Comparative Study of Analytic and Numerical Solutions of Steady-state Temperature Distribution on Semi-circular Plate Using Laplace Equation

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Abstract: *In this paper numerical methods have been used to solve two dimensional steady state heat flow problem in polar coordinates with Dirichlet boundary conditions inside a semi-circular plate and the work focuses on the numerical methods for solving Laplace equation; finite difference schemes and Gauss elimination method. The numerical solution is compared with exact solution of the same problem. Finally, we analyze the absolute error in different number of iterations to check the accuracy of schemes.*

Keywords: Finite difference, Laplace equation, Numerical method, Semi-circular plate

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1 Introduction

The Laplace equation, which bears the name of French mathematician Pierre-Simon Laplace (1740 - 1827), is one of the most important partial differential equation in physics. This equation governs a variety of equilibrium physical phenomena such as temperature distribution in solids, electrostatics, inviscid and irrotational two-dimensional flow. Laplace equation arises in application to fluid dynamics and potential theory, so that it is also called potential equation. The Laplace equation in two dimensions is [6, 7].

$$\nabla^2 f = 0 \ \text{(vector form)}$$

where,

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{bmatrix},$$

That is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

where $f = f(x, y)$ is the temperature.

The function $f(x, y)$ which satisfies Laplace equation is called harmonic (In 1848 William Thomson introduced the term harmonic function). Laplace equation has no dependence on time, just on the spatial variables x, y. This means that it describes steady state situation that is $\frac{\partial f}{\partial t} = 0$ and no initial condition required, only require boundary conditions. The average value over a spherical surface is equal to the value at the center of the sphere according to the Laplace equation's solution.

In this paper, we discuss the approximate solution of Laplace equation in polar coordinates. There are many circles, semi-circles in real life, it is important to be able to solve partial differential equations that are written in polar coordinates. This Laplace equation reduced into polar coordinates, which of course are the just polar coordinates (r, θ) replacing (x, y) where $x = r\cos\theta,\ y = r\sin\theta$. Then, Laplace equation in polar coordinates form is

$$\frac{1}{r}(rf_r)_r + \frac{1}{r^2} f_{\theta\theta} = 0$$

That is,

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$$

1.1 Steady-state temperature distribution inside a semi-circular plate

Let R be radius of a circle and the temperature distribution along the interior of the semi-circle with radius R is maintained as $g(\theta)$ on the boundary (on the curve surface) and zeros on the diameter and f is the steady-state temperature at (r, θ). Then Laplace equation in polar coordinates form with Dirichlet boundary value problem is [6, 9]

$$\frac{1}{r}(rf_r)_r + \frac{1}{r^2} f_{\theta\theta} = 0 \tag{1}$$

with boundary conditions

$$f(r, 0) = f(r, \pi) = 0, f(1, \theta) = \sin\theta \tag{2}$$

on the curve surface and $f(0, \theta) = 0$ at the center, where, $R = 1, g(\theta) = \sin\theta, 0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. Analytic solution of Laplace equation (1) along the interior of the semi-circle with boundary conditions (2) is [1, 7] $f(r, \theta) = r\sin\theta, \quad 0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$.

1.2 Numerical method: Finite difference scheme

Numerical methods are mathematical procedures for tackling problems that are either impossible to solve or difficult to solve. The numerical solution is a numerical value that approximates the solution. Many numerical approaches involve performing calculations iteratively until the desired accuracy is obtained. Nowadays, numerical methods are used in fast electronics digital computers that make it possible to execute many tedious and repetitive calculation that produce approximate solution in a very short time. For every type of mathematical problem there are several numerical techniques that is Gauss elimination method, Bisection method, Newton's method and so on that can be used. The techniques differ in accuracy, length of calculation and difficulty in programming. In this paper, we use finite difference schemes and Gauss elimination method as well as MATLAB programming [13].

1.2.1 Finite difference grid of semi-circular plate

The continuous differential equation is converted to a discrete form by using numerical techniques to locate the solution at grid locations in space. The closed solution domain $D(r, \theta)$ is $r\theta$-plane for a two dimensional problem and the solution domain must be covered by two dimensional grid of lines, called the finite difference grid. The solution to the PDE with finite difference method is obtained at the intersections of these grid lines. Let these grid lines be equally spaced lines along r and θ directions having uniform spacing $\Delta r = h$ and $\Delta\theta = k$. The set of grid points are denoted by $(r_i, \theta_j), i = 1, 2, \cdots, M$ and $j = 0, 1, 2, \cdots, N$ where $r_0 = 0$ and $\theta_N = \pi$. On the grid point (r_i, θ_j), a continuous function $f(r, \theta)$ which is varying on (r_i, θ_j) is denoted by $f_{i,j}$ (discrete function).

