Higher Order Convergent Newton Type Iterative Methods

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Abstract: *Newton method is one of the most widely used numerical methods for solving nonlinear equations. McDougall and Wotherspoon [Appl. Math. Lett., 29 (2014), 20-25] modified this method in predictor-corrector form and get an order of convergence* $1+\sqrt{2}$. *In this paper, we use this modified Newton method on Ujević, Erceg and Lekić method [Appl. Math.Comput., 192(2007), 311-318] and obtain a new Newton type iterative method having order of convergence* $\frac{3+\sqrt{17}}{2} \approx 3.5615$. *We also derive a hybrid method combining our method and the standard secant method. The resulting method turns out to be of order of convergence* $2+2\sqrt{2} \approx 4.82$. *Finally numerical comparisons are implemented to demonstrate the performance of the developed methods.*

Keywords: Newton method, Secant method, Predictor-corrector method, Nonlinear equation, Order of convergence.

1 Introduction

Finding zeros of the single variable nonlinear equations efficiently is one of the interesting and most important problem in numerical analysis and has wide range of application in all fields of science and engineering. Most of the time, it is not possible to solve these equations analytically. Therefore iterative methods are employed to get approximate solutions of nonlinear equations. The best known and the most widely used among these type of methods for solving numerical solution of nonlinear equation $f(x) = 0$ is the classical Newton method [1] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

It converges quadratically for simple zero. In literature, large number of its modifications have been appeared each one claim to be better than the other in some or the other aspect (see [2]-[8]). McDougall and Wotherspoon [6] obtained a method with a slight modification in the standard Newton method and achieved order of convergence $1+\sqrt{2}$.

Their method is following:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0, \\ x_1 &= x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_0 - \frac{f(x_0)}{f'(x_0)}. \end{aligned}\right\} \tag{1.2}$$

Subsequently for $n \geq 1$, the iterations can be obtained as

$$\left.\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. \end{aligned}\right\} \tag{1.3}$$

One of the methods for solving nonlinear equations given by Ujević, Erceg and Lekić [7] is

$$\left.\begin{aligned} x_{n+1} &= x_n + (z_n - x_n)\frac{f(x_n)}{f(x_n) - f(z_n)}, \\ \text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned}\right\} \tag{1.4}$$

The order of convergence of above method is 3. In this paper, we modify this method by using modified Newton method given by McDougall and Wotherspoon instead of classical Newton method.

2 The Iterative Method and the Convergence

We suggest the following method as a variant of Ujević, Erceg and Lekić method:
If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_1 &= x_0 + \frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)}, \\ \text{where} \qquad z_0 &= x_0 - \frac{f(x_0)}{f'(\frac{x_0^*+x_0}{2})} = x_0 - \frac{f(x_0)}{f'(x_0)}. \end{aligned}\right\} \tag{2.1}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n + \frac{(z_n^* - x_n)f(x_n)}{f(x_n) - f(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_{n+1} &= x_n + \frac{(z_n - x_n)f(x_n)}{f(x_n) - f(z_n)}, \\ \text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. \end{aligned}\right\} \tag{2.2}$$

Below we prove the convergent result for the method (2.1)-(2.2).

Theorem 2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (2.1)-(2.2) is convergent with order of convergence $\frac{3+\sqrt{17}}{2} \approx 3.5615$.*

Proof. Let e_n and e_n^* denote respectively the errors in the terms x_n and x_n^*. Also, we denote $c_j = \frac{f(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. Then from (2.1) $x_0^* = x_0$ implies $e_0^* = e_0$. We now proceed to calculate the error e_1 in x_1. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned} z_0 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= \alpha + e_0 - \frac{f(\alpha + e_0)}{f'(\alpha + e_0)} \\ &= \alpha + c_2 e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4) \end{aligned}.$$

