On the Uniqueness of Reserving Pythagorean Orthogonality in Terms of Non-zero Linear Operators

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Abstract: *This paper is concerned with the connection between reserving Birkhoff-James orthogonality and the new particular case of the Carlsson orthogonality concerning non-zero linear operators in a complete normed linear space. The purpose of this article is to find the interrelation between reserving Pytharegorean and Birkhoff-James orthogonality. It is proved that if a non-zero linear operator reserves Pythagorean orthogonality, then it also reserves Birkhoff's orthogonality. It has also been proved that the unique property of orthogonality concerning the reserving Pythagorean orthogonality.*

Keywords: Normed space, Reserving Birkhoff-James orthogonality, Reserving Pythagorean orthogonality

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1 Introduction

A non-zero linear operator $T : U \to U$ is said to have orthogonality preserving property if and only if for every $u, v \in U$, u is orthogonal to v implies that $T(u)$ is orthogonal to $T(v)$. Chmielinski [3] discussed that any non-zero linear operator preserving orthogonality has a linear similarity. Chmielinski [3] also discussed the orthogonality preserving property in both directions in the sense that for any $u, v \in U$, u is orthogonal to v if and only if $T(u)$ is orthogonal to $T(v)$ and proved that the orthogonality preserving property, linear similarity, and orthogonality preserving property in both directions are equivalent and analogous result is true for the mapping T from a normed space U to V.

The most widely used orthogonality in normed linear space is the Birkhoff orthogonality introduced by Birkhoff in 1935 [1]. Different generalizations of Birkhoff orthogonality and the connection between other orthogonalities have been studied by different researchers. The notion of the Birkhoff orthogonality coincides with inner product orthogonality if the underlying space is a Hilbert space. Dragomir [6] defined approximate Birkhoff-James orthogonality and later Chmielinski [3] slightly modified the definition of approximate Birkhoff-James orthogonality by taking $\sqrt{1-\epsilon^2}$ in place of $1-\epsilon$ for $\epsilon \in [0, 1)$. Based on approximate orthogonality, the notion of approximate parallelism was introduced by Zamani and Moshelian [10].

By categorizing Birkhoff-James orthogonality into exclusive and exhaustive cases, Mall et al. [6] generalized the orthogonality in the space of bounded linear operators. In the space of bounded linear operators between finite-dimensional normed spaces, they characterize the strong Birkhoff-James orthogonality [6]. Moreover, a new geometric notion based on the operator theoretic results has been introduced in [6] involving Birkhoff-James orthogonality and strong Birkhoff-James orthogonality and proved that $0 \neq z \in Z$ is a semi-rotund point of Z if there is an element $v \in Z$ under the condition that z is strongly orthogonal to v.

A relation between the exposed point and the semi-rotund point is discussed in the paper [6] by concluding that if z is an exposed point of the unit ball $B_z = \{z \in Z : \|z\| \leq 1\}$ of the normed linear space Z is a semi-rotund point, however the converse may not be the true, considering dimension of the space $Z > 2$. For instance, the point $z_1 = (1, 1, 0) \in l^3_\infty$ is a semi-rotund point, however it is not an exposed point of B_z. It is also proved that a non-zero compact linear operator $T : U \to V$, where U is reflexive complete normed linear space and V is strictly convex complete normed linear space is a semi-rotund point of the corresponding operator space. Mall et al. [6] also proved that the set of all bounded linear operators from a normed linear space U to V is a semi-rotund space but not strictly convex if the space U and V are finite-dimensional normed linear spaces.

2 Notations and Introductory Results

Through the paper, the letters U and V denotes the normed linear spaces, $B(U,V)$ is the set of bounded linear operators form a normed linear space U to V. The set $M_T = \{u \in S_V : \|Tu\| = \|T\|\}$ denotes the norm attainment set of T and $S_U = \{u \in U : \|u\| = 1\}$ denoted the unit sphere of U. An element $u \in U$ is said to be norm parallel to $v \in U$ if $\|u+\alpha v\| = \|u\| + \|v\|$ for some $\alpha \in W = \{\alpha \in K :| \alpha |= 1\}$, denoted by $u\|v$. For any $\epsilon \in [0,1)$ and $u, v \in U$, u is called approximate parallel to v if $inf\{\|u+\xi v\| : \xi \in K\} \leq \epsilon\|u\|$, denoted by $u\|^{\epsilon}\|$, where $K \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.1. [1] Let $(U, \|.\|)$ be a normed space and $u, v \in U$. Then u is said to be orthogonal to v in the sense of Birkhoff if and only if

$$\|u\| \leq \|u + \lambda v\|$$

for all $\lambda \in \mathbb{K}$, where $\mathbb{K}$ is a scalar field.