1.2.2 Finite difference schemes of first and second order partial differential equations in polar coordinate form

The finite difference method is one of methods for solving partial differential equations (PDEs) or ordinary differential equations (ODEs) numerically. Finite-difference approaches place a regular grid over the area of interest and approximate Laplace equation at each grid point. Iteration is used to solve the equations that resulted. The finite difference method was first developed by A. Thom in 1920s under the title the method of square to solve non-linear hydrodynamic equations. If we replace partial derivatives of PDEs by finite difference by using Taylor series, we obtain the finite difference schemes of PDEs. Aim of finite difference schemes is to approximate the values of the continuous function $f(r, \theta)$ on a set of grid points in $r\theta-$plane. The finite difference technique is a numerical approximation approach solving finite difference schemes that uses iteration methods or computational algorithm to provide an approximate solution of the

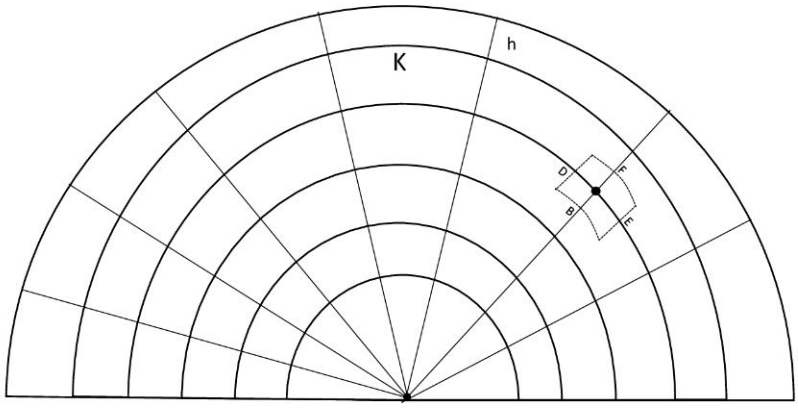

Figure 1: Finite Difference Grid in Semi-circular Plate.

PDE.
Some important finite difference schemes [12, 13]

$$\text{i)}\ \frac{\partial f}{\partial r}(r_i,\theta_j) \approx \frac{f_{i+1,j}-f_{i,j}}{\Delta r}\ \text{(Forward scheme)}$$

$$\text{ii)}\ \frac{\partial f}{\partial r}(r_i,\theta_j) \approx \frac{f_{i,j}-f_{i-1,j}}{\Delta r}\ \text{(Backward scheme)}$$

$$\text{iii)}\ \frac{\partial f}{\partial r}(r_i,\theta_j) \approx \frac{f_{i+1,j}-f_{i-1,j}}{2\Delta r}\ \text{(Central scheme)}$$

$$\text{iv)}\ \frac{\partial^2 f}{\partial r^2}(r_i,\theta_j) \approx \frac{f_{i+1,j}-2f_{i,j}+f_{i-1,j}}{(\Delta r)^2}\ \text{(Second-order difference scheme)}$$

Now,

$$\begin{aligned}\frac{1}{r}(rf_r)_r &\approx \frac{1}{r_i}[\frac{(rf_r)_{i+1/2,j}-(rf_r)_{i-1/2,j}}{\Delta r}]\\ &\approx \frac{1}{r_i\Delta r}[r_{i+1/2}(\frac{f_{i+1,j}-f_{i,j}}{\Delta r})-r_{i-1/2}(\frac{f_{i,j}-f_{i-1,j}}{\Delta r})]\\ &\approx \frac{1}{r_i(\Delta r)^2}[r_{i+1/2}(f_{i+1,j}-f_{i,j})-r_{i-1/2}(f_{i,j}-f_{i-1,j})],\end{aligned}$$

and

$$\frac{1}{r^2}f_{\theta\theta} \approx \frac{1}{r_i^2(\Delta\theta)^2}(f_{i,j+1}-2f_{i,j}+f_{i,j-1}).$$