So that after some calculation, we get

$$z_0 - x_0 = -e_0 + c_2 e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4),$$

$$f(z_0) = f'(\alpha)[c_2 e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4)],$$

$$f(x_0) - f(z_0) = e_0 f'(\alpha)[1 + 2c_2^2 e_0^2 - c_3 e_0^2 + O(e_0^3)],$$

$$\text{and} \qquad \frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)} = -e_0 + c_2^2 e_0^3 + O(e_0^4).$$

Hence from (2.1),

$$\alpha + e_1 = \alpha + e_0 - e_0 + c_2^2 e_0^3 + O(e_0^4)$$

$$\therefore \qquad e_1 = ae_0^3, \tag{2.3}$$

where $a = c_2^2$ and we have neglected the higher power of e_n. Again,

$$x_1^* = x_1 + \frac{(z_1^* - x_1)f(x_1)}{f(x_1) - f(z_1^*)} \tag{2.4}$$

Here

$$z_1^* - x_1 = x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0 + x_0^*)]} - x_1 = -\frac{f(x_1)}{f'(x_0)},$$

so that

$$(z_1^* - x_1)f(x_1) = -\frac{[f(x_1)]^2}{f'(x_0)}.$$

Since

$$\begin{aligned} f(x_1) &= f(\alpha + e_1) \\ &= f'(\alpha)[e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)], \end{aligned}$$

therefore

$$\begin{aligned} \frac{[f(x_1)]^2}{f'(x_0)} &= \frac{[f(\alpha + e_1)]^2}{f'(\alpha + e_0)} \\ &= e_1f'(\alpha)[e_1 - 2c_2e_0e_1 + O(e_0^5). \end{aligned}$$

Also

$$\begin{aligned} f(z_1^*) &= f\left[x_n - \frac{f(x_n)}{f'(x_n)}\right] \\ &= f'(\alpha)[2c_2e_0e_1 + 3c_3{e_0}^2e_1 - 4c_2^2e_0^2e_1 + O(e_0^6)] \end{aligned}$$

so that

$$f(x_1) - f(z_1^*) = e_1f'(\alpha)[1 - (2c_2e_0 + 3c_3e_0^2 - 4c_2^2e_0^2) + O(e_0^3)]$$

and

$$(z_1^* - x_1)\frac{f(x_1)}{f(x_1) - f(z_1^*)} = -e_1 + (4c_2^2 - 3c_3)e_0^2e_1 + O(e_0^6) \tag{2.5}$$

Now, using 2.5, the error e_1^* in x_1^* in equation 2.4 can be calculated as

$$\begin{aligned} e_1^* &= e_1 + [-e_1 + (4c_2^2 - 3c_3)e_0^2e_1 + 0(e_0^6)] \\ &= (4c_2^2 - 3c_3)e_0^2e_1 \\ &= abe_0^5 \end{aligned} \tag{2.6}$$

where b $=4c_2^2 - 3c_3$ and we have neglected the higher power terms of e_0. Now, we compute the error e_2 in the term

$$x_2 = x_1 + (z_1 - x_1)\frac{f(x_1)}{f(x_1) - f(z_1)},$$

where

$$z_1 = x_1 - \frac{f(x_1)}{f'\left(\frac{x_1 + x_1^*}{2}\right)}.$$

Now

$$\begin{aligned} f'\left(\frac{x_1 + x_1^*)}{2}\right) &= f'(\alpha + \frac{e_1 + e_1^*}{2}) \\ &= f'(\alpha)(1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3) \end{aligned}$$

so that

$$\begin{aligned} \frac{f(x_1)}{f'\left(\frac{x_1 + x_1^*}{2}\right)} &= (e_1 + c_2e_1^2 + O(e_1^3))(1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3))^{-1} \\ &= e_1 + \frac{1}{4}c_3e_1^3 - c_2e_1e_1^* \end{aligned}$$

and therefore

$$z_1 = \alpha - \frac{1}{4}c_3e_1^3 + c_2e_1e_1^*$$

where the higher power terms are neglected.Thus

$$f(z_1) = f'(\alpha)[c_2e_1e_1^* + c_2^2e_1^2e_1^* - \frac{1}{4}c_3e_1^3]$$

and

$$f(x_1) - f(z_1) = e_1f'(\alpha)(1 + c_2e_1 + c_3e_1^2 - c_2e_1^* - c_2^2e_1e_1^* + \frac{5}{4}c_3e_1^2).$$

