Absolute $|A,\delta|_k$ and $|A,\gamma;\delta|_k$ Summability for n-tupled Triangle Matrices

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Abstract: *In this study, new sequence spaces $(\mathcal{A}_k,\delta)$ & $(\mathcal{A}_k,\gamma;\delta)$ have been introduced to establish two theorems on minimal set of the sufficient conditions for a n-tupled triangle T to be a bounded operator on sequence spaces $(\mathcal{A}_k^n,\delta)$ & $(\mathcal{A}_k^n,\gamma;\delta)$. Generalized summability method $|A,\delta|_k$ & $|A,\gamma;\delta|_k$ have been applied for determining the sufficient conditions, where $k\geq 1$, $\delta\geq 0$ and γ is real number. Further, a set of new and well-known applications has been deduced from the main result under suitable conditions, which shows the importance of the main result.*

Keywords: Absolute Summability, Boundness of Matrix, Infinite Series, Sequence Space
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1 Introduction

Let $\sum a_n$ be a given infinite series such that

$$s_k = a_0 + a_1 + a_2 + \cdots + a_k = \sum_{l=0}^{k} a_l,$$

where s_k denotes the k^{th} partial sum of the series $\sum a_n$ and $\{s_n\}$ defines the sequence of partial sums. The n^{th} term of sequence-to-sequence transformation of $\{s_n\}$ is defined by

$$t_n = \sum_{k=0}^{\infty} t_{nk}s_k = \sum_{k=0}^{\infty} t_{n,n-k}s_{n-k}.$$

The sequence $\{t_n\}$ of the matrix means of sequence $\{s_n\}$ is generated by the sequence of coefficients $\{t_{nk}\}$. The series $\sum a_n$ is said to be summable to the sum S by matrix mean if $lim_{n\to\infty}t_n$ exists and equal to S [12], then we can write,

$$t_n \longrightarrow S(T), \text{ as } n \longrightarrow \infty.$$

A sequence of the partial sums $\{s_n\}$ is said to be of bounded variation if the series $|s_1-s_0|+|s_2-s_1|+\cdots+|s_n-s_{n-1}|$ converges, i.e.,

$$\sum_n |\bigtriangleup s_n| < \infty.$$

The sequence $\{s_n\}$ is absolute summable by the method A (A-summable) to the limit s if $\lim_{n\to\infty} t_n = s$ and the sequence $\{t_n\}$ is of bounded variation:

$$\sum_{n=1}^{\infty} |\, t_n - t_{n-1}| < \infty.$$

For the infinite series $\sum a_n$, if $\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k$ converges, then $\sum_{n=0}^{\infty} a_n \in |A|_k$, i.e., the series is absolutely $|A|_k$-summable of degree $k\geq 1$. Liu [8] studied the absolute Cesàro summability of the Fourier series. He worked on unsolved problem given by Pati [9]. Das [5] defined the concept of absolute conservation by transforming the sequence $\{s_n\}$ into $\{t_n\}$. Let T be such sequence to sequence transformation, whenever $\{s_n\}$ converges absolutely, $\{t_n\}$ converges absolutely, then T is called absolutely conservative and if the absolute convergence of $\{s_n\}$ implies absolute convergence of $\{t_n\}$ to the same limit, T is called absolutely regular. For some given $k \geq 1$, if $T \in B(\mathcal{A}_k)$; i.e., if $\{s_0, s_1, \cdots, s_n\}$ satisfying

$$\sum_{n=1}^{\infty} n^{k-1}|s_n - s_{n-1}|^k < \infty, \tag{1}$$

implies $\sum\limits_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty.$

Then, T is said to be absolutely k^{th} power conservative. Note that when $k > 1$, (1) does not necessarily imply the convergence of $\{s_n\}$. Hirokawa [7] also worked on conservative property of summability method. He found the relation between two summability methods and presented the condition for a summability method to be absolutely conservative. There exists a sequence space $\mathcal{A}_k$ which is given by

$$\mathcal{A}_k = \Big\{\{s_n\} : \sum_{n=1}^{\infty} n^{k-1}|a_n|^k < \infty, a_n = s_n - s_{n-1}\Big\}.$$

