\{f_i\}_{i\in\mathbb{N}}$ be a basis of a Banach space X, and $\{g_i\}_{i\in\mathbb{N}}$ be a sequence of vectors in X. If there exists a constant $\lambda \in [0, 1)$ such that*

$$\Big\|\sum_{i\in\mathbb{N}} c_i(f_i - g_i)\Big\| \le \lambda\Big\|\sum_{i\in\mathbb{N}} c_i f_i\Big\|$$

for all finite sequences $\{c_i\}_{i\in\mathbb{N}}$ of scalars, then $\{g_i\}_{i\in\mathbb{N}}$ is also a basis for X.

The following theorem is a Paley–Wiener type stability theorem for K-g-frames in Hilbert C^*-modules.

Theorem 2.3. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $End^*_{\mathcal{A}}(H, H_i)$ and $\Gamma_i \in End^*_{\mathcal{A}}(H, H_i)$, for all $i \in I$. If there exist constants $0 \le \mu, \nu < 1$ such that for all $x \in H$*

$$\|\sum_{i\in I}\langle(a_i\Lambda_i - b_i\Gamma_i)x, (a_i\Lambda_i - b_i\Gamma_i)x\|^{\frac{1}{2}} \le \mu\|\sum_{i\in I}\langle a_i\Lambda_i x, a_i\Lambda_i x\rangle\|^{\frac{1}{2}} + \nu\|\sum_{i\in I}\langle b_i\Gamma_i x, b_i\Gamma_i x\rangle\|^{\frac{1}{2}}.$$

Then, $\{\Gamma_i\}_{i\in I}$ is a $K-g-$frame for $End^*_{\mathcal{A}}(H, H_i)$ where $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are positively confined sequences.

Proof. Let $x \in H$, we have

$$\begin{aligned}
\|\sum_{i\in I}\langle b_i\Gamma_i x, b_i\Gamma_i x\rangle\|^{\frac{1}{2}} &= \|\{b_i\Gamma_i x\}_{i\in I}\| \\
&= \|\{b_i\Gamma_i x - a_i\Lambda_i x\}_{i\in I} + \{a_i\Lambda_i x\}_{i\in I}\| \\
&\le \|\{b_i\Gamma_i x - a_i\Lambda_i x\}_{i\in I}\| + \|\{a_i\Lambda_i x\}_{i\in I}\| \\
&= \|\sum_{i\in I}\langle(a_i\Lambda_i - b_i\Gamma_i)x, (a_i\Lambda_i - b_i\Gamma_i)x\|^{\frac{1}{2}} + \|\sum_{i\in I}\langle a_i\Lambda_i x, a_i\Lambda_i x\rangle\|^{\frac{1}{2}} \\
&\le (1+\mu)\|\sum_{i\in I}\langle a_i\Lambda_i x, a_i\Lambda_i x\rangle\|^{\frac{1}{2}} + \nu\|\sum_{i\in I}\langle b_i\Gamma_i x, b_i\Gamma_i x\rangle\|^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$(1-\nu)\|\sum_{i\in I}\langle b_i\Gamma_i x, b_i\Gamma_i x\rangle\|^{\frac{1}{2}} \le (1+\mu)\|\sum_{i\in I}\langle a_i\Lambda_i x, a_i\Lambda_i x\rangle\|^{\frac{1}{2}}.$$

Hence,

$$(1-\nu)(\inf_{i\in I}|b_i|)\|\sum_{i\in I}\langle \Gamma_i x, \Gamma_i x\rangle\|^{\frac{1}{2}} \le (1+\mu)(\sup_{i\in I}|a_i|)\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\|^{\frac{1}{2}}.$$

Therefore,

$$\|\sum_{i\in I}\langle\Gamma_i x,\Gamma_i x\rangle\| \leq B\left(\frac{(1+\mu)(\sup_{i\in I}|a_i|)}{(1-\nu)(\inf_{i\in I}|b_i|)}\right)^2\|x\|^2. \tag{2.2}$$

On the other hand for each $x\in H$, we have

$$\begin{aligned}\|\sum_{i\in I}\langle a_i\Lambda_i x,a_i\Lambda_i x\rangle\|^{\frac{1}{2}} &= \|\{a_i\Lambda_i x\}_{i\in I}\|\\ &\leq \|\{a_i\Lambda_i x-b_i\Gamma_i x\}_{i\in I}\|+\|\{b_i\Gamma_i x\}_{i\in I}\|\\ &\leq (1+\nu)\|\sum_{i\in I}\langle b_i\Gamma_i x,b_i\Gamma_i x\rangle\|^{\frac{1}{2}}+\mu\|\sum_{i\in I}\langle a_i\Lambda_i x,a_i\Lambda_i x\rangle\|^{\frac{1}{2}},\end{aligned}$$

