Inner Finite Order Automorphisms of Type A

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Abstract: *Let $\mathfrak{g}$ be a complex simple Lie algebra, and let* $\mathrm{aut}(\mathfrak{g})$ *be the group of all automorphisms on $\mathfrak{g}$. The finite order members of* $\mathrm{aut}(\mathfrak{g})$ *have been classified by Kac, up to conjugation by $\mathfrak{g}$-automorphisms. In other words, this classification does not distinguish two finite order $\mathfrak{g}$-automorphisms σ, τ if there exists $u \in \mathrm{aut}(\mathfrak{g})$ such that $\sigma = u^{-1}\tau u$. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and σ, τ inner, we provide a finer classification up to conjugation by inner $\mathfrak{g}$-automorphisms, namely we do not distinguish σ, τ only if u is an inner automorphism.*

Keywords: complex simple Lie algebras, inner automorphisms.

1 Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $\mathrm{aut}(\mathfrak{g})$ denote the $\mathfrak{g}$-automorphisms. The finite order $\mathfrak{g}$-automorphisms play important roles in algebra, such as the construction of Kac-Moody algebras. For this reason, they are well-studied and classified by Kac [4], which prompts generalization in the super setting by Chuah [1]. Kac's classification is up to conjugation by $\mathfrak{g}$-automorphism, namely it does not distinguish $\sigma, \tau \in \mathrm{aut}(\mathfrak{g})$ if there exists $u \in \mathrm{aut}(\mathfrak{g})$ such that $\sigma = u^{-1}\tau u$. Here $\mathrm{aut}(\mathfrak{g})$ is a Lie group, and we let $\mathrm{int}(\mathfrak{g})$ denote the inner $\mathfrak{g}$-automorphisms, namely the $\mathfrak{g}$-automorphisms in the identity component of $\mathrm{aut}(\mathfrak{g})$. One may consider the classification under a stricter notion of conjugation, by requiring $u \in \mathrm{int}(\mathfrak{g})$ in above. Obviously this concerns only $\mathfrak{g}$ of types A, D and E_6, since they are the only cases where $\mathrm{int}(\mathfrak{g}) \neq \mathrm{aut}(\mathfrak{g})$ (this is equivalent to the existence of nontrivial symmetry on the Dynkin diagram of $\mathfrak{g}$). In this article, we classify the finite order inner automorphisms on $\mathfrak{sl}(n, \mathbb{C}) = A_{n-1}$ up to conjugation by inner automorphisms.

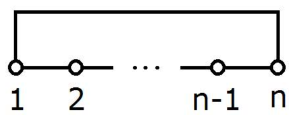

Figure 1: Extended Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$.

Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Its extended Dynkin diagram is as given in Figure 1. Let D denote the diagram in Figure 1, with vertices labeled $1, ..., n$ as indicated. A Kac diagram c is an assignment of nonnegative integers c_j to the vertices j of D, such that $c_1, ..., c_n$ have no nontrivial common factor. Let $\mathcal{K}$ denote all the Kac diagrams.

Let c be a Kac diagram. Let $m = \sum_1^n c_j$, and let $\omega = \exp(2\pi i)/m \in \mathbb{C}$. The vertices $\alpha \in D$ represent the union of a simple system and its lowest roots.

Theorem 1.1. (Kac) [2, Ch.X-5,Thm.5.15][4, Ch.8] *A Kac diagram c represents a $\mathfrak{g}$-automorphism σ of order m, where σ acts as multiplication by ω^{c_α} on the root space represented by $\alpha \in D$. Up to conjugation by* $\mathrm{aut}(\mathfrak{g})$*, all finite order inner $\mathfrak{g}$-automorphisms are obtained in this way.*

The general Kac's theorem deals with all finite order automorphisms on complex simple Lie algebras. Here we focus only on finite order inner automorphisms on $\mathfrak{sl}(n, \mathbb{C})$.