Let, $\Delta r = h$, $\Delta\theta = k$. Now, substituting in Laplace equation in polar coordinates (1), we obtain

$$\frac{1}{r_ih^2}[r_{i+1/2}(f_{i+1,j}-f_{i,j})-r_{i-1/2}(f_{i,j}-f_{i-1,j})]+\frac{1}{r_i^2k^2}(f_{i,j+1}-2f_{i,j}+f_{i,j-1}) = 0 \quad (3)$$

$$\Rightarrow \frac{r_{i+1/2}}{r_ih^2}f_{i+1,j}+\frac{r_{i-1/2}}{r_ih^2}f_{i-1,j}+\frac{1}{r_i^2k^2}f_{i,j+1}+\frac{1}{r_i^2k^2}f_{i,j-1}-(\frac{r_{i+1/2}}{r_ih^2}+\frac{r_{i-1/2}}{r_ih^2}+\frac{1}{r_i^2k^2})f_{i,j} = 0 \quad (4)$$

Multiplying both sides by $r_i^2h^2$, we obtain,

$$r_{i+1/2}r_i\ \ f_{i+1,j}+r_{i-1/2}r_i\ \ f_{i-1,j}+\tfrac{h^2}{k^2}(f_{i,j+1}+f_{i,j-1})-(r_{i+1/2}r_i+r_{i-1/2}r_i+\tfrac{2h^2}{k^2})f_{i,j}=0$$

Let, $r_{i+1/2}r_i=\alpha_i$, $\quad r_{i-1/2}r_i=\beta_i$, $\quad \frac{h^2}{k^2}=\gamma$.
Then,

$$\alpha_i\ \ f_{i+1,j}+\beta_i\ \ f_{i-1,j}+\gamma(f_{i,j+1}+f_{i,j-1})-(\alpha_i+\beta_i+2\gamma)f_{i,j}=0 \quad (5)$$

This is a five-point Laplacian in polar coordinates [8, 11].

Let, $h = 0.2$ and $k = \frac{\pi}{4}$ and from (2), discrete boundary conditions for the semi-circular plate are [5] $f_{0,j} = 0, j = 1, 2, 3, \cdots, M-1,\ f_{i,0} = 0,\ f_{i,\pi} = 0$. That is, $f_{1,0} = f_{2,0} = \cdots = f_{M-1,0} = 0$ and $f_{1,N-1} = f_{2,N-1} = \cdots = f_{M-1,N-1} = 0$.

and

$$f_{M-1,j} = sin(jk)(i.e. f_{5,1} = 0.7071, f_{5,2} = 1, f_{5,3} = 0.7071) \tag{6}$$

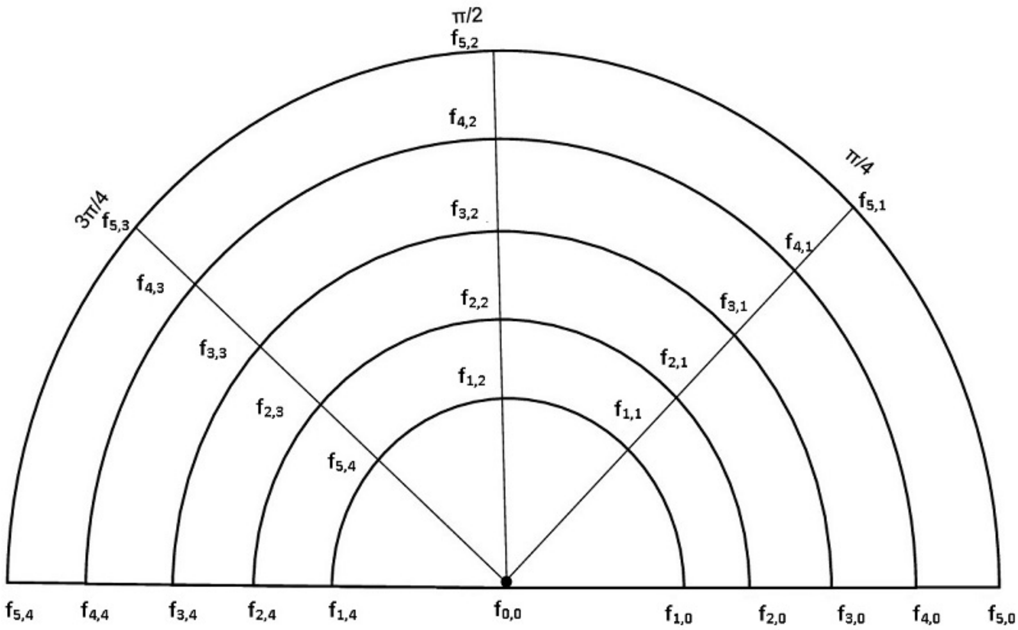

Figure 2: Finite Difference Grid Points in Semi-circular Plate.