then,

$$(1-\mu)\|\sum_{i\in I}\langle a_i\Lambda_i x,a_i\Lambda_i x\rangle\|^{\frac{1}{2}} \leq (1+\nu)\|\sum_{i\in I}\langle b_i\Gamma_i x,b_i\Gamma_i x\rangle\|^{\frac{1}{2}},$$

hence,

$$(1-\mu)(\inf_{i\in I}|a_i|)\|\sum_{i\in I}\langle\Lambda_i x,\Lambda_i x\rangle\|^{\frac{1}{2}} \leq (1+\nu)(\sup_{i\in I}|b_i|)\|\sum_{i\in I}\langle\Gamma_i x,\Gamma_i x\rangle\|^{\frac{1}{2}},$$

therfore,

$$A\left(\frac{(1-\mu)(\inf_{i\in I}|a_i|)}{(1+\nu)(\sup_{i\in I}|b_i|)}\right)^2\|K^*x\|^2 \leq \|\sum_{i\in I}\langle\Gamma_i x,\Gamma_i x\rangle\|. \tag{2.3}$$

From inequality (2.2) and (2.3), we conclude that $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$. □

The following corollaries are consequences of Theorem 2.3.

Corollary 2.4. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$ and $\Gamma_i\in End_{\mathcal{A}}^*(H,H_i)$ for all $i\in I$. Then, the following statements hold.*

(1) If there exist $0<\mu<1$, such that for every $x\in H$

$$\|\sum_{i\in I}\langle(\Lambda_i-\Gamma_i)x,(\Lambda_i-\Gamma_i)x\rangle\|^{\frac{1}{2}} \leq \mu\|\sum_{i\in I}\langle\Lambda_i x,\Lambda_i x\rangle\|^{\frac{1}{2}},$$

then $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$.

(2) If $\sup_{i\in I}\|\Lambda_i\|<\frac{\sqrt{2}}{2}$, then the sequence $\{\Lambda_i+\Lambda_i^2\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$.

Corollary 2.5. *Let $T\in End_{\mathcal{A}}^*(H)$ with $\|T\|<\frac{\sqrt{2}}{2}$, H_i be invariant for T and $\Lambda_i\in End_{\mathcal{A}}^*(H,H_i)$ for all $i\in I$. Then, the following assertions are equivalent.*

(1) $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$.

(2) For every $n\in \mathrm{N}$, the sequence $\{\Lambda_i+T^n\Lambda_i\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$.

(3) There exists $n\in \mathrm{N}$, such that the sequence $\{\Lambda_i+T^n\Lambda_i\}_{i\in I}$ is a K-g-frame for $End_{\mathcal{A}}^*(H,H_i)$.

In the sequel, we show that K-g-frame is invariant under a adjointable operator.

Theorem 2.6. *Let K be surjective, $T\in End_{\mathcal{A}}^*(H)$ and $\{\Lambda_i T\}_{i\in I}$ be a K-g-frame for H. Then, the following statements hold:*

(1) T is injective.

(2) If T is self-adjoint and has closed range, then T is invertible and $\{\Lambda_i\}$ is a $T^{-1}K$-g-frame for H.

(3) If T is self-adjoint and $TK=KT$, then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H.

Proof. (1) There exists two constants $A, B > 0$ such that

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i Tx, \Lambda_i Tx\rangle \leq B\langle x, x\rangle.$$

Then, $\mathcal{N}(T) \subseteq \mathcal{N}(K^*)$. And we have $\mathcal{N}(K^*) = \mathcal{R}(K)^\perp$, since K is surjective then, $\mathcal{N}(K^*) = \{0\}$, therefore $\mathcal{N}(T) = \{0\}$ i.e, T is injective.

(2) Since T is self-adjoint and has closed range, then $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp = \mathcal{N}(T)^\perp = \{0\}^\perp = H$. So, T is surjective, hence invertible.

To complete the proof, let $x \in H$, we have

$$\begin{aligned}\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle &= \sum_{i\in I}\langle \Lambda_i TT^{-1}x, \Lambda_i TT^{-1}x\rangle \\ &\leq B\langle T^{-1}x, T^{-1}x\rangle \\ &\leq B\|T^{-1}\|^2\langle x, x\rangle. \end{aligned} \tag{2.4}$$

On the other hand, let $x \in H$, then there exists $y \in H$ such that $x = Ty$. So,

$$\begin{aligned}\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle &= \sum_{i\in I}\langle \Lambda_i Ty, \Lambda_i Ty\rangle \\ &\geq A\langle K^*y, K^*y\rangle \\ &= A\langle K^*T^{-1}x, K^*T^{-1}x\rangle \\ &= A\langle (T^{-1}K)^*x, (T^{-1}K)^*x\rangle. \end{aligned} \tag{2.5}$$

From inequality (2.4) and (2.5), we conclude that $\{\Lambda_i\}_{i\in I}$ is a T^{-1} K-g-frame for H. □

Now, we prove that if $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H and $\{\Gamma_i\}_{i\in I}$ is a g-Bessel sequence for H, such that for every $i \in I$, $\mathcal{R}(\Lambda_i) \perp \mathcal{R}(\Gamma_i)$, then $\{\Lambda_i T_1 + \Gamma_i T_2\}_{i\in I}$ is a T_1^* K-g-frame, where $T_1, T_2 \in End^*_{\mathcal{A}}(H)$.