In the above theorem, "up to conjugation" means that it does not distinguish σ, τ if they are related by $\sigma = u^{-1}\tau u$ for some $u \in \mathrm{aut}(\mathfrak{g})$. It also does not distinguish two Kac diagrams $c, d \in \mathcal{K}$ if they are related by a symmetry on D. In this article, we study a stricter condition, namely we do not distinguish τ and

$u^{-1}\tau u$ only if u is an inner automorphism. We shall see in the theorem below that a stricter condition is imposed on $\mathcal{K}$, namely we do not distinguish $c, d \in \mathcal{K}$ only if they are related by the cyclic group $\mathbb{Z}_n$ in the symmetry group of D.

The symmetry group of D is the dihedral group $\mathbb{D}_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, where the normal subgroup $\mathbb{Z}_n$ acts as rotations and $\mathbb{Z}_2$ contains a nontrivial reflection. It induces a $\mathbb{D}_n$-action on $\mathcal{K}$ by

$$(gc)_j = c_{g(j)} \ ; \ g \in \mathbb{D}_n, c \in \mathcal{K}.$$

Theorem 1.2. *Two inner finite order $\mathfrak{g}$-automorphisms are conjugate by inner $\mathfrak{g}$-automorphisms if and only if their Kac diagrams are in the same $\mathbb{Z}_n$-orbit.*

Theorem 1.2 is the main result of this article. In Section 2, we prove Theorem 1.2. In Section 3, we provide some examples to illustrate the ideas.

2 Kac Diagrams of $\mathfrak{sl}(n, \mathbb{C})$

Recall that D is the extended Dynkin diagram of $\mathfrak{sl}(n, \mathbb{C})$ with vertices labeled as $1, ..., n$ as in Figure 1. Let $\mathcal{K}$ be the Kac diagrams on D. In Kac's Theorem 1.1, he does not distinguish two Kac diagrams which are related by a symmetry on D. However we consider two such diagrams as distinct members of $\mathcal{K}$. So for example Figure 3(b) and Figure 3(c) are distinct members of $\mathcal{K}$. The dihedral group $\mathbb{D}_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ acts on D, and hence on $\mathcal{K}$. Given $c \in \mathcal{K}$, we let $\mathbb{Z}_n(c) \subset \mathcal{K}$ denote its $\mathbb{Z}_n$-orbit.

The group of $\mathfrak{g}$-automorphisms has a semi-direct product

$$\text{aut}(\mathfrak{g}) = \text{int}(\mathfrak{g}) \rtimes \text{aut}(D_0), \tag{1}$$

where $\text{int}(\mathfrak{g})$ consists of the inner automorphisms and is a normal subgroup, and $\text{aut}(D_0)$ is the automorphism group of the Dynkin diagram D_0 of $\mathfrak{g}$. We let D_0 denote the Dynkin diagram, because the notation D has been used for the extended Dynkin diagram. See for instance [5, Thm.7.8]. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $\Delta \subset \mathfrak{h}^*$ be its root system. We have the root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_\Delta \mathfrak{g}_\alpha$. Each member σ of $\text{aut}(\mathfrak{g})$ which stabilizes $\mathfrak{h}$ induces an automorphism θ on Δ, given by $\sigma\mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$. So by considering $\mathfrak{g}$-automorphisms which stabilize $\mathfrak{h}$, (1) leads to

$$\text{aut}(\Delta) = W \rtimes \text{aut}(D_0),$$

where W is the Weyl group of Δ.

Given $\alpha \in \Delta$, it defines a reflection $R_\alpha : \Delta \longrightarrow \Delta$, where R_α maps α to $-\alpha$, and $R_\alpha(\beta) = \beta$ if β is perpendicular to α. Then W is generated by the reflections $\{R_\alpha \, ; \alpha \in \Delta\}$ [3, III-9].