2 Numerical Solution of Laplace Equation Inside Semi-circular Plate

For numerical solution of Laplace equation in the semi-circular disc, using five-point Laplacian (3) and discrete boundary conditions (4), we obtain the following system of equations [2]:
When $i = 1,\ j = 1$, then we obtain

$$f_{1,1} + 0.0648 f_{1,2} + 0.06 f_{2,1} = 0, \tag{7}$$

where, $\alpha_1 = r_{3/2} r_1 = 0.3 \times 0.2 = 0.06$, $\beta_1 = r_{1/2} r_1 = 0.1 \times 0.2 = 0.02$ and $\gamma = \frac{0.2^2 \times 16}{\pi^2} = 0.0648$.

Similarly,

$$\begin{aligned}
0.0648f_{1,1} - 0.2096f_{1,2} + 0.0648f_{1,3} + 0.06f_{2,2} &= 0 \quad (8)\\
0.0648f_{1,2} - 0.2096f_{1,3} + 0.06f_{2,3} &= 0 \quad (9)\\
0.12f_{1,1} - 0.4496f_{2,1} + 0.0648f_{2,2} + 0.2f_{3,1} &= 0 \quad (10)\\
0.12f_{1,2} + 0.0648f_{2,1} - 0.4496f_{2,2} + 0.0648f_{2,3} + 0.2f_{3,2} &= 0 \quad (11)\\
0.12f_{1,3} + 0.0648f_{2,2} - 0.4496f_{2,3} + 0.2f_{3,3} &= 0 \quad (12)\\
0.3f_{2,1} - 0.8496f_{3,1} + 0.0648f_{3,2} + 0.42f_{4,1} &= 0 \quad (13)\\
0.3f_{2,2} + 0.0648f_{3,1} - 0.8496f_{3,2} + 0.0648f_{3,3} + 0.42f_{4,2} &= 0 \quad (14)\\
0.3f_{2,3} + 0.0648f_{3,2} - 0.8496f_{3,3} + 0.42f_{4,3} &= 0 \quad (15)\\
0.56f_{3,1} - 1.4096f_{4,1} + 0.0648f_{4,2} &= -0.5091 \quad (16)\\
0.56f_{3,2} + 0.0648f_{4,1} - 1.4096f_{4,2} + 0.0648f_{4,3} &= -0.72 \quad (17)\\
0.56f_{3,3} + 0.0648f_{4,2} - 1.4096f_{4,3} &= -0.5091 \quad (18)
\end{aligned}$$

From the system of linear equations (5)-(16), we obtain a linear sparse system [12, 13]

$$AF = B$$

where,

$$\begin{aligned}
F &= \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{2,1} & f_{2,2} & f_{2,3} & f_{3,1} & f_{3,2} & f_{3,3} & f_{4,1} & f_{4,2} & f_{4,3} \end{bmatrix}^T,\\
B &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5091 & -0.72 & -0.5091 \end{bmatrix}^T,
\end{aligned}$$

and A is a sparse matrix with tridiagonal blocks;

$$\begin{bmatrix}
-0.2096 & 0.0648 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.0648 & -0.2096 & 0.0648 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.0648 & -0.2096 & 0 & 0 & 0.06 & 0 & 0 & 0 & 0 & 0 & 0\\
0.12 & 0 & 0 & -0.4496 & 0.0648 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0\\
0 & 0.12 & 0 & 0.0648 & -0.4496 & 0.0648 & 0 & 0.2 & 0 & 0 & 0 & 0\\
0 & 0 & 0.12 & 0 & 0.0648 & -0.4496 & 0 & 0 & 0.2 & 0 & 0 & 0\\
0 & 0 & 0 & 0.3 & 0 & 0 & -0.8496 & 0.0648 & 0 & 0.42 & 0 & 0\\
0 & 0 & 0 & 0 & 0.3 & 0 & 0.0648 & -0.8496 & 0.0648 & 0 & 0.42 & 0\\
0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0.0648 & -0.8496 & 0 & 0 & 0.42\\
0 & 0 & 0 & 0 & 0 & 0 & 0.56 & 0 & 0 & -1.4096 & 0.0648 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.56 & 0 & 0.0648 & -1.4096 & 0.0648\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.56 & 0 & 0.0648 & -1.4096
\end{bmatrix}$$