Definition 2.2. [4] Let $(U, \|.\|)$ be a normed space and $u, v \in U$. We say u is isosceles orthogonal to v if and only if

$$\|u - v\| = \|u + v\|.$$

Definition 2.3. [4] Let $(U, \|.\|)$ be a normed space and $u, v \in U$. We say u is orthogonal to v in the sense of Pythagorean if and only if

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2.$$

Definition 2.4. [2] Let $(U, \|.\|)$ be a normed space and $l_k, m_k, n_k, k = 1, 2 \cdots, r$ a fixed collection of real numbers satisfying

$$\sum_{k=1}^{r} l_k m_k n_k = 1, \sum_{k=1}^{r} l_k m_k^2 = \sum_{k=1}^{r} l_k n_k^2 = 0.$$

We say an element $u \in U$ is Carlsson orthogonal to $v \in U$ if and only if

$$\sum_{k-1}^{r} l_k \|m_k u + n_k v\|^2 = 0.$$

Definition 2.5. [6] Let $\epsilon \in [0,1)$ and $u, v \in U$. Then we say u is an approximate Bifkhoff orthogonal to v if and only if $(1-\epsilon)\|u\| \leq \|u + \xi v\|$ for every all $\xi \in \mathbb{K}$.

Definition 2.6. [7] For any normed space U, an element $u \in U$ is said to be orthogonal to v if and only if the following equality holds

$$\|u + \frac{1}{2}v\|^2 + \|u - \frac{1}{2}v\|^2 = \frac{1}{2}\|\sqrt{2}u + v\|^2 + \|u\|^2. \tag{1}$$

The different concepts of the approximate Birkhoff-James orthogonality was given by Dragomir in the paper [6], which is denoted by $u \perp_D^{\epsilon} v$ and defined as follows:

Definition 2.7. [6] For any elements $u, v \in U$ and $\xi \in [0,1)$, u is said to be an approximate Birkhoff-James orthogonal to v if

$$\|u + \mu v\| \geq (1-\xi)\|u\|, \forall \mu \in \mathbb{K}.$$

Theorem 2.8. [6] *Let $(U, \|.\|)$ and $(V, \|.\|)$ be normed spaces and T be a bounded linear operator from U to V. If u is an element of the set M_T, then for any $\xi \in [0,1)$ and $v \in U$, $Tu \perp_D^{\xi} Tv \Rightarrow u \perp_D^{\xi} v$.*

Definition 2.9. [9] Let $T : U \to V$ be a non-zero linear operator form a normed space U to V. Then, T is said to reserve the Birkhoff orthogonality if

$$\forall u, v \in U, u \perp_B v \Rightarrow T(u) \perp_B T(v).$$

Definition 2.10. [8] Let $u, v \in U$, where U is a normed linear space. Then u is called strongly orthogonal to v in the sense of Birkhoff-James if

$$\forall \eta \neq 0, \|u\| < \|u + \eta v\|.$$

Theorem 2.11. [6] *Let T be a bounded linear operator from a normed linear space U to V and u is an element of the set M_T. Then, for any real number $\xi \in [0,1)$ and $v \in U$, $u\|^{\xi} v$ implies $T(u)\|^{\xi}T(v)$.*

Theorem 2.12. [6] *Let T_1 and T_2 are compact linear operators from a reflexive complete normed linear space U to any normed linear space V. Then the operator T_1 is norm parallel to the operator T_2 if and only if there exists an element u in the set $M_{T_1} \cap M_{T_2}$ with $T_1(u)$ is norm parallel to $T_2(u)$.*