Theorem 2.7. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for H and $\{\Gamma_i\}_{i\in I}$ be a g-Bessel sequence for H. If for every $i \in I$, $\mathcal{R}(\Lambda_i) \perp \mathcal{R}(\Gamma_i)$, then $\{\Lambda_i T_1 + \Gamma_i T_2\}_{i\in I}$ is a T_1^*K-g-frame, where $T_1, T_2 \in End^*_{\mathcal{A}}(H)$.*

Proof. Let $\{\Lambda_i\}$ be a K-g-frame for H and $\{\Gamma_i\}_{i\in I}$ be a g-Bessel sequence for H, then there exist constants $A_1, B_1, B_2 > 0$ such that

$$A_1\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B_1\langle x, x\rangle, \quad \forall x \in H,$$

and

$$\sum_{i\in I}\langle \Gamma_i x, \Gamma_i x\rangle \leq B_2\langle x, x\rangle, \quad \forall x \in H.$$

Since $\mathcal{R}(\Lambda_i) \perp \mathcal{R}(\Gamma_i)$, then $\mathcal{R}(\Lambda_i T_1) \perp \mathcal{R}(\Gamma_i T_2)$, $\forall i \in I$, so for each $x \in H$, we have

$$\begin{aligned}\sum_{i\in I}\langle (\Lambda_i T_1 + \Gamma_i T_2)x, (\Lambda_i T_1 + \Gamma_i T_2)x\rangle &= \sum_{i\in I}\langle \Lambda_i T_1 x, \Lambda_i T_1 x\rangle + \sum_{i\in I}\langle \Gamma_i T_2 x, \Gamma_i T_2 x\rangle \\ &\leq B_1\langle T_1 x, T_1 x\rangle + B_2\langle T_2 x, T_2 x\rangle \\ &\leq B_1\|T\|^2\langle x, x\rangle + B_2\|T_2\|^2\langle x, x\rangle \\ &= (B_1\|T_1\|^2 + B_2\|T_2\|^2)\langle x, x\rangle. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned}A_1\langle (T_1^*K)^*x, (T_1^*K)x\rangle = A_1\langle K^*T_1^*x, K^*T_1 x\rangle &\leq \sum_{i\in I}\langle \Lambda_i T_1 x, \Lambda_i T_1 x\rangle \\ &\leq \sum_{i\in I}\langle \Lambda_i T_1 x, \Lambda_i T_1 x\rangle + \sum_{i\in I}\langle \Gamma_i T_2 x, \Gamma_i T_2 x\rangle \\ &= \sum_{i\in I}\langle (\Lambda_i T_1 + \Gamma_i T_2)x, (\Lambda_i T_1 + \Gamma_i T_2)x\rangle. \end{aligned} \tag{2.7}$$

From inequlity (2.6) and (2.7), we conclude that $\{\Lambda_i T_1 + \Gamma_i T_2\}_{\in I}$ is a K-g-frame for H. □

The following corollary is consequence of Theorem 2.7.

Corollary 2.8. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for H and $\{\Gamma_i\}$ be a g-Bessel sequence for H, such that $\mathcal{R}(\Lambda_i) \perp \mathcal{R}(\Gamma_i)$, $\forall i \in I$. Then, the following statements hold:*

(1) The sequence $\{\Lambda_i T + \Gamma_i T\}_{i\in I}$ is a (T^*K)-g-frame for H.

(2) The sequence $\{\Lambda_i + \Gamma_i\}_{i\in I}$ and $\{\Lambda_i - \Gamma_i\}_{i\in I}$ are K-g-frames for H.

(3) If $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are two positively confined sequences then the sequence $\{a_i\Lambda_i + b_i\Gamma_i\}_{i\in I}$ is a $K-g-$frame for H.

Proof. (1) Take in the theorem 2.7, $T_1 = T_2 = T$.
(2) Take in the theorem 2.7, $T_1 = I_H$ and $T_2 = -I_H$.
(3) Since $\mathcal{R}(\Lambda_i) \perp \mathcal{R}(\Gamma_i)$, then $\mathcal{R}(a_i\Lambda_i) \perp \mathcal{R}(b_i\Gamma_i)$ and apply (2). □

We conclude the section with characterizing the K-g-frame by quotient maps.