Flett [6] considered a further extension of absolute summability in which he introduced a parameter δ. The series $\sum a_n$ is said to be $|A, \delta|_k$-summable, $k \geq 1$, $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|t_n - t_{n-1}|^k < \infty.$$

A series $\sum a_n$ is $|A, \gamma; \delta|_k$ summable [11], where $k \geq 1, \delta \geq 0$ and γ is real number, if

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)}|t_n - t_{n-1}|^k < \infty.$$

Bor [1, 2, 3, 4] developed some theorems on absolute summability factor. In [1], he established a theorem for infinite series with the help of absolute Cesáro summability. In [3], he obtained the result for infinite series by using the absolute summability of index k which is generalization of result [2]. After this, he used more generalized summability method $|C, \alpha, \gamma; \beta|_k$ for infinite series [4]. Based on the concept of Das [5] for absolute conservation, we can say that for some given $k \geq 1$ and $\delta \geq 0$, if $T \in B(\mathcal{A}_k, \delta)$, i.e., if $\{s_0, s_1, \cdots, s_n\}$ satisfy

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|s_n - s_{n-1}|^k < \infty, \tag{2}$$

implies $\sum\limits_{n=1}^{\infty} n^{\delta k+k-1}|t_n - t_{n-1}|^k < \infty.$

Then, T is said to be absolutely (k, δ) conservative. The sequence space $(\mathcal{A}_k, \delta)$ is given by

$$(\mathcal{A}_k, \delta) = \Big\{\{s_n\} : \sum_{n=1}^{\infty} n^{\delta k+k-1}|a_n|^k < \infty, a_n = s_n - s_{n-1}\Big\}.$$

A matrix is called a bounded linear operator on $(\mathcal{A}_k, \delta)$, i.e., $T \in B(\mathcal{A}_k, \delta)$, if

$$T : (\mathcal{A}_k, \delta) \rightarrow (\mathcal{A}_k, \delta).$$

If $T \in B(\mathcal{A}_k, \gamma; \delta)$; i.e., if $\{s_0, s_1, \cdots, s_n\}$ satisfy

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)}|s_n - s_{n-1}|^k < \infty, \tag{3}$$

implies $$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)}|t_n - t_{n-1}|^k < \infty.$$

Then, T is said to be absolutely $(k,\gamma;\delta)$ conservative. The sequence space $(\mathcal{A}_k,\gamma;\delta)$ is given by

$$(\mathcal{A}_k,\gamma;\delta) = \Big\{\{s_n\} : \sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)}|a_n|^k < \infty, a_n = s_n - s_{n-1}\Big\}.$$

A matrix is called a bounded linear operator on $(\mathcal{A}_k,\gamma;\delta)$, i.e., $T \in B(\mathcal{A}_k,\gamma;\delta)$, if

$$T : (\mathcal{A}_k,\gamma;\delta) \to (\mathcal{A}_k,\gamma;\delta).$$

Notation: Let T be the infinite matrix for the series $\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty} a_{N_1,N_2,\cdots,N_n}$ and there exists two infinite matrices $\bar{T}$ and $\hat{T}$ with T as follows:

$$\bar{t}^{i_1,i_2,\cdots,i_n}_{N_1,N_2,\cdots,N_n} = \sum_{\mu_1=i_1}^{N_1}\sum_{\mu_2=i_2}^{N_2}\cdots\sum_{\mu_n=i_n}^{N_n} t^{\mu_1,\mu_2,\cdots,\mu_n}_{N_1,N_2,\cdots,N_n}$$

$$\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1} = \triangle_{11\cdots n\,times}\bar{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}$$
$$N_1,N_2,\cdots,N_n,i_1,i_2,\cdots,i_n = 0,1,2,\cdots$$

2 Known results

A triangle $C = [c_{ij}]$ is defined as a lower triangle matrix such that,

$$C = \begin{cases} c_{ij} \neq 0, & i \geq j \\ c_{ij} = 0, & i < j \end{cases}$$

Savaş and Şevli [10] obtained the following result for boundness of the double triangle on $\mathcal{A}_k$.