Let $c \in \mathcal{K}$, and let $m = \sum_D c_j$. As discussed in Theorem 1.1, it represents some $\mathfrak{g}$-automorphisms σ of order m. This means that there exists $\Pi \subset \Delta$, which is a simple system together with its lowest root, such that the vertices of D represent Π, and σ has eigenvalue ω^{c_j} on the root space of c_j. Here $\omega = \exp(2\pi i)/m$.

Suppose $c, d \in \mathcal{K}$ both represent σ, but with respect to Π, Π'. Then we have $\Pi' = g\Pi$ for some $g \in \text{aut}(\Delta)$, because $\text{aut}(\Delta)$ acts transitively on the family of simple systems. If g is induced by an inner automorphism, then $g \in W$. And since the reflections generate W, g is the composite of a sequence of reflections $R_1, ..., R_n$. In this case we write $c \sim d$. We shall investigate the necessary and sufficient conditions for $c \sim d$.

Proposition 2.1. *If $\mathbb{Z}_n(c) = \mathbb{Z}_n(d)$, then $c \sim d$.*

Proof: Let $\alpha_1, ..., \alpha_n$ be the roots which are represented by the vertices of D. Let R_j be the root reflection which maps α_j to $-\alpha_j$ and fixes the roots perpendicular to α_j. We claim that

$$\begin{array}{ll} \text{(a)} & R_2 \cdot ... \cdot R_n(\alpha_j) = \alpha_{j+1} \text{ for } j = 1, ..., n-1; \\ \text{(b)} & R_2 \cdot ... \cdot R_n(\alpha_n) = \alpha_1. \end{array} \tag{2}$$

For $j = 2, ..., n-1$,

$$\begin{aligned} R_2 \cdot ... \cdot R_n(\alpha_j) &= R_2 \cdot ... \cdot R_{j+1}(\alpha_j) \\ &= R_2 \cdot ... \cdot R_j(\alpha_j + \alpha_{j+1}) \\ &= R_2 \cdot ... \cdot R_{j-1}((-\alpha_j) + (\alpha_{j+1} + \alpha_j)) \\ &= R_2 \cdot ... \cdot R_{j-1}(\alpha_{j+1}) = \alpha_{j+1}. \end{aligned} \tag{3}$$

Also,

$$\begin{aligned} R_2 \cdot ... \cdot R_n(\alpha_1) &= R_2 \cdot ... \cdot R_{n-1}(\alpha_1 + \alpha_n) \\ &= R_2 \cdot ... \cdot R_{n-2}(\alpha_1 + \alpha_{n-1} + \alpha_n) \\ &= R_2 \cdot ... \cdot R_{n-3}(\alpha_1 + (\alpha_{n-2} + \alpha_{n-1}) + \alpha_n) \\ &= ... = R_2(\alpha_1 + \alpha_3 + ... \alpha_n) \\ &= (\alpha_2 + \alpha_1) + (\alpha_2 + \alpha_3) + \alpha_4 + ... + \alpha_n. \end{aligned} \tag{4}$$

Since $\sum_1^n \alpha_j = 0$, the last line of (4) becomes α_2. Thus (3) and (4) lead to (2)(a). We can also prove (2)(b) directly, or simply note that (2)(a) implies (2)(b). This proves (2) as claimed.

Let $\theta \in \mathbb{Z}_n$ be the 1-step counter-clockwise rotation on D. Namely it rotates the vertices by

$$\theta : 1 \to 2 \to ... \to n \to 1. \tag{5}$$

By (2), $\theta = R_2 \cdot ... \cdot R_n$. Hence $\theta c \sim c$. Since θ generates $\mathbb{Z}_n$, the proposition follows. □