Definition 2.9. Let $T_1, T_2 \in End^*_{\mathcal{A}}(H)$, the map $[T_1/T_2] : \mathcal{R}(T_2) \to \mathcal{R}(T_1)$ defined by $[T_1/T_2](T_2(x)) = T_1(x)$ is called the quotient map.

Remark 2.10. $[T_1/T_2]$ is a linear operator if and only if $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$.

Theorem 2.11. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-Bessel sequence for H with the frame operator S and $K \in End^*_{\mathcal{A}}(H)$. Then, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H if and only if the quotient operator $[K^*/S^{\frac{1}{2}}]$ is a bounded linear operator. In this case if $\overline{\mathcal{R}(S^{\frac{1}{2}})}$ is orthogonally complemented and $\mathcal{R}(K) \subseteq \mathcal{R}(S^{\frac{1}{2}})$, then $K = S^{\frac{1}{2}}X$ for some $X \in End^*_{\mathcal{A}}(H)$.*

Proof. Assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H, then there exists constant $A > 0$, such that

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle, \forall x \in H,$$

and we have

$$\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle = \langle Sx, x\rangle.$$

So,

$$A\langle K^*x, K^*x\rangle \leq \langle Sx, x\rangle = \langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}x\rangle.$$

Therefore, $\mathcal{N}(S^{\frac{1}{2}}) \subseteq \mathcal{N}(K^*)$ which implies that the quotient map $[K^*/S^{\frac{1}{2}}]$ is bounded linear operator.
Conversely, suppose that $[K^*/S^{\frac{1}{2}}]$ is a bounded linear operator, then there exists $C > 0$, such that for each $x \in H$, $\|K^*x\| \leq C\|S^{\frac{1}{2}}x\|$, so

$$\frac{1}{C}\|K^*x\|^2 \leq \|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\|.$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H.
To complete the proof, if $\overline{\mathcal{R}(S^{\frac{1}{2}})}$ is orthogonally complemented and $\mathcal{R}(K) \subseteq \mathcal{R}(S^{\frac{1}{2}})$, then by lemma 1.3, there exist $X \in End^*_{\mathcal{A}}(H)$ such that $K = S^{\frac{1}{2}}X$. □

Corollary 2.12. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for H. Then $\{\Lambda_i\}_{i\in I}$ is a K^n-g-frame for H.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for H, then $[K^*/S^{\frac{1}{2}}]$ is a bounded linear operator by theorem 2.11. Hence there exists $A > 0$, such that for every $x \in H$, $\|K^*x\| \leq A\|S^{\frac{1}{2}}x\|$. Then,

$$\begin{aligned}\|(K^n)^*x\| &= \|(K^{n-1})^*K^*x\| \\ &\leq \|(K^{n-1})^*\|\|K^*x\| \\ &\leq A\|(K^{n-1})^*\|\|S^{\frac{1}{2}}x\|.\end{aligned}$$

So, $[(K^n)^*/S^{\frac{1}{2}}]$ is a bounded linear operator, hence $\{\Lambda_i\}_{i\in I}$ is a K^n-g-frame for H. □

3 Dual of K-g-Frame

This section is devoted to the study of the dual K-g-Bessel sequence of K-g-frame in Hilbert C^*-modules. We firstly define the notions of g-complete and g-orthonormal bases for Hilbert C^*-modules.

Definition 3.1. We say that $\{\Lambda_i \in End^*_{\mathcal{A}}(H, H_i), i \in I\}$ is g-complete if $\{x : \Lambda_i x = 0, i \in I\} = \{0\}$.

Lemma 3.2. *$\{\Lambda_i \in End^*_{\mathcal{A}}(H, H_i), i \in I\}$ is g-complete if and only if $\overline{Span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$.*

Proof. Let $\{\Lambda_i\}_{i\in I}$ be g-complete. Since $\overline{Span}\{\Lambda_i^*(H_i)\}_{i\in I} \subseteq H$ it is enough to proof that if $x \in H$ and $x \perp Span\{\Lambda_i^*(H_i)\}_{i\in I}$, then $x = 0$. Let $x \in H$ and $x \perp Span\{\Lambda_i^*(H_i)\}_{i\in I}$, since for any $i \in I$, $x \perp \Lambda_i^*\Lambda_i x$, then for all $i \in I$,

$$\langle \Lambda_i x, \Lambda_i x\rangle = \langle x, \Lambda_i^*\Lambda_i x\rangle = 0.$$

So, $\Lambda_i x = 0$. Therefore, $x = 0$. Conversely, let $\overline{Span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$, let $x \in H$ and suppose that $\Lambda_i x = 0$, $\forall i \in I$. Then, for each $y \in H_i$

$$\langle \Lambda_i x, y\rangle = \langle x, \Lambda_i^* y\rangle = 0.$$

Hence, $x \perp Span\{\Lambda_i^*(H_i)\}_{i\in I}$. Therefore, $x \perp \overline{Span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$. It shows that $x = 0$, thus $\{\Lambda_i\}_{i\in I}$ is g-complete. □