Theorem 2.1. *Let $T = \Big(t_{mnij}\Big)$ be a double triangle satisfying*

(i) $\sum_{i=0}^{m}\sum_{j=0}^{n}\Big|t_{ijij}\Big|\Big|\hat{t}_{m-1,n-1,i,j}\Big| = O\Big(\Big|t_{mnmn}\Big|\Big)$ *and*

(ii) $\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\Big(mn\Big|t_{mnmn}\Big|\Big)^{k-1}\Big|\hat{t}_{m-1,n-1,i,j}\Big| = O\Big(ij\Big|t_{ijij}\Big|\Big)^{k-1}$.

Then, $T \in B\big(\mathcal{A}_k^2\big), k \geq 1$.

3 Main results

The following result has been presented for the boundness of the operator using summability $|A,\delta|_k, k \geq 1$ and $\delta \geq 0$.

Theorem 3.1. *Let $T = \Big(t^{i_1,i_2,\cdots,i_n}_{N_1,N_2,\cdots,N_n}\Big)$ be n-tupled triangle satisfying*

(i) $\Bigg(\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|\Bigg)^{k-1} = O\Big(\Big|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\Big|\Big)^{\delta k+k-1}$ *and*

(ii) $$\sum_{N_1=i_1}^{\infty}\sum_{N_2=i_2}^{\infty}\cdots\sum_{N_n=i_n}^{\infty}\Big(N_1N_2\cdots N_n\Big|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\Big|\Big)^{\delta k+k-1}\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|$$
$$= O\Bigg(\Big(i_1i_2\cdots i_n\Big)^{\delta k+k-1}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{k-1}\Bigg).$$

Then, $T \in B\big(\mathcal{A}_k^n,\delta\big), k \geq 1$ and $\delta \geq 0$.

The following Theorem 3.2 has been developed for the boundness of the operator using the more general summability $|A,\gamma;\delta|_k$, $k\geq 1$, $\delta\geq 0$ and γ is a real number.

Theorem 3.2. *Let $T=\left(t^{i_1,i_2,\cdots,i_n}_{N_1,N_2,\cdots,N_n}\right)$ be n-tupled triangle satisfying*

(i) $\left(\sum\limits_{i_1=0}^{N_1}\sum\limits_{i_2=0}^{N_2}\cdots\sum\limits_{i_n=0}^{N_n}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|\right)^{k-1}=O\left(\left|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\right|\right)^{\gamma(\delta k+k-1)}$ *and*

(ii) $$\sum_{N_1=i_1}^{\infty}\sum_{N_2=i_2}^{\infty}\cdots\sum_{N_n=i_n}^{\infty}\left(N_1N_2\cdots N_n\left|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\right|\right)^{\gamma(\delta k+k-1)}\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|$$
$$=O\left(\left(i_1i_2\cdots i_n\right)^{\gamma(\delta k+k-1)}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{k-1}\right).$$

Then, $T\in B\left(\mathcal{A}^n_k,\gamma;\delta\right)$ for $k\geq 1,\delta\geq 0$ and γ is a real number.