i.e.,

$$A = \begin{bmatrix} B_1 & B_2 & \bigcirc & \bigcirc\\ C_1 & C_2 & C_3 & \bigcirc\\ \bigcirc & D_2 & D_3 & D_4\\ \bigcirc & \bigcirc & E_3 & E_4 \end{bmatrix}$$

where $\bigcirc$ is zero matrix, $B_1,\ C_2,\ D_3,\ E_4$ are tridiagonal, symmetric, diagonally dominant and negative definite (diagonal elements are negative) matrices and $B_2,\ C_1,\ C_3,\ D_2,\ D_4,\ E_3$ are diagonal matrices

$$\bigcirc = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.2096 & 0.0648 & 0\\ 0.0648 & -0.2096 & 0.0648\\ 0 & 0.0648 & -0.2096 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -0.4496 & 0.0648 & 0 \\ 0.0648 & -0.4496 & 0.0648 \\ 0 & 0.0648 & -0.4496 \end{bmatrix}, D_3 = \begin{bmatrix} -0.8496 & 0.0648 & 0 \\ 0.0648 & -0.8496 & 0.0648 \\ 0 & 0.0648 & -0.8496 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} -1.4096 & 0.0648 & 0 \\ 0.0648 & -1.4096 & 0.0648 \\ 0 & 0.0648 & -1.4096 \end{bmatrix}, B_2 = \begin{bmatrix} 0.06 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.06 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.12 & 0 & 0 \\ 0 & 0.12 & 0 \\ 0 & 0 & 0.12 \end{bmatrix}, C_3 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, D_2 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0.42 & 0 & 0 \\ 0 & 0.42 & 0 \\ 0 & 0 & 0.42 \end{bmatrix}, E_3 = \begin{bmatrix} 0.56 & 0 & 0 \\ 0 & 0.56 & 0 \\ 0 & 0 & 0.56 \end{bmatrix}$$

Using Gaussian elimination method in MATLAB [3, 4, 10], we obtain

$$\begin{aligned} F &= \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{2,1} & f_{2,2} & f_{2,3} & f_{3,1} & f_{3,2} & f_{3,3} & f_{4,1} & f_{4,2} & f_{4,3} \end{bmatrix}^T \\ &= \begin{bmatrix} 0.1473 & 0.2083 & 0.1473 & 0.2895 & 0.4095 & 0.2895 & 0.4299 & 0.6079 & 0.4299 & 0.5689 & 0.8046 & 0.5689 \end{bmatrix}^T \end{aligned}$$

3 Error Analysis

Whenever any mathematical calculation is performed it is not always possible to use exact and accurate values, so we need approximate values. Due to approximation, the result is inaccurate and we can say that error is introduced in the calculations. There are three main sources of error in computation that is human errors, truncation errors and round-off errors. Absolute error in computation is the absolute value of difference between the problem's exact solution and its approximation. The error quantities are useful for evaluating the accuracy of different numerical methods when the problem's accurate solution is known. When the exact solution of the problem is unknown, the true error cannot (in most cases) be calculated, other means are used for evaluating the accuracy of a numerical solution [3, 4]. In this paper, we use simple absolute error. The absolute errors of the exact and numerical solutions (values) of the given problem is shown in the following table:

Table 1: Comparing between analytic and numerical solutions

Values of r, θ	Numerical Values	Exact Values	Absolute Errors
$r = 0.2, \theta = \pi/4$	0.1473	0.1414	0.0059
$r = 0.4, \theta = \pi/4$	0.2895	0.2828	0.0067
$r = 0.6, \theta = \pi/4$	0.4299	0.4243	0.0056
$r = 0.8, \theta = \pi/4$	0.5689	0.5657	0.0032
$r = 0.2, \theta = \pi/2$	0.2083	0.2000	0.0083
$r = 0.4, \theta = \pi/2$	0.4095	0.4000	0.0095
$r = 0.6, \theta = \pi/2$	0.6079	0.6000	0.0079
$r = 0.8, \theta = \pi/2$	0.8046	0.8000	0.0046
$r = 0.2, \theta = 3\pi/4$	0.1473	0.1414	0.0059
$r = 0.4, \theta = 3\pi/4$	0.2895	0.2828	0.0067
$r = 0.6, \theta = 3\pi/4$	0.4299	0.4243	0.0056
$r = 0.8, \theta = 3\pi/4$	0.5689	0.5657	0.0032

4 Comparative Study Through MATLAB Figures

4.1 Contour and surface plots analytical solution

The plots on Fig. 3 are the steady-state temperature distribution on semi-circular plate with given boundary values analytically [3, 4, 10].