Definition 3.3. $\{\Lambda_i\}_{i\in I}$ is called a g-orthonormal bases for H with respect to $\{H_i\}_{i\in I}$ if

$$\langle \Lambda_i^* x_i, \Lambda_j^* x_j\rangle = \delta_{i,j}\langle x_i, x_j\rangle, \quad \forall i, j \in I, \forall x_i \in H_i, \forall x_j \in H_j$$

and

$$\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle = \langle x, x\rangle, \quad \forall x \in H.$$

Lemma 3.4. *Let $\{\theta_i\}_{i\in I} \in \{End^*_{\mathcal{A}}(H, H_i), i \in I\}$ be a g-orthonormal bases for H with respect to $\{H_i\}_{i\in I}$. Then $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence if and only if there exists a unique adjointable operator $V : H \to H$ such that $\Lambda_i = \theta_i V^*, \forall i \in I$.*

Proof. Since $\{\theta_i\}_{i\in I}$ is a g-orthonormal bases for H, $\{\theta_i x\}_{\in I} \in l^2(\{H_i\}_{i\in I})$, for all $x \in H$. If $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for H with bound B, let $x \in H$, we have

$$\begin{aligned}
\|\sum_{i\in I}\Lambda_i^*\theta_i x\| &= \sup_{\|y\|=1}\|\langle\sum_{i\in I}\Lambda_i^*\theta_i x, y\rangle\| \\
&= \sup_{\|y\|=1}\|\sum_{i\in I}\langle\theta_i x, \Lambda_i y\rangle\| \\
&\leq \sup_{\|y\|=1}\|\sum_{i\in I}\langle\theta_i x, \theta_i x\rangle\|^{\frac{1}{2}}\|\sum_{i\in I}\langle\Lambda_i y, \Lambda_i y\rangle\|^{\frac{1}{2}} \\
&\leq \sqrt{B}\|x\|,
\end{aligned}$$

then we can defined the bounded operator V on H by

$$Vx = \sum_{i\in I}\Lambda_i^*\theta_i x,$$

V is adjointable because, for each $x, y \in H$

$$\begin{aligned}
\langle Vx, y\rangle &= \langle\sum_{i\in I}\Lambda_i^*\theta_i x, y\rangle \\
&= \sum_{i\in I}\langle x, \theta_i^*\Lambda_i y\rangle \\
&= \langle x, \sum_{i\in I}\theta_i^*\Lambda_i y\rangle.
\end{aligned}$$

So, $V^*x = \sum_{i\in I}\theta_i^*\Lambda_i x$, $\forall x\in H$.

Also by definition of g-orthonormal bases we have for each $x\in H$, $\theta_i\theta_j^*x = \delta_{i,j}x$. So,

$$V\theta_j^*x = \sum_{i\in I}\Lambda_i^*\theta_i\theta_j^*x = \Lambda_j^*x, \quad \forall x\in H, \forall j\in I.$$

Hence, $V\theta_j^* = \Lambda_j^*$ which implies that $\theta_j V^* = \Lambda_j$.
Suppose that $V_1, V_2\in End_{\mathcal{A}}^*(H)$, $\theta_i V_1^* = \theta_i V_2^* = \Lambda_i, \quad \forall i\in I$, then for each $x\in H$, $y_i\in H_i$, we have

$$\langle\theta_i V_1^*x, y_i\rangle = \langle\theta_i V_2^*x, y_i\rangle,$$

that is

$$\langle V_1^*x, \theta_i^*y_i\rangle = \langle V_2^*x, \theta_i^*y_i\rangle.$$

Since, $\overline{Span}\{\Lambda_i^*(H_i)\}_{i\in I} = H$, by lemma 3.2, $V_1^*x = V_2^*x$. So, $V_1 = V_2$, thus the operator V is unique.
Conversely, since $\Lambda_i = \theta_i V^*$, $\forall i\in I$, for any $x\in H$, we have

$$\begin{aligned}\sum_{i\in I}\langle\Lambda_i x, \Lambda_i x\rangle &= \sum_{i\in I}\langle\theta_i V^*x, \theta_i V^*x\rangle\\ &= \langle V^*x, V^*x\rangle\\ &\le \|V^*\|^2\langle x, x\rangle.\end{aligned}$$

So, $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for H. □

Remark 3.5. Given the g-orthonormal bases $\{\theta_i\}_{i\in I}$, the operator V in lemma 3.4 is called the g-preframe operator associated with $\{\Lambda_i\}_{i\in I}$.