4 Proof of the Theorems

If $A_{N_1,N_2,\cdots,N_n}$ denotes the $N_1N_2\cdots N_n$-term of the T-transform of a sequence $\left\{s_{N_1,N_2,\cdots,N_n}\right\}$, then

$$\begin{aligned}
A_{N_1,N_2,\cdots N_n} &= \sum_{\mu_1=0}^{N_1}\sum_{\mu_2=0}^{N_2}\cdots\sum_{\mu_n=0}^{N_n} t^{\mu_1,\mu_2,\cdots\mu_n}_{N_1,N_2,\cdots N_n}s_{\mu_1,\mu_2,\cdots\mu_n}\\
&= \sum_{\mu_1=0}^{N_1}\sum_{\mu_2=0}^{N_2}\cdots\sum_{\mu_n=0}^{N_n} t^{\mu_1,\mu_2,\cdots,\mu_n}_{N_1,N_2,\cdots,N_n}\sum_{i_1=0}^{\mu_1}\sum_{i_2=0}^{\mu_2}\cdots\sum_{i_n=0}^{\mu_n}a_{i_1,i_2,\cdots,i_n}\\
&= \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}a_{i_1,i_2,\cdots,i_n}\sum_{\mu_1=i_1}^{N_1}\sum_{\mu_2=i_2}^{N_2}\cdots\sum_{\mu_n=i_n}^{N_n}t^{\mu_1,\mu_2,\cdots,\mu_n}_{N_1,N_2,\cdots,N_n}\\
&= \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}a_{i_1,i_2,\cdots,i_n}\bar{t}^{i_1,i_2,\cdots,i_n}_{N_1,N_2,\cdots,N_n}.
\end{aligned}$$

Then follows

$$\begin{aligned}
\widetilde{A}_{N_1,N_2,\cdots,N_n} &= \triangle_{11\cdots n\text{ times}}A_{N_1-1,N_2-1,\cdots,N_n-1}\\
&= A_{N_1-1,N_2-1,\cdots,N_n-1}-\Big(A_{N_1,N_2-1,\cdots,N_n-1}+\cdots+A_{N_1-1,N_2-1,\cdots,N_{n-1}-1,N_n}\Big)\\
&\quad+\Big(A_{N_1,N_2,N_3-1,\cdots,N_n-1}+A_{N_1-1,N_2,N_3,N_4-1,\cdots,N_n-1}+\cdots\Big)\\
&\quad+\cdots+(-1)^nA_{N_1,N_2,\cdots,N_n}\\
&= \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\Big[\bar{t}_{N_1-1,N_2-1,\cdots,N_n-1}-\Big(\bar{t}_{N_1,N_2-1,\cdots,N_n-1}+\cdots+\bar{t}_{N_1-1,\cdots,N_{n-1}-1,N_n}\Big)\\
&\quad+\Big(\bar{t}_{N_1,N_2,N_3-1,\cdots,N_n-1}+\cdots+\bar{t}_{N_1,N_2-1,N_3-1,N_4-1,\cdots,N_{n-1}-1,N_n}\Big)\\
&\quad+\cdots+(-1)^n\bar{t}_{N_1,N_2,\cdots,N_n}\Big]a_{i_1,i_2,\cdots,i_n}\\
&= \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\left(\triangle_{11\cdots n\ times}\,\bar{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}a_{i_1,i_2,\cdots,i_n}\right)\\
&= \sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}a_{i_1,i_2,\cdots,i_n}.
\end{aligned}$$

Proof of the Theorem 3.1: For $|A,\delta|_k$ summable,

$$\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\delta k+k-1}\left|\widetilde{A}_{N_1,N_2,\cdots,N_n}\right|^k=O(1).$$

Using Hölder's inequality and condition (i) of Theorem 3.1, we get

$$\begin{aligned}
&\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\delta k+k-1}\left|\widetilde{A}_{N_1,N_2,\cdots,N_n}\right|^k\\
&=\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\delta k+k-1}\left|\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}a_{i_1,i_2,\cdots,i_n}\right|^k\\
&\leq\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2...N_n\right)^{\delta k+k-1}\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|\\
&\times\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{1-k}\left|a_{i_1,i_2,\cdots,i_n}\right|^k\left(\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|\right)^{k-1}\\
&=O(1)\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2...N_n\left|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\right|\right)^{\delta k+k-1}\\
&\times\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{1-k}\left|a_{i_1,i_2,\cdots,i_n}\right|^k\\
&=O(1)\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{1-k}\left|a_{i_1,i_2,\cdots,i_n}\right|^k\\
&\times\sum_{N_1=i_1}^{\infty}\sum_{N_2=i_2}^{\infty}\cdots\sum_{N_n=i_n}^{\infty}\left(N_1N_2\cdots N_n\left|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\right|\right)^{\delta k+k-1}\left|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\right|
\end{aligned}$$