Theorem 2.13. [6] *Let T_1 and T_2 are bounded linear operators from a normed linear space U to V. Then, the operator T_1 is said to be norm parallel to the operator T_2 if and only if there exists a sequence $\{u_n\}$ in S_U such that*

$$\lim_{n\to\infty} \|T_1(u_n)\| = \|T_1\|, \quad \lim_{n\to\infty} \|T_2(u_n)\| = \|T_2\| \quad \textit{and}$$
$$\lim_{n\to\infty} \|T_1(u_n) + \alpha T_2(u_n)\| = \|T_1\| + \|T_2\|.$$

Theorem 2.14. [6] *Let T be a bounded linear operator from a normed space U to V and u is an element of the set M_T. Then, for any $\xi \in [0,1)$ and $y \in U$, $T(u)$ is approximate orthogonal to $T(v)$ implies u is orthogonal to v.*

Theorem 2.15. [6] *Let T_1 and T_2 are compact linear operators form a reflexive complete normed linear space U to any normed linear space V with the condition $T_1 \perp_B T_2$ but $T_1 \not\perp_{SB} T_2$. Then there is an element u in the set M_T such that $T_1(u) \perp_B T_2(u)$.*

Theorem 2.16. [3] *Let U be a Minkowski plane that is either smooth and not strictly convex or strictly convex but not smooth. Then there are no nontrivial linear operators reserving orthogonality.*

Corollary 2.17. [3] *T be a non-zero linear operator reserving orthogonality in a Minkowski plane U. Then U is strictly convex and smooth or neither strictly convex nor smooth.*

Theorem 2.18. [3] *For any smooth normed space U with $dim(U) > 3$, there exists a non-zero linear operator $T : U \to U$ satisfying the reserving orthogonality if and only if U is an inner product space.*

Theorem 2.19. [3] *For any non-zero linear operator $T : U \to U$ in a Minkowski plane, the operator T is said to reserve orthogonality satisfying the condition "$u \perp v \Rightarrow T(u) \perp T(v)$ for all $u, v \in U$" if and only if with some positive constant ξ, one of the following equivalent conditions hold:*

1. $\|Tu\|_\beta = \xi\|u\|, \quad u \in U$;
2. $\|Tu\| = \xi\|u\|_\beta, \quad u \in U$;
3. $\|Tu\|_\beta = \xi, \quad u \in S$;
4. $\|Tv\| = \xi, \quad v \in S_\beta$,

where S and S_β denote unit spheres with respect to the norm and anti-norm respectively.

3 Main Result

Motivated by the concept of reserving Birkhoff orthogonality, the approximate reserving Birkhoff orthogonality in terms of the non-zero linear operator can be defined in the following ways:

Definition 3.1. Let $T : U \to V$ be a non-zero linear operator from a normed space U to V. Then, we say that the operator T reserves an approximate Birkhoff orthogonality if and only if

$$\forall u, v \in U \quad \text{and} \quad \xi \in [0,1), \|u + \eta v\| \geq (1-\xi)\|u\| \Rightarrow \|T(u) + \eta T(v)\| \geq (1-\xi)\|T(u)\|, \forall \eta \in \mathbb{K}.$$

Definition 3.2. Let $T : U \to V$ be a non-zero linear operator from a normed space U to V. Then, we say that the operator T reserves the strong Birkhoff orthogonality if

$$\forall \eta \neq 0, \|u\| < \|u + \eta v\| \Rightarrow \|T(u)\| < \|T(u) + \lambda T(v)\|.$$

Theorem 3.3. *Let $T : U \to V$ be a non-zero linear operator from a normed space U to V reserving Pythagorean orthogonality, then T reserve the Birkhoff-James orthogonality.*

Proof. Assume T reserve the Pythagorean orthogonality, then

$$\forall u, v \in U, \|u + \eta v\|^2 = \|u\|^2 + \|\eta v\|^2 \Rightarrow \|T(u) + \eta T(v)\|^2 = \|T(u)\|^2 + \|\eta T(v)\|^2 \\ \geq \|T(u)\|^2.$$

Therefore, $\|T(u)+\eta T(v)\|^2 \geq \|T(u)\|^2 \Rightarrow \|T(v)+\eta T(v)\| \geq \|T(u)\|$. This shows that T reserve the Birkhoff orthogonality. □