Lemma 3.6. *Suppose $\{\theta_i\}_{i\in I}$ is a g-orthonormal bases for H, $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for H, and V and S are the g-preframe operator and g-frame operator associated with $\{\Lambda_i\}_{i\in I}$, respectively. Then, $S = VV^*$.*

Proof. We have $\Lambda_i = \theta_i V^*, \quad \forall i\in I$, so for each $x\in H$, we have

$$\begin{aligned}Sx &= \sum_{i\in I}\Lambda_i^*\Lambda_i x\\ &= \sum_{i\in I}(\theta_i V^*)^*(\theta_i V^*)x\\ &= V\sum_{i\in I}\theta_i^*\theta_i V^*x\\ &= VV^*x.\end{aligned}$$

Therefore, $S = VV^*$. □

Now, we introduce the concept of dual K-g-Bessel sequence for Hilbert C^*-modules.

Definition 3.7. Suppose that $K\in End_{\mathcal{A}}^*(H)$ and $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H. A g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for H is said to be a dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$ if

$$Kx = \sum_{i\in I}\Lambda_i^*\Gamma_i x, \quad \forall x\in H.$$

In the following, we give some sufficient and necessary conditions of K-g-frame by utilizing the associated g-preframe operators.

Theorem 3.8. *Let $K\in End_{\mathcal{A}}^*(H)$ and $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for H. The associated g-preframe operator with $\{\Lambda_i\}_{i\in I}$ is V and $\{\theta_i\}_{i\in I}$ is the g-orthonormal bases for H with respect to $\{H_i\}_{i\in I}$. Then V is a co-isometry if and only if $\{\Lambda_i K^*\}_{i\in I}$ is a Parseval K-g-frame.*

Proof. From the definition of g-orthonormal basis we obtain

$$\sum_{i\in I}\langle\Lambda_i K^*x,\Lambda_i K^*x\rangle=\sum_{i\in I}\langle\theta_i V^*K^*x,\theta_i V^*K^*x\rangle=\langle V^*K^*x,V^*K^*x\rangle,\forall x\in H,$$

which implies that the conclusion is obvious. □

In the sequel, we give some constructions of dual K-g-Bessel in conjunction with g-preframe operators.

Theorem 3.9. *Let $K\in End^*_{\mathcal{A}}(H)$ and $\{\theta_i\}_{i\in I}$ be a g-orthonormal bases for H with respect to $\{H_i\}_{i\in I}$. $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H with the g-preframe operator V. U is the g-preframe operator of g-Bessel sequence $\{\Gamma_i\}_{i\in I}$. If U is invertible and U^{-1} is the right inverse of V, then $\{\Gamma_i\}_{i\in I}$ is a K-g-frame for H.*

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H, there exists a constant $A>0$, such that for each $x\in H$,

$$\begin{aligned}A\langle K^*x,K^*x\rangle&\le\sum_{i\in I}\langle\Lambda_i x,\Lambda_i x\rangle\\&=\sum_{i\in I}\langle\theta_i V^*x,\theta_i V^*x\rangle\\&=\sum_{i\in I}\langle\theta_i U^*(U^*)^{-1}V^*x,\theta_i U^*(U^*)^{-1}V^*x\rangle,\end{aligned}$$

we have $VU^{-1}=I_H$, then $(U^*)^{-1}V^*=I_H$. So,

$$A\langle K^*x,K^*x\rangle\le\sum_{i\in I}\langle\theta_i U^*x,\theta_i U^*x\rangle=\sum_{i\in I}\langle\Gamma_i x,\Gamma_i x\rangle.$$

□

Theorem 3.10. *Let $K\in End^*_{\mathcal{A}}(H)$ and $\{\theta_i\}_{i\in I}$ be a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$. $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H with g-preframe operator V. U is the g-preframe operator of g-Bessel sequence $\{\Gamma_i\}_{i\in I}$. Then, $\{\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$ if and only if $K=VU^*$.*

Proof. Suppose $\{\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$, then for each $x\in H$, we have $Kx=\sum_{i\in I}\Lambda_i^*\Gamma_i x$, hence by lemma 3.4, for all $x\in H$

$$Kx=\sum_{i\in I}(\theta_i V^*)(\theta_i U^*)x=V\sum_{i\in I}\theta_i^*\theta_i U^*x=VU^*x,$$

we obtain $K=VU^*$. Conversely, since $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are both g-Bessel sequences, then by lemma 3.4, there exist a unique $V,U\in End^*_{\mathcal{A}}(H)$, such that $\Lambda_i=\theta_i V^*$ and $\Gamma_i=\theta_i U^*$. Hence for each $x\in H$

$$\sum_{i\in I}\Lambda_i^*\Gamma_i x=\sum_{i\in I}(\theta_i V^*)(\theta_i U^*)x=V\sum_{i\in I}\theta_i^*\theta_i U^*x=VU^*x=Kx.$$

Therefore, $\{\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$. □

Theorem 3.11. *Let $K\in End^*_{\mathcal{A}}(H)$. $\{\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$. Suppose that $T\in End^*_{\mathcal{A}}(H)$, if T is a co-isometry, then $\{T^*\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{T^*\Lambda_i\}_{i\in I}$.*