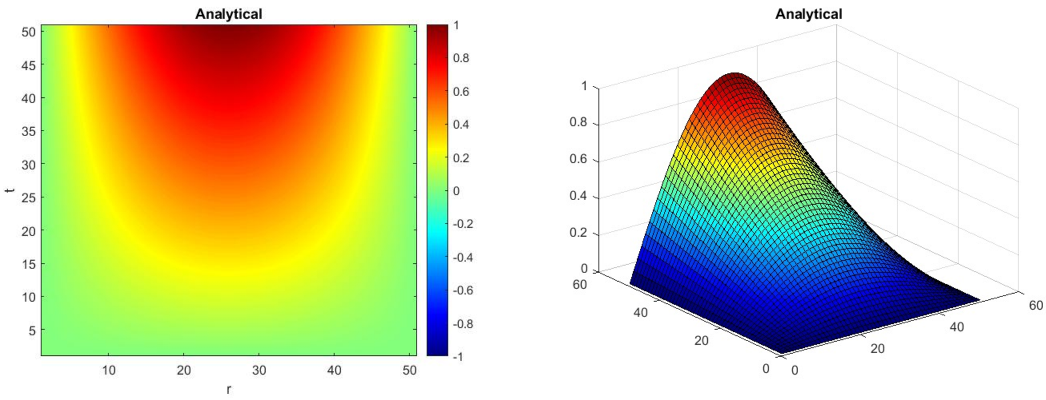

Figure 3: Temperature distribution on a semi-circular plate analytically.

4.2 Contour and surface plots of the numerical solution and error

The plots in the Fig. 4, Fig. 5, Fig. 6 and Fig. 7 represent the steady-state temperature distribution on a semi-circular plate with Dirichlet's boundary condition including errors. In the Fig. 4, we took 100 iterations. Here we observe that there is some error present. As we increase the number of iterations to 245 and 500 respectively, the error further decreases as shown in the Fig. 5 and Fig. 6. For the last Fig. 7, we took 1000 iterations and the temperature distribution on a semi-circular plate almost similar to analytical as shown in Fig. 3 and error tends to zero [3, 4, 10].

I. When numbers of iterations is 100:

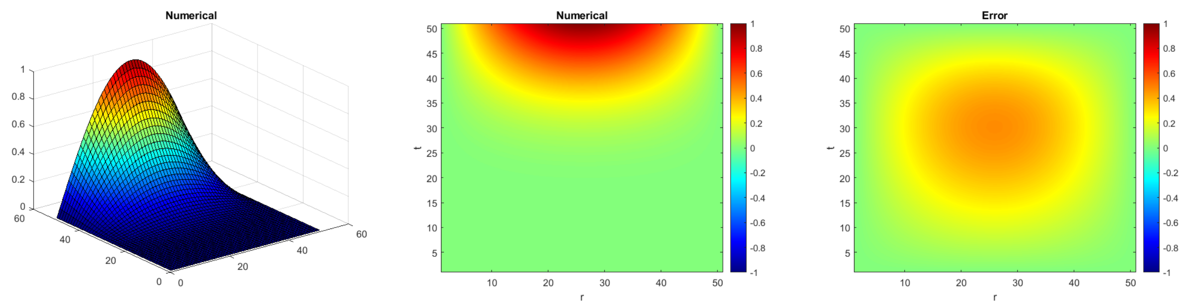

Figure 4: Temperature distribution on a semi-circular plate numerically with Error of Iteration 100.