We shall illustrate the arguments of Proposition 2.1 by an example later (see Figure 2). Fix a positive integer $m > 1$. Let

$$\omega = \exp(2\pi i/m).$$

We shall work on $\mathfrak{g}$-automorphisms of order m, and they are represented by Kac diagrams c such that $\sum_1^n c_j = m$. The diagram c represents the automorphism which acts as multiplication by ω^{c_j} on the root space of vertex j. Define

$$\phi : \mathcal{K} \longrightarrow \mathbb{C} \ ; \ \phi(c) = \sum_{j=1}^{n} \omega^{c_n + ... + c_j}.$$

Thus $\phi(c) = \omega^{c_n + ... + c_1} + ... + \omega^{c_n} = 1 + \omega^{c_n + ... + c_2} + ... + \omega^{c_n}$. Let

$$S(c) = \{\omega^{c_1 + ... + c_j} \phi(c) \ ; \ j = 1, ..., n\} \subset \mathbb{C}.$$

The next proposition shows that $S(c)$ is an invariance of rotations and equivalence relation on $\mathcal{K}$.

Proposition 2.2.
(a) *$S(c) = \phi(\mathbb{Z}_n(c))$.*
(b) *If $d \in \mathbb{Z}_n(c)$, then $S(c) = S(d)$.*
(c) *If $c \sim d$, then $S(c) = S(d)$.*

Proof: Let $\theta \in \mathbb{Z}_n$ be the rotation (5), and let $d = \theta c$. Then $d_1 = c_n$ and $d_r = c_{r-1}$ for all $r > 1$. Therefore,

$$\begin{aligned} \phi(\theta c) &= 1 + \omega^{c_{n-1} + ... + c_1} + \omega^{c_{n-1} + ... + c_2} + ... + \omega^{c_{n-1}} \\ &= \omega^{-c_n}(\omega^{c_n} + 1 + \omega^{c_n + ... + c_2} + ... + \omega^{c_n + c_{n-1}}) \\ &= \omega^{-c_n} \phi(c). \end{aligned} \tag{6}$$

Then $\phi(\theta^2 c) = \omega^{-(\theta c)_n} \phi(\theta c) = \omega^{-c_{n-1} - c_n} \phi(c)$. Continue this process and get

$$\phi(\theta^j c) = \omega^{-c_{n-j+1} - ... - c_n} \phi(c) = \omega^{c_1 + ... + c_{n-j}} \phi(c).$$

Hence $S(c) = \{\omega^{c_1 + ... + c_j}\}_j \cdot \phi(c) = \{\phi(\theta^j c)\}_{n-j} = \phi(\mathbb{Z}_n(c))$, and part (a) follows.

By (6),

$$S(d) = \{\omega^{d_1 + ... + d_r}\}_{r=1}^n \cdot \phi(d) = (\{\omega^{c_n}\} \cup \{\omega^{c_n + c_1 + ... + c_r}\}_{r=1}^{n-1}) \cdot \omega^{-c_n} \phi(c) = S(c).$$

Since θ generates $\mathbb{Z}_n$, this proves part (b) of the proposition.

Next we prove part (c). If c is a Kac diagram with respect to $\Pi \subset \Delta$ (where Π is a simple system and its lowest root), we let $F_j c$ be the resulting Kac diagram with respect to $R_j\Pi$. We claim that

$$S(F_j c) = S(c)\ ;\ j = 1, ..., n. \tag{7}$$

Let $j = 2, ..., n-1$ and let $d = F_j c$. Then

$$\begin{aligned} &d_r = c_r \text{ for all } r = 1, ..., j-2, j+2, ..., n; \\ &d_{j\pm 1} = c_{j\pm 1} + c_j\ ,\ d_j = -c_j. \end{aligned} \tag{8}$$

By (8), $d_{j-1} + d_j + d_{j+1} = c_{j-1} + c_j + c_{j+1}$. Hence

$$d_r + ... + d_n = c_r + ... + c_n \text{ for all } r \leq j-1 \text{ or } r > j+1. \tag{9}$$