Proof. Suppose T is a co-isometry, then $TT^*=I_H$, so, for each $x\in H$

$$\begin{aligned}\sum_{i\in I}(T^*\Lambda_i)^*(T^*\Gamma_i)x&=\sum_{i\in I}\Lambda_i^*TT^*\Gamma_i x\\&=\sum_{i\in I}\Lambda_i^*\Gamma_i x\\&=Kx.\end{aligned}$$

Then, $\{T^*\Gamma_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{T^*\Lambda_i\}_{i\in I}$. □

Theorem 3.12. *Let $K \in End_{\mathcal{A}}^{*}(H)$ be with closed range, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H and $\mathcal{R}(K), S(\mathcal{R}(K))$ are orthogonally complemented. Then, $\{\Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i P_{\mathcal{R}(K)}\}_{i\in I}$, where S_{Λ} is defined by* (1.3).

Proof. Since S_{Λ} is bounded, it is easy to check that $\{\Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K\}_{i\in I}$ is a g-Bessel sequence. Since S_{Λ} is self-adjoint and invertible we have for each $x \in H$,

$$\begin{aligned}
Kx &= (S_{\Lambda}^{-1}S_{\Lambda})^{*}Kx \\
&= S_{\Lambda}^{*}(S_{\Lambda}^{-1})^{*}Kx \\
&= S_{\Lambda}^{*}P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}Kx \\
&= P_{\mathcal{R}(K)}S_{\Lambda}^{*}P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}Kx \\
&= P_{\mathcal{R}(K)}\sum_{i\in I}\Lambda_i^{*}\Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}Kx \\
&= \sum_{i\in I}(\Lambda_i P_{\mathcal{R}(K)})^{*}(\Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K)x.
\end{aligned}$$

□

Theorem 3.13. *Let $K \in End_{\mathcal{A}}^{*}(H)$ be with closed range, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for H and $\mathcal{R}(K), S(\mathcal{R}(K))$ are orthogonally complemented. Then $\{\phi_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i P_{\mathcal{R}(K)}\}_{i\in I}$ if and only if $\forall i \in I$, $\phi_i = \Gamma_i + \theta_i G$, where $\Gamma_i = \Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K$ and $\{\theta_i\}_{i\in I}$ is the g-orthonormal bases of H with respect to $\{H_i\}_{i\in I}$, $G \in End_{\mathcal{A}}^{*}(H)$ and V is the g-preframe operator of $\{\Lambda_i\}_{i\in I}$ such that $P_{\mathcal{R}(K)}VG = 0$.*

Proof. Suppose that $G \in End_{\mathcal{A}}^{*}(H)$ and $P_{\mathcal{R}(K)}VG = 0$. It is obvious that $\{\phi_i\}_{i\in I} = \{\Gamma_i + \theta_i G\}_{i\in I}$ is a g-Bessel sequence, then

$$\begin{aligned}
\sum_{i\in I}(\Lambda_i P_{\mathcal{R}(K)})^{*}\phi_i x &= \sum_{i\in I}(\Lambda_i P_{\mathcal{R}(K)})^{*}\Gamma_i x + \sum_{i\in I}(\Lambda_i P_{\mathcal{R}(K)})^{*}\theta_i Gx \\
&= Kx + P_{\mathcal{R}(K)}\sum_{i\in I}V\theta_i^{*}\theta_i Gx \\
&= Kx + P_{\mathcal{R}(K)}VGx = Kx.
\end{aligned}$$

Hence, $\{\phi_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i P_{\mathcal{R}(K)}\}_{i\in I}$.
Conversely, define the operator $G = U^{*} - V^{*}P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K$, where $U \in End_{\mathcal{A}}^{*}(H)$ is the g-preframe operator associated with $\{\phi_i\}_{i\in I}$, then $G \in End_{\mathcal{A}}^{*}(H)$, by theorem 3.10 and lemma 3.6, we know $VU^{*} = K$ and $S = VV^{*}$, hence

$$\begin{aligned}
P_{\mathcal{R}(K)}VGx &= P_{\mathcal{R}(K)}VU^{*}x - P_{\mathcal{R}(K)}VV^{*}P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}Kx \\
&= Kx - (S_{\Lambda})^{*}(S_{\Lambda}^{-1})^{*}Kx = 0,
\end{aligned}$$

and we have,

$$\theta_i G = \theta_i U^{*} - \theta_i V^{*}P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K = \phi_i - \Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K,$$

hence,

$$\Gamma_i + \theta_i G = \Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K + \phi_i - \Lambda_i P_{S(\mathcal{R}(K))}(S_{\Lambda}^{-1})^{*}K = \phi_i.$$

