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Double Integral Inequalities of Hermite-Hadamard-Type for ϕ_h-Convex Functions on Linear Spaces

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Abstract: *The concept of ϕ_h-convexity is extended for functions defined on closed ϕ_h-convex subsets of linear spaces. Consequently, some double integral inequalities of Hermite-Hadamard type defined on time-scaled linear spaces are established for ϕ_h-convex functions.*

Keywords: Time scales, ϕ_h-convex function, diamond-ϕ_h, Hermite-Hadamard inequality.

1 Introduction

A well celebrated, fundamental inequality for a convex function f is the classical Hermite-Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $a < b$.

It was first suggested by Hermite in 1881. But this result was nowhere mentioned in literature and was not widely known as Hermite's result. A leading expert on the history and theory of complex functions, Beckenbach [2], wrote that the inequality (1.1) was proven by Hadamard in 1893. In general, (1.1) is now known as the Hermite-Hadamard inequality. It has several extensions and generalizations for univariate and multivariate convex functions and its classes on classical intervals.

The concept of the theory of time scales was initiated by Stefen Hilger (see [10]) in order to unify and extend the theory of difference and differential calculus in a consistent way. In this theory, the delta and nabla calculus are introduced. A linear combination of these delta and nabla dynamics, the diamond-α calculus on time scales was developed by Sheng et al.[12]. Since the advent of this notion, several authors have extended the classical Hermite-Hadamard inequality (1.1) to time scales via the diamond-alpha dynamic calculus for univariate convex functions (see Dinu[5]) and the references therein.

Recently, Fagbemigun and Mogbademu [6] introduced the time-scaled version of some classes of convex functions, including a more generalized class of ϕ_h-convex function on time scales as given below:

Definition 1.1. [6] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and ϕ be a given real valued function with $\phi(t) = t$. A mapping $f : I_{\mathbb{T}} \to \mathbb{R}$ is said to be ϕ_h-convex on time scales if

$$f(\lambda\phi(x) + (1-\lambda)\phi(y)) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(\phi(x)) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(\phi(y)), \tag{1.2}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and $x, y \in I_{\mathbb{T}}$.

Remark 1.1.

(i) If $s = 1$, $h(\lambda) = 1$ and $\phi(x) = x$, then $f \in SX(I_{\mathbb{T}})$, i.e, f is convex on time scales (see [5]).

(ii) If $s = 1$, $h(\lambda) = 1$, where $\lambda = \frac{1}{2}$ and $\phi(x) = x$, then $f \in J(I_{\mathbb{T}})$ is mid-point convex on time scales (see [6]).

(iii) If $s = 0$ and $\phi(x) = x$, then $f \in P(I_{\mathbb{T}})$ is P-convex on time scales (see [6]).

(iv) If $h(\lambda) = \lambda^{\frac{s}{s+1}}$ and $\phi(x) = x$, then $f \in SX(h, I_{\mathbb{T}})$ is h-convex on time scales (see [6]).

(v) If $s = 1, h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$ and $\phi(x) = x$, then $f \in MT(I_{\mathbb{T}})$ is MT-convex on time scales (see [6]).

In a more recent paper, the authors [7] introduced a more general calculus of diamond-ϕ_h dynamics for a single variable function on time scales as follows;

Definition 1.2. [7] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a real valued function, with the property that $h(t) > 0$ for all $t \geq 0$. The *diamond-ϕ_h dynamic derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ in $t \in \mathbb{T}$ is defined to be the number denoted by $f^{\diamond_{\phi_{h-s,T}}}(t)$(when it exists), with the property that for any $\epsilon > 0$, there is a neighbourhood U of m such that, for all $n \in U$, $0 \leq s \leq 1$ and $0 \leq \lambda \leq 1$, with $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$, where $m, n \in \mathbb{T}^k_k$, then,

$$\left| \left(\frac{\lambda}{h(\lambda)}\right)^s [f(\sigma(m)) - f(n)]\nu_{mn} \quad + \quad \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s [f(\rho(m)) - f(n)]\mu_{mn} \right.$$
$$\left. - f^{\diamond_{\phi_h}}(\phi(t))\mu_{mn}\nu_{mn} \right| < \epsilon|\mu_{mn}\nu_{mn}|. \tag{1.3}$$

Definition 1.3. [7] The *diamond-ϕ_h integral* of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;

$$\int_a^b f(\phi(t)) \diamond_{\phi_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(\phi(t))\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(\phi(t))\nabla t, \tag{1.4}$$

for all $s \in [0,1]$, $\lambda \in [0,1]$ and $h(t) > 0$ $\forall t \geq 0$ provided that f has a delta and nabla integral on $[a,b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Remark 1.2.

(i) The inequality (1.4) reduces to the diamond-α integral defined by Sheng et al. [12], if $\phi(x) = x, s = 1$, $h(\lambda) = 1$ and $\lambda = \alpha$. Thus, every diamond-α integrable function on $\mathbb{T}$ is diamond-ϕ_h integrable but the converse is not true (see [7]).

(ii) If f is diamond-ϕ_h integrable for $0 \leq s \leq 1$, and $0 \leq \lambda \leq 1$, then f is both Δ and ∇ integrable.

The inequality (1.1) was equally extended to time scales by the authors [7], using the new class of univariate ϕ_h-convex function of [6] to obtain several generalizations of the Hermite-Hadamard inequality on time scales. We present one of such results

Theorem 1.1. [7] Suppose that
(i) $f : I_{\mathbb{T}} \to \mathbb{R}$ is a continuous ϕ_h-convex function on $I_{\mathbb{T