Theorem 3.4. *Let $T : U \to V$ be a non-zero linear operator from a normed space U to V such that $T(u) \perp_P T(\frac{v}{2})$ and T reserve orthogonality relation (1). Then,*

$$\|T(u) + T(v)\|^2 \geq 2\|T(u)\|\|T(v)\|$$

Proof. Assume T reserve orthogonality relation (1). Then,

$$\begin{aligned}
&\forall u, v \in U,\ \|u + \frac{1}{2}v\|^2 + \|u - \frac{1}{2}v\|^2 = \frac{1}{2}\|\sqrt{2}u + v\|^2 + \|u\|^2 \\
&\Rightarrow \|T(u) + \frac{1}{2}T(v)\|^2 + \|T(u) - \frac{1}{2}T(v)\|^2 = \frac{1}{2}\|\sqrt{2}T(u) + T(v)\|^2 + \|T(u)\|^2 \\
&\Rightarrow \|T(u) + \frac{1}{2}T(v)\|^2 + \|T(u) - \frac{1}{2}T(v)\|^2 \geq \|T(u)\|^2 \\
&\Rightarrow 2\|T(u)\|^2 + \|T(v)\|^2 \geq \|T(u)\|^2 \\
&\Rightarrow T(u)\|^2 + \|T(v)\|^2 \geq 0 \\
&\Rightarrow \|T(u) + T(v)\|^2 \geq 2\|T(u)\|\|T(v)\|.
\end{aligned}$$

□

Theorem 3.5. *Let $T : U \to V$ be non-zero and bounded linear operator reserving the Pythagorean orthogonality and $T(\xi u) = \xi T(u)$. Then, there exists a non-zero real number ξ such that $T(u) \perp_P T(\xi u + v)$.*

Proof. The theorem is proved as a similar concept discussed by James [4] (pp. 299-300). Since $T \neq 0$ reserves the Pythagorean orthogonality, then

$$\forall u, v \in U, u \perp_P v \Rightarrow T(u) \perp_P T(v)$$

Let us define

$$\begin{aligned}
T(\xi) =& \|T(u)\|^2 + \|T(\xi u) + T(v)\|^2 - \|T(u) - T(\xi u + v)\|^2 \\
=& \|T(u)\|^2 + \|T(\xi u) + T(v)\|^2 - \|(\xi - 1)T(u) - T(v)\|^2
\end{aligned}$$

We know that

$$\frac{(\xi - 1)^2}{\xi^2} + \frac{2\xi - 1}{\xi^2} = 1.$$

using this identity, we obtain

$$\begin{aligned}
T(\xi) =& \|T(u)\|^2 + \frac{2\xi - 1}{\xi^2}\|T(\xi u) + T(v)\|^2 + [\|(\xi - 1)T(u) + (\frac{\xi - 1}{\xi})T(v)\|^2 - \|(\xi - 1)T(u) + T(v)\|^2] \\
=& \|T(u)\|^2 + (2\xi - 1)\|T(u) + \frac{1}{\xi}T(v)\|^2 + [\|(\xi - 1)T(u) + (\frac{\xi - 1}{\xi})T(v)\| - \|(\xi - 1)T(u) + T(v)\|] \\
&\times [\|(\xi - 1)T(u) + (\frac{\xi - 1}{\xi})T(v)\| + \|(\xi - 1)T(u) + T(v)\|] \\
\geq& \|T(u)\|^2 + (2\xi - 1)\|T(v) + \frac{1}{\xi}T(v)\|^2 - \mid \frac{1}{\xi} \mid \|T(v)\| \quad [\|(\xi - 1)T(u) \\
&+ (\frac{\xi - 1}{\xi})T(v)\| + \|(\xi - 1)T(u) + T(v)\|].
\end{aligned}$$