Using condition (ii) of Theorem 3.1

$$\begin{aligned}
&=O(1)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{1-k}\left|a_{i_1,i_2,\cdots,i_n}\right|^k\left(i_1i_2\cdots i_n\right)^{\delta k+k-1}\left|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\right|^{k-1}\\
&=O(1)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\left(i_1i_2\cdots i_n\right)^{\delta k+k-1}\left|a_{i_1,i_2,\cdots,i_n}\right|^k\\
&=O(1)
\end{aligned}$$

Since $\left(s_{N_1,N_2,\cdots,N_n}\right)\in\left(\mathcal{A}^n_k,\delta\right)$.
This completes the proof.

Proof of the theorem 3.2: For $|A,\gamma;\delta|_k$ summable,

$$\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\gamma(\delta k+k-1)}\left|\widetilde{A}_{N_1,N_2,\cdots,N_n}\right|^k=O(1).$$

Using Hölder's inequality and condition (i) of Theorem 3.2, we get

$$\begin{aligned}
&\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\gamma(\delta k+k-1)}\left|\widetilde{A}_{N_1,N_2,\cdots,N_n}\right|^k\\
&=\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\left(N_1N_2\cdots N_n\right)^{\gamma(\delta k+k-1)}\left|\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,\cdots,N_n-1}a_{i_1,i_2,\cdots,i_n}\right|^k
\end{aligned}$$

$$\leq \sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\Big(N_1N_2\cdots N_n\Big)^{\gamma(\delta k+k-1)}\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|$$
$$\times\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{1-k}\Big|a_{i_1,i_2,\cdots,i_n}\Big|^k\left(\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|\right)^{k-1}$$
$$= O(1)\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\Big(N_1N_2\cdots N_n\Big|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\Big|\Big)^{\gamma(\delta k+k-1)}$$
$$\times\sum_{i_1=0}^{N_1}\sum_{i_2=0}^{N_2}\cdots\sum_{i_n=0}^{N_n}\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{1-k}\Big|a_{i_1,i_2,\cdots,i_n}\Big|^k$$
$$= O(1)\sum_{N_1=1}^{\infty}\sum_{N_2=1}^{\infty}\cdots\sum_{N_n=1}^{\infty}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{1-k}\Big|a_{i_1,i_2,\cdots,i_n}\Big|^k$$
$$\times\sum_{N_1=i_1}^{\infty}\sum_{N_2=i_2}^{\infty}\cdots\sum_{N_n=i_n}^{\infty}\Big(N_1N_2\cdots N_n\Big|t^{N_1,N_2,\cdots,N_n}_{N_1,N_2,\cdots,N_n}\Big|\Big)^{\gamma(\delta k+k-1)}\Big|\hat{t}^{i_1,i_2,\cdots,i_n}_{N_1-1,N_2-1,\cdots,N_n-1}\Big|$$

Using condition (ii) of theorem 3.2

$$= O(1)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{1-k}\Big|a_{i_1,i_2,\cdots,i_n}\Big|^k\Big(i_1i_2\cdots i_n\Big)^{\gamma(\delta k+k-1)}\Big|t^{i_1,i_2,\cdots,i_n}_{i_1,i_2,\cdots,i_n}\Big|^{k-1}$$
$$= O(1)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}\Big(i_1i_2\cdots i_n\Big)^{\gamma(\delta k+k-1)}\Big|a_{i_1,i_2,\cdots,i_n}\Big|^k$$
$$= O(1)$$

Since $\Big(s_{N_1,N_2,\cdots,N_n}\Big)\in\Big(\mathcal{A}^n_k,\gamma;\delta\Big)$.
This completes the proof.