II. When numbers of iterations is 245:

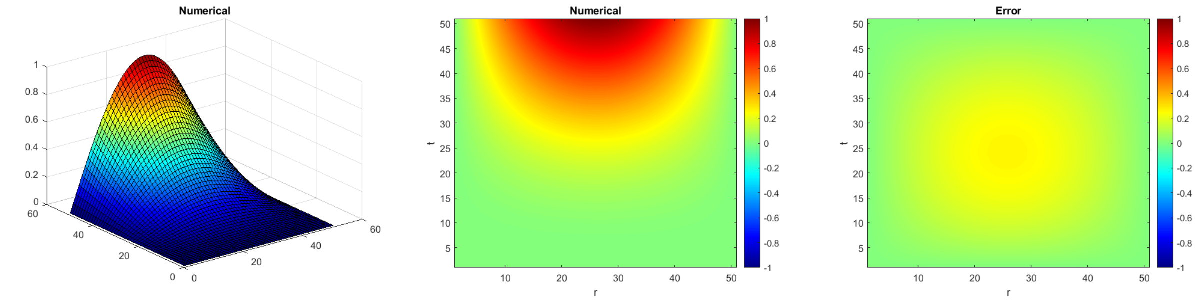

Figure 5: Temperature distribution on a semi-circular plate numerically with Error of Iteration 245.

III. When numbers of iterations is 500:

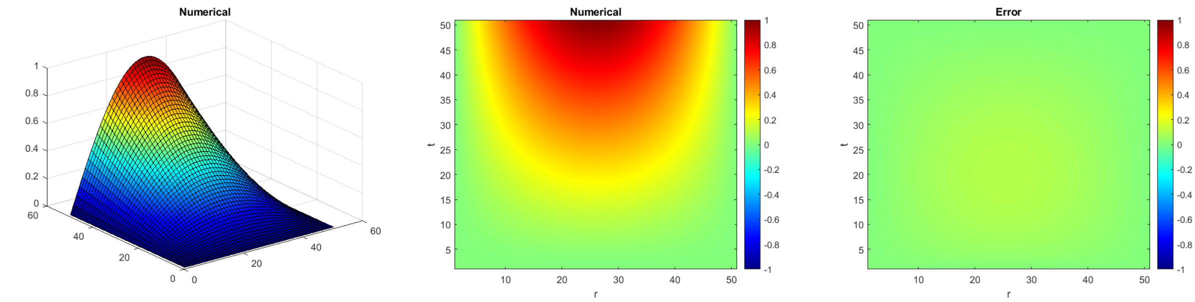

Figure 6: Temperature distribution on a semi-circular plate numerically with Error of Iteration 500.

IV. When numbers of iterations is 1000:

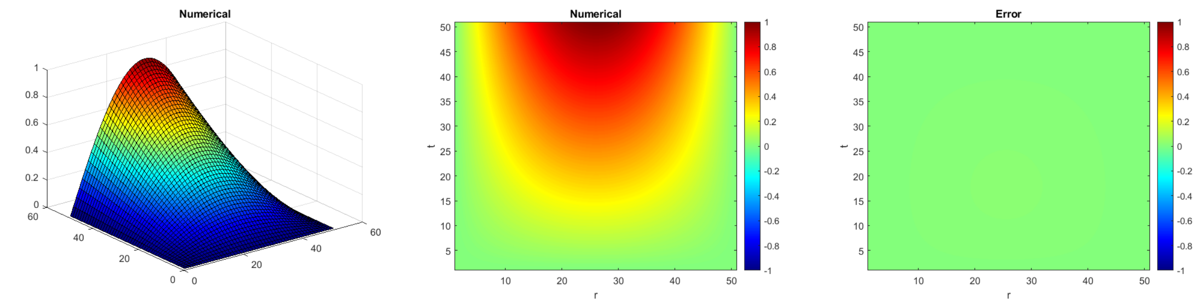

Figure 7: Temperature distribution on a semi-circular plate numerically with Error of Iteration 1000.

5 Conclusions

Steady-state temperature distribution inside a semi-circular disc plays the important role in the applied field. The analytic solution of Laplace equation in polar coordinate is not easy because of the use of various lengthy theory like variable separable method, Sturm-Liouville equation, Cauchy-Euler equation. Comparatively numerical method is both easy and short for the same problem. In this paper, we used Gauss-elimination algorithm using MATLAB software for the numerical solution of the problem of Laplace equation in polar coordinates inside the semi-circular disc with Dirichlet boundary conditions. We compare analytic solution with corresponding numerical solution then we found the error. If we increase the numbers of iterations the error decreased. At 1000 iteration, the error was found nearly tending to zero. Hence, if we increase the number of iteration, numerical solution of the given problem is more accurate. It observed that numerical solution by using finite difference schemes and Gauss elimination technique in MATLAB give near to the exact solution.

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