□

Theorem 3.14. *Let $K \in End_{\mathcal{A}}^{*}(H)$ and $\{\Gamma_i\}_{i\in I}$ be the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$ whose g-preframe operator is V. Suppose the g-preframe operator of the g-Bessel sequence $\{\phi_i\}_{i\in I}$ is U, then $VU^{*} = 0$ if and only if $\{\Gamma_i + \phi_i\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$.*

Proof. Suppose that $VU^* = 0$, then for each $x \in H$

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*\phi_i x &= \sum_{i\in I}(\theta_i V^*)^*(\theta_i U^*)x\\ &= V\sum_{i\in I}\theta_i^*\theta_i U^* x\\ &= VU^*x = 0,\end{aligned}$$

so,

$$\sum_{i\in I}\Lambda_i^*(\Gamma_i+\phi_i)x = \sum_{i\in I}\Lambda_i^*\Gamma_i x = Kx.$$

Conversely, for each $x \in H$,

$$\sum_{i\in I}\Lambda_i^*(\Gamma_i+\phi_i)x = Kx,$$

$$\Rightarrow \sum_{i\in I}\Lambda_i^*\Gamma_i x + \sum_{i\in I}\Lambda_i^*\phi_i x = Kx,$$

$$\Rightarrow \sum_{i\in I}\Lambda_i^*\phi_i x = 0,$$

$$\Rightarrow \sum_{i\in I}(\theta_i V^*)^*(\theta_i U^*)x = 0,$$

$$\Rightarrow VU^*x = 0$$

□

Theorem 3.15. *Let $K \in End_{\mathcal{A}}^*(H)$. $\{\Gamma_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$ are both the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$, respectively. Operators P_1 and P_2 are two linear operators on H, if $P_1 + P_2 = I_H$ then $\{\Gamma_i P_1 + \phi_i P_2\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$.*

Proof. Since $P_1 + P_2 = I_H$, we obtain for each $x \in I$,

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*(\Gamma_i P_1+\phi_i P_2)x &= \sum_{i\in I}\Lambda_i^*\Gamma_i P_1 x + \sum_{i\in I}\Lambda_i^*\phi_i P_2 x\\ &= KP_1x + KP_2x = K(P_1+P_2)x = Kx.\end{aligned}$$

□

Corollary 3.16. *Let $K \in End_{\mathcal{A}}^*(H)$ and it is invertible. $\{\Gamma_i\}_{i\in I}$ and $\{\phi_i\}_{i\in I}$ are both the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$, respectively. Operators P_1 and P_2 are two linear operators on H, then $\{\Gamma_i P_1+\phi_i P_2\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$ if and only if $P_1 + P_2 = I_H$*

Proof. By theorem 3.15, we can know necessity holds.
Suppose $\{\Gamma_i P_1 + \phi_i P_2\}_{i\in I}$ is the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$, then for each $x \in H$,

$$\begin{aligned}Kx &= \sum_{i\in I}\Lambda_i^*(\Gamma_i P_1+\phi_i P_2)x\\ &= \sum_{i\in I}\Lambda_i^*\Gamma_i P_1 x + \sum_{i\in I}\Lambda_i^*\phi_i P_2 x\\ &= KP_1x + KP_2x = K(P_1+P_2)x.\end{aligned}$$

So, $K = K(P_1 + P_2)$, since K is invertible, we obtain $P_1 + P_2 = I_H$. □

Theorem 3.17. *Let $K \in End^*_{\mathcal{A}}(H)$ have closed range and $\{\Gamma_i\}_{i\in I}$ be the dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$. Suppose that α is a complex number and $\mathcal{R}(K) \subset S(\mathcal{R}(K))$, then the sequence $\{\Delta_i\}_{i\in I}$ defined by*

$$\Delta_i = \alpha\Gamma_i + (1-\alpha)\Lambda_i S_\Lambda^{-1} K$$

is a dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in I}$ for H, where S_Λ is defined by (1.3).

Proof. By the definition of dual K-g-Bessel sequences, $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are both g-Bessel sequences, and then it is easy to check the sequence $\{\Delta_i\}_{i\in I}$ is also g-Bessel sequence. And we have for each $x \in H$,

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*\Delta_i x &= \sum_{i\in I}\Lambda_i^*\alpha\Gamma_i x + \sum_{i\in I}\Lambda_i^*(1-\alpha)\Lambda_i S_\Lambda^{-1}Kx\\ &= \alpha Kx + (1-\alpha)\sum_{i\in I}\Lambda_i^*\Lambda_i S_\Lambda^{-1}Kx\\ &= \alpha Kx + (1-\alpha)Kx\\ &= Kx.\end{aligned}$$

□

4 Conclusions

In this work, we constructed some new K-g-frames and we characterized the concept of K-g-frames by quotient maps in Hilbert C^*-module. Moreover, we investigated the notion of dual K-g-Bessel sequence in Hilbert C^*-modules.

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