Also

$$(z_1 - x_1)f(x_1) = -\frac{[f(x_1)]^2}{f'[\frac{1}{2}(x_1 + x_1^*)]}$$

So that

$$(z_1 - x_1)f(x_1) = -e_1f'(\alpha)(e_1 + c_2e_1^3 - c_2e_1e_1^* + \frac{5}{4}c_3e_1^3)$$

Using above considerations, the error e_2 in x_2 is given by

$$\begin{aligned} e_2 &= -3c_2^2e_1^2e_1^* + c_2^2e_1(e_1^*)^2 \\ &= -3c_2^2e_1^2e_1^* \\ &= ce_1^2e_1^* \end{aligned}$$

where $c = -3c_2^2$. In fact it can be workout for $n \geq 1$, the following relation holds:

$$e_{n+1} = ce_n^2e_n^* \tag{2.7}$$

In order to compute e_{n+1} explicitly, we need e_n^*. We already find e_1^*. We now compute e_2^*. We have

$$x_2^* = x_2 + (z_2^* - x_2)\frac{f(x_2)}{f(x_2) - f(z_2^*)},$$

$$\text{where} \qquad z_2^* = x_2 - \frac{f(x_2)}{f'(\frac{(x_1+x_1^*)}{2})}$$

Similar as above, it can be calculated the error e_2^* is given by

$$e_2^* = de_1^2e_2$$

where $d = \frac{4}{3}c_3$ and, again, it can be checked that in general for $n \geq 2$, the following relation holds:

$$e_n^* = de_{n-1}^2e_n \tag{2.8}$$

In the view of (2.7) and (2.8), the error at each stage in x_n^* and x_{n+1} are calculated which are tabulated below:

n	e_n	e_n^*
0	e_0	e_0
1	ae_0^3	abe_0^5
2	$a^3bce_0^{11}$	$a^5bcde_0^{17}$
3	$a^{11}b^3c^4de_0^{39}$	$a^{17}b^5c^6d^2e_0^{61}$
4	$a^{39}b^{11}c^{15}d^4e_0^{139}$	$a^{61}b^{17}c^{23}d^7e_0^{217}$
5	$a^{139}b^{39}c^{44}d^{15}e_0^{495}$	...
$\vdots$	$\vdots$	$\vdots$

It is observed that the powers of e_0 in the errors at each iterate from a sequence

$$3,\ 11,\ 39,\ 139,\ 495,\ 1763,\ 6279,\ 22363,\ ... \tag{2.9}$$

and the sequence of their successive ratios is

$$\frac{11}{3},\ \frac{39}{11},\ \frac{139}{39},\ \frac{495}{139},\ \frac{1763}{495},\ \frac{6279}{1763},\ \frac{22363}{6279},\ ...$$

or,

$$3.67,\ 3.5454,\ 3.5641,\ 3.5611,\ 3.5616,\ 3.5615,\ 3.5615, ...$$

This sequence seems to converge the number 3.5615 approximately. The numbers α_i in the sequence (2.9) are related by the relation

$$\alpha_i = 3\alpha_{i-1} + 2\alpha_{i-2},\ i = 2, 3, 4... \tag{2.10}$$

If we set the limit

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{\alpha_{i-1}}{\alpha_{i-2}} = R,$$

Then dividing (3.1) by α_{i-1}, we obtain

$$R^2 - 3R - 2 = 0,$$

which has its positive root as $R = \frac{3+\sqrt{17}}{2} \approx 3.5615$.
Hence we conclude that the order of convergence of method is at least 3.5615. □

3 Method with Faster Convergence

In this section, we obtain a new method by combining the iteration of method (2.1)-(2.2) with secant method and show that order of convergence of resulting method is increased by more than one. Precisely, we purpose the following method:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0 + \frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)}, \\ \text{where} \qquad z_0 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}). \end{aligned}\right\} \tag{3.1}$$