Also,

$$\omega^{d_j + d_{j\pm 1}} + \omega^{d_{j\pm 1}} = \omega^{-c_j + (c_{j\pm 1} + c_j)} + \omega^{c_{j\pm 1} + c_j} = \omega^{c_{j\pm 1}} + \omega^{c_{j\pm 1} + c_j}. \tag{10}$$

By (9) and (10), we have $\{\omega^{d_1 + ... d_r}\}_{r=1}^n = \{\omega^{c_1 + ... c_r}\}_{r=1}^n$ as well as

$$\phi(d) = \sum_1^n \omega^{d_r + ... + d_n} = \sum_1^n \omega^{c_r + ... + c_n} = \phi(c).$$

This implies that

$$S(d) = \{\omega^{d_1 + ... + d_r}\}_{r=1}^n \cdot \phi(d) = \{\omega^{c_1 + ... + c_r}\}_{r=1}^n \cdot \phi(c) = S(c).$$

This proves (7) for $j = 2, ..., n-1$.

One can also check directly (7) for $j = 1$ and $j = n$. Alternatively, for $j = 1, n$, there exists a rotation g and $k \in \{2, ..., n-1\}$ such that $F_j c = g^{-1} F_k g c$. Then by part (b) of this proposition, $S(F_j c) = S(g^{-1} F_k g c) = S(c)$. This completes the proof of (7), which leads to part (c) of this proposition. □

An element of $\mathbb{D}_n \backslash \mathbb{Z}_n$ of order 2 is called a reflection. We say that a Kac diagram c is *symmetric* if there exists a reflection r such that $rc = c$. Equivalently for any $z \in \mathbb{D}_n \backslash \mathbb{Z}_n$, there exists $g \in \mathbb{Z}_n$ such that $zc = gc$. Let $\overline{S(c)}$ denote the complex conjugation on the elements of $S(c) \subset \mathbb{C}$.

Proposition 2.3. *If $S(c) = \overline{S(c)}$, then c is symmetric.*

Proof: Use the polar coordinates to write $re^{it} \in \mathbb{C}^\times$, where $r > 0$ and $t \in [0, 2\pi)$. Define a partial order on $\mathbb{C}^\times$ by $re^{it} \succeq r'e^{it'}$ if $t \geq t'$.

Suppose that $S(c) = \overline{S(c)}$. By Proposition 2.2(b), if $c' \in \mathbb{Z}_n(c)$, then $S(c') = S(c)$. Furthermore c is symmetric if and only if c' is symmetric. Therefore, replacing c by another member in $\mathbb{Z}_n(c)$ if necessary, we may assume that $\phi(c) \preceq \phi(c')$ for all $c' \in \mathbb{Z}_n(c)$. The elements of $S(c)$ are

$$\phi(c) \preceq \omega^{c_1}\phi(c) \preceq \omega^{c_1+c_2}\phi(c) \preceq ... \preceq \omega^{c_1 + ... + c_{n-1}}\phi(c). \tag{11}$$

Case 1: $\phi(c) \in \mathbb{R}$.

Since $\omega^{c_1 + ... + c_n} = 1$, by (11), the elements of $\overline{S(c)}$ are

$$\phi(c) \preceq \omega^{c_n}\phi(c) \preceq \omega^{c_n + c_{n-1}}\phi(c) \preceq ... \preceq \omega^{c_n + ... + c_2}\phi(c). \tag{12}$$

Since $S(c) = \overline{S(c)}$, by (11) and (12), we have $c_1 = c_n$, $c_2 = c_{n-1}$ and more generally $c_j = c_{n-j+1}$. Then c is symmetric by the reflection $r(j) = n - j + 1$. *Case 2:* $\phi(c) \notin \mathbb{R}$.