5 Corollary

Corollary 5.1. *If $T\in(t_{nv})$ be a triangle satisfying*

(i) $\left(\sum\limits_{v=0}^{n}|t_{vv}||\hat{t}_{n-1,v}|\right)^{k-1} = O(|t_{nn}|)^{\delta k+k-1}$,

(ii) $\sum\limits_{n=v}^{\infty}(n|t_{nn}|)^{\delta k+k-1}|\hat{t}_{n-1,v}| = O\Big(v^{\delta k+k-1}|t_{vv}|^{k-1}\Big)$.

Then, $T\in B(\mathcal{A}_k;\delta), k\geq 1$ and $\delta\geq 0$.

Proof: We can obtain the above corollary from Theorem 3.2 of the main result. For the one-dimensional problem and $|A|_k$ summability, take $n=1$ and $\gamma=1$.
Let $T\in B(\mathcal{A}_k), k\geq 1$, then we will get the sufficient conditions with the help of conditions (i) and (ii) of both the Theorems of main results,

$$\left(\sum_{i_1=0}^{N_1}|t^{i_1}_{i_1}||\hat{t}^{i_1}_{N_1-1}|\right)^{k-1} = O\Big(|t^{N_1}_{N_1}|\Big)^{\delta k+k-1}$$

and

$$\sum_{N_1=i_1}^{\infty}(N_1|t^{N_1}_{N_1}|)^{\delta k+k-1}|\hat{t}^{i_1}_{N_1-1}| = O\Big(i_1^{\delta k+k-1}|t^{i_1}_{i_1}|^{k-1}\Big),$$

where $t^{N_1}_{N_1}=t_{N_1N_1}$. Hence this completes the proof.

Corollary 5.2. *If $T \in (t_{nv})$ be a triangle satisfying*

(i) $\sum\limits_{v=0}^{n} |t_{vv}||\hat{t}_{n-1,v}| = O(|t_{nn}|)$,

(ii) $\sum\limits_{n=v}^{\infty} (n|t_{nn}|)^{k-1}|\hat{t}_{n-1,v}| = O(v|t_{vv}|)^{k-1}$.

Then, $T \in B(\mathcal{A}_k), k \geq 1$.

Proof: We can obtain the above corollary from both Theorems of the main result. For the one-dimensional problem and $|A|_k$ summability, take $n = 1$; $\delta = 0$ in the theorem 3.1 and $n = 1$; $\gamma = 1$; $\delta = 0$ in the Theorem 3.2 of main result.

Let $T \in B(\mathcal{A}_k), k \geq 1$, then we will get the sufficient conditions with the help of conditions (i) and (ii) of both the theorems of main results,

$$\sum_{i_1=0}^{N_1} |t_{i_1}^{i_1}||\hat{t}_{N_1-1}^{i_1}| = O(|t_{N_1}^{N_1}|)$$

and

$$\sum_{N_1=i_1}^{\infty} (N_1|t_{N_1}^{N_1}|)^{k-1}|\hat{t}_{N_1-1}^{i_1}| = O(i_1|t_{i_1}^{i_1}|)^{k-1},$$

where $t_{N_1}^{N_1} = t_{N_1N_1}$. Hence this completes the proof.

6 Conclusion

The main result of this study is an attempt to formulate the problem of absolute summability factor of infinite series to develop a much efficient filter. Through the investigation, we concluded that under certain sufficient conditions, a n-tupled triangle T is a bounded operator on sequence spaces $(\mathcal{A}_k^n, \delta)$ & $(\mathcal{A}_k^n, \gamma; \delta)$ by applying $|A, \delta|_k$ & $|A, \gamma; \delta|_k$ summability method. This study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nut shell, the absolute summability methods are a motivation for the engineers and researchers working in the area of filters for signal processing.

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