Subsequently, for $n \geq 1$, the iteration can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n + \frac{(z_n^* - x_n)f(x_n)}{f(x_n) - f(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n - \frac{(z_n - x_n)f(x_n)}{f(x_n) - f(z_n)}, \\ \text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]} \\ x_{n+1} &= x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \end{aligned}\right\} \tag{3.2}$$

For the convergence of this method, we prove the following:

Theorem 3.1. *Let f be a function having sufficient number of smooth derivatives in a neighborhood of α which is a simple root of the equation $f(x) = 0$. Then method (3.1)-(3.2) to approximate the root α is convergent with order of convergence $2 + 2\sqrt{2} \approx 4.828$.*

Proof. We prove this theorem on the line of the proof of Theorem 2.1 and error equation of standard secant method . In particular, the errors e_0^*, e_0^{**} and e_1 respectively in x_0^*, x_0^{**} and x_1 in equation (3.1) are given by

$$e_0^* = e_0$$
$$e_0^{**} = ae_0^3, \qquad \text{where} \qquad a = c_2^2$$
$$e_1 = c_2 e_0^* e_0^{**} = \lambda a e_0^4, \qquad \text{where} \qquad \lambda = c_2$$

Also the errors e_1^* in x_1^* in equation(3.2) is given by

$$e_1^* = \lambda a b e_0^6, \qquad \text{where} \qquad b = 4c_2^2 - 3c_3$$

and the error e_1^{**} in x_1^{**} in equation (3.2)is given by

$$e_0^{**} = c e_1^2 e_1^* = c\lambda^3 a^3 b e_0^{14}, \qquad \text{where} \qquad c = -3c_2^2$$

In fact, it can be workout that for $n \geq 1$, the following relation holds:

$$e_n^{**} = c e_n^2 e_n^* \tag{3.3}$$

In order to compute e_n^{**} explicitly, we need to compute e_n and e_n^*. We have already computed e_1 and e_1^*. From the proof of Theorem 2.1,

$$e_2^* = d e_1^2 e_2$$

where $d = \frac{4}{3}c_3$ and again it can be verified that following relation holds:

$$e_n^* = d e_{n-1}^2 e_n \tag{3.4}$$

Also from 3.1, it can be shown that

$$e_2 = \lambda e_1^* e_1^{**} = \lambda^5 a^4 b^2 c e_0^{20}$$

Thus, for $n \geq 1$, it can be shown that the error e_{n+1} in x_{n+1} in the method (3.1)-(3.2) satisfy the following recursion formula

$$e_{n+1} = \lambda e_n^* e_n^{**} \tag{3.5}$$

Using the above information, the error at each stage in x_n^*, x_n^{**}, and x_n are obtained and calculated as follows.

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	ae_0^3
1	$\lambda a e_0^4$	$\lambda a b e_0^6$	$\lambda^3 a^3 b e_0^{14}$
2	$\lambda^5 a^4 b^2 c e_0^{20}$	$\lambda^7 a^6 b^2 c d e_0^{28}$	$\lambda^{17} a^{14} b^6 c^4 d e_0^{68}$
3	$\lambda^{25} a^{20} b^8 c^5 d^2 e_0^{96}$	$\lambda^{35} a^{28} b^{12} c^7 d^3 e_0^{136}$	$\lambda^{75} a^{68} b^{28} c^{18} d^7 e_0^{328}$
4	$\lambda^{111} a^{96} b^{40} c^{25} d^{10} e_0^{464}$	$\lambda^{161} a^{136} b^{56} c^{35} d^{15} e_0^{656}$	$\lambda^{383} a^{328} b^{136} c^{86} d^{25} e_0^{1584}$
5	$\lambda^{545} a^{464} b^{192} c^{121} d^{40} e_0^{2240}$	...	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We construct the analysis of the table as done in [6]. Note that Powers of e_0 in the error at each iterate form the sequence

$$4,\ 20,\ 96,\ 464,\ 2240,\ \ldots \tag{3.6}$$

and sequence of their successive ratios is

$$\frac{20}{4}, \frac{96}{20}, \frac{464}{96}, \frac{2240}{464}, \ldots$$

or,

$$5,\ 4.8,\ 4.84,\ 4.82,\ ...$$

If the terms of the sequence(3.6) are denoted by α_i, then it can be seen that

$$\alpha_i = 4\alpha_{i-1} + 4\alpha_{i-2}$$

Thus as in Theorem 2.1, the rate of convergence of method (3.1)-(3.2) is at least $2 + 2\sqrt{2} \approx 4.828$. □