Since $\omega^{c_1 + ... + c_n} = 1$, by (11), the elements of $\overline{S(c)}$ are

$$\omega^{c_n}\overline{\phi(c)} \preceq \omega^{c_n + c_{n-1}}\overline{\phi(c)} \preceq \omega^{c_n + ... + c_2}\overline{\phi(c)} \preceq \overline{\phi(c)}. \tag{13}$$

Since $S(c) = \overline{S(c)}$, by (11) and (13), we have $\phi(c) = \omega^{c_n}\overline{\phi(c)}$, $\omega^{c_1}\phi(c) = \omega^{c_n + c_{n-1}}\overline{\phi(c)}$ and more generally $\omega^{c_1 + ... + c_j}\phi(c) = \omega^{c_n + ... + c_{n-j}}\overline{\phi(c)}$. It implies that $c_1 = c_{n-1}$, $c_2 = c_{n-2}$ and so on. Then c is symmetric by the reflection $r(j) = n - j$. □

Proposition 2.4. *For any $z \in \mathbb{D}_n \backslash \mathbb{Z}_n$, we have $S(zc) = \overline{S(c)}$.*

Proof: A specific reflection r is given by $1 \leftrightarrow n$, $2 \leftrightarrow n-1$, and more generally

$$r(j) = n - j + 1. \tag{14}$$

For such r and any $c \in \mathcal{K}$,

$$\begin{aligned} \phi(rc) &= \omega^{c_1+\ldots+c_n} + \omega^{c_1+\ldots+c_{n-1}} + \ldots + \omega^{c_1} \\ &= 1 + \omega^{-c_n} + \omega^{-c_n-c_{n-1}} + \ldots + \omega^{-c_n-\ldots-c_2} \\ &= 1 + \overline{\omega^{c_n}} + \overline{\omega^{c_n+c_{n-1}}} + \ldots + \overline{\omega^{c_n+\ldots+c_2}} \\ &= \overline{\phi(c)}. \end{aligned} \tag{15}$$

Recall that $S(c) = \{\omega^{c_1+\ldots+c_j}\phi(c)\}_{j=1}^n$. We have

$$\begin{aligned} \overline{S(c)} &= \{\omega^{c_{j+1}+\ldots+c_n}\overline{\phi(c)}\}_{j=1}^n && \text{as } \omega^{c_1+\ldots+c_n} = 1 \\ &= \{\omega^{(rc)_1+\ldots+(rc)_j}\phi(rc)\}_{j=1}^n && \text{by (14) and (15)} \\ &= S(rc). \end{aligned} \tag{16}$$

An arbitrary element of $\mathbb{D}_n \backslash \mathbb{Z}_n$ is of the form rg, where $g \in \mathbb{Z}_n$. By Proposition 2.2(b) and (16), $S(rgc) = \overline{S(gc)} = \overline{S(c)}$. □

Proof of Theorem 1.2:

Let $c, d \in \mathcal{K}$ be related by the diagram symmetry $\mathbb{D}_n$. By Proposition 2.1, if there exists a rotation $g \in \mathbb{Z}_n$ such that $gc = d$, then $c \sim d$.

Conversely, suppose that $rc = d$ for some $r \in \mathbb{D}_n \backslash \mathbb{Z}_n$, but c and d are not related by the rotation group $\mathbb{Z}_n$. Then c is not symmetric, so by Proposition 2.3, $S(c) \neq \overline{S(c)}$. By Proposition 2.4, $S(c) \neq S(d)$. By Proposition 2.2, $c \not\sim d$. This proves Theorem 1.2. □

3 Examples

We provide an example to illustrate the concepts discussed earlier. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. Its extended Dynkin diagram D is given by Figure 2(a), namely a triangle. We denote its vertices by x, y, z, and they represent a simple system with lowest root as indicated in Figure 2(a).

Consider the Kac diagram c in Figure 2(b), where $c_x = 1$, $c_y = 2$ and $c_z = 0$. It represents a $\mathfrak{g}$-automorphism σ of order $c_x + c_y + c_z = 3$. Let $\omega = \exp(2\pi i/3)$. Then all the root spaces are eigenspaces of σ, with eigenvalues 1, ω or ω^2. We indicate these eigenvalues with the roots in Figure 2(b).