By using the inequality

$$[\|T(u)\| - \|T(\frac{u}{\xi})\|]^2 \leq \|T(u) + T(\frac{v}{\xi})\|^2$$

and

$$\|(\xi-1)T(u) + \frac{\xi-1}{\xi}T(v)\| + \|(\xi-1)T(u) + T(v)\| \leq 2(\xi-1)\|T(u)\| + \frac{2\xi-1}{\xi}\|T(v)\|.$$

we obtain,

$$\begin{aligned}
T(\xi) \geq & 2\xi\|T(u)\|^2 + \frac{2\xi-1}{\xi^2}\|T(v)\|^2 - 2(\frac{2\xi-1}{\xi^2})\|T(u\|\|T(v)\| \\
& - \|T(v)\|[2(\frac{\xi-1}{\xi})\|T(u)\| + \frac{2\xi-1}{\xi^2}\|T(v)\|] \\
= & 2\xi\|T(u)\| - 2(\frac{3\xi-2}{\xi})\|T(u)\|\|T(v)\| \\
= & 2\|T(u)\|[\xi\|T(u)\| - (\frac{3\xi-2}{\xi})\|T(v)\|] \\
\geq & 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

If we take

$$T(-\xi) = T(u)\|^2 + \|T(\xi u) - T(v)\|^2 - \|(\xi+1)T(u) - T(v)\|^2$$

and use the identity

$$\Big[\frac{(\xi+1)^2}{\xi}\Big]^2 - \frac{(2\xi+1)}{\xi^2} = 1$$

we get,

$$\begin{aligned}
T(-\xi) \leq & \|T(u)\|^2 - (2\xi+1)\|T(u) - T(\frac{v}{\xi})\|^2 + \|T(v)\| \quad [\|(\frac{\xi+1}{\xi})T(u) - (\frac{\xi-1}{\xi^2})T(v)\| \\
& + \|(\frac{\xi+1}{\xi})T(u) - T(\frac{v}{\xi})\|]
\end{aligned}$$

Setting $n > 0$, it can be written as

$$\begin{aligned}
T(-\xi) \leq -2\|T(u)\| \quad [\xi\|T(u)\| - \frac{3\xi+2}{\xi}\|T(v)\|] \\
\Rightarrow T(-\xi) < 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Since $T(\xi)$ being continuous function leads to the result $T(\xi) = 0$ for some value $T(\xi)$ of ξ; $T(u) \perp_P T(\xi u + v)$. □

4 Conclusions

In terms of non-zero linear operators, reserving Birkhoff orthogonality in the complete normed linear space has been studied and characterized by different mathematicians, during the last decades. There may be some chances of introducing reserving Pythagorean orthogonality, reserving new particular cases of the Carlsson orthogonality, and reserving isosceles orthogonalities with the help of non-zero linear operators in the complete normed linear spaces.

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References

[1] Birkhoff, G., 1935, Orthogonality in linear metric spaces, *Duke Mathematical Journal*, 1(1), 169-172.

[2] Carlson, S., 1962, Orthogonality in Normed Linear Spaces *Arkiv för Matematik* 1(1), 297-318.

[3] Chmielinski, J., 2016, Operator reserving orthogonality in normed linear spaces, *Advanced Operator Theory*, 1(1), 8-14.

[4] James, R. C., 1945, Orthogonality in normed linear spaces *Duke Mathematical Journal*, 12(1), 291-302.

[5] James, R. C., 1947, Orthogonality and linear functionals in normed linear spaces, *Transactions of the American Mathematical Society*, 61(2), 265-292.

[6] Mall, A., Sain, D. and Paul, K., 2019, Some geometric properties of operator spaces *Banach Journal of Mathematical Analysis*, 13(1), 174-191.

[7] Ojha, B. P. and Bajracharya, P. M., 2021, Orthogonality in terms 2-HH norm and bounded linear operators in Banach spaces, *Applications and Applied Mathematics: An International Journal (AAM)*, 16(1), 631-642.

[8] Paul, K., Sain, D. and Jha, K., 2013, On strong orthogonality and strictly convex normed linear spaces, *Journal of Inequalities and Applications*, 242 (1), 1-7.

[9] Wu, S., He, S. and Yang, G., 2017, Orthogonalities, linear operators, and characterization of inner product spaces, *Aequationes Mathematicae*, 91(422), 969-978.

[10] Zamani, A., Sain, D. and Moslehian, M. S., 2015, Exact and approximate operator parallelism *Canadian Mathematical Bulletin*, 58(1), 175-182.