4 Numerical Examples

In order to check the performance of the newly introduced method (2.1)-(2.2), the test functions and their roots α which are used as numerical examples are as follows:
(i) $f_1 = (x-1)^8 - 1$, $\alpha = 2$
(ii) $f_2 = \sin^2 x - x^2 + 1$, $\alpha = 1.40449164821534$
(iii) $f_3 = \cos x - xe^x + x^2$, $\alpha = 0.639154069332008$

Numerical computations have been performed using Matlab software and stopping criteria $|x_{n+1} - x_n| < (10)^{12}$ and $|f(x_n)| < (10)^{14}$. We also compare the result of this method with Newton method and Ujević, Erceg and Lekić (UEL)method

Table 1: $f_1 = (x-1)^8 - 1$ and initial guess $x_0 = 3$

n	Newton Method	UEL method	Present method (2.1)-(2.2))
1	2.750976562500000	2.621212292220119	2.621212292220119
2	2.534581615819526	2.321482528817460	2.240259790619724
3	2.348995976046720	2.106434089229419	2.029715679791304
4	2.195747198046065	2.009090545951117	2.000022929984292
5	2.082041836760382	2.000008831906093	2.000000000000000
6	2.018764916659598	2.000000000000008	
7	2.001166173395949	2.000000000000000	
8	2.000004743257317		
9	2.000000000078744		
10	2.000000000000000		

Table 2: $f_2 = \sin^2 x - x^2 + 1$ and initial guess $x_0 = 1$

n	Newton Method	UEL method	Present method (2.1)-(2.2)
1	1.649190196932272	1.320546154049013	1.320546154049013
2	1.439042347687187	1.404061768716632	1.404460568207670
3	1.405385086160459	1.404491648166524	1.404491648215341
4	1.404492272936243	1.404491648215341	
5	1.404491648215647		
6	1.404491648215341		

Table 3: $f_3 = \cos x - xe^x + x^2$ and initial guess $x_0 = 1$

n	Newton Method	UEL method	Present method (2.1)-(2.2)
1	0.724644697567095	0.660764858475215	0.660764858475215
2	0.644658904870270	0.639160213376992	0.639154122061457
3	0.639177807467281	0.639154096332008	0.639154096332008
4	0.639154096773051		
5	0.639154096332008		

Again let us take the same test function $f_3 = \cos x - xe^x + x^2$ to check the performance of the method(3.1)-(3.2). The comparison table is given below.

Table 4: $f_3 = \cos x - xe^x + x^2$ and initial guess $x_0 = 1$

n	Newton Method	UEL method	Present method (2.1)-(2.2)	Present method (3.1)-(3.2)
1	0.724644697567095	0.660764858475215	0.660764858475215	0.644691946674196
2	0.644658904870270	0.639160213376992	0.639154122061457	0.639154096332009
3	0.639177807467281	0.639154096332008	0.639154096332008	
4	0.639154096773051			
5	0.639154096332008			

5 Conclusion

In this paper, we have obtained two new higher order Newton type iterative methods for solving nonlinear equations. The method (2.1)-(2.2) needs one more function evaluation than Ujević, Erceg and Lekić method and two more functions evaluation than Newton method. However numerical examples are showed that this method is easily compete with cited methods. Also we derived new hybrid method (3.1)-(3.2) by combining method (2.1)-(2.2) with secant method. It is shown that resulting method is of order 4.828 and the computational cost is comparable with that of the methods cited in the table.

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