Consider the reflection R_x, which maps x to $-x$ and fixes the hyperplane $x^\perp$. The hyperplane is just the dotted line in Figure 2(b), so R_x is the reflection about the dotted line. Figure 2(c) show the resulting eigenvalues after performing R_x. The corresponding Kac diagram is given in Figure 2(c). By comparing the Kac diagrams of Figure 2(b) and Figure 2(c), we see that they are related by a rotation on the triangle. This observation verifies Proposition 2.1, namely the Kac diagrams in Figure 2(b) and Figure 2(c) are related by rotation (i.e. they are in the same $\mathbb{Z}_3$-orbit), and indeed they are equivalent because they are related by performing R_x to the root system.

Next we classify all inner automorphisms on $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ of order 3, up to conjugation by inner automorphisms. All Kac diagrams are assignments of nonnegative integers $\{c_1, c_2, c_3\}$ without nontrivial common factor such that $c_1 + c_2 + c_3 = 3$. So the only possibilities are $\{1, 1, 1\}$ and $\{0, 1, 2\}$.

Figure 3 provides all possibilities, where two diagrams are not distinguished if they are related by a rotation $\mathbb{Z}_3$. So there are three such $\mathfrak{g}$-automorphisms. Note that Figure 3(b) and Figure 3(c) are related by a diagram reflection but not rotation, so they represent order-3 automorphisms $\sigma, \tau \in \mathrm{aut}(\mathfrak{g})$ such that $\sigma = u^{-1}\tau u$ for some $u \in \mathrm{aut}(\mathfrak{g})$, but $u \notin \mathrm{int}(\mathfrak{g})$. So σ, τ are not distinguished by Kac's Theorem 1.1, but are distinguished by our stricter Theorem 1.2. Diagram rotations are not needed, as discussed in Figure 2 above. So Figure 3 exhausts all possibilities up to conjugation by inner automorphisms.

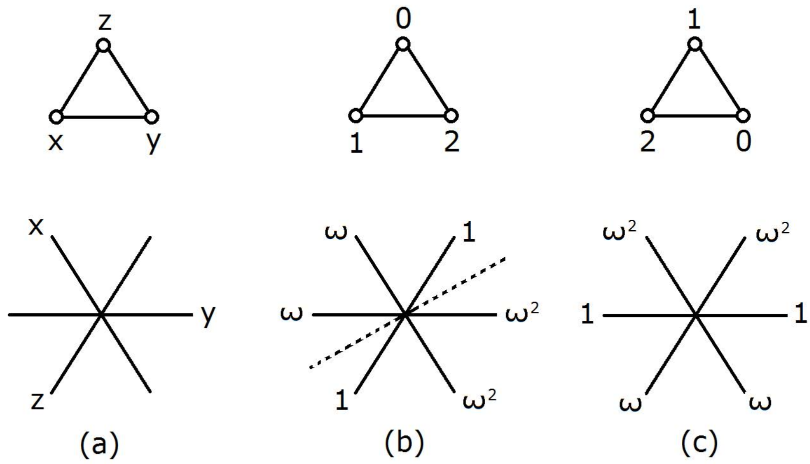

Figure 2: Reflection and Kac diagrams.

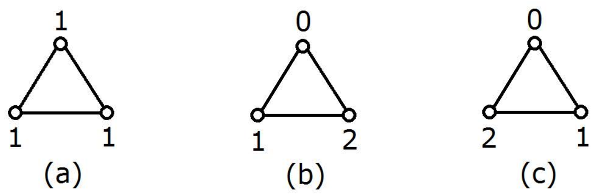

Figure 3: Automorphisms of order 3 on $\mathfrak{sl}(3, \mathbb{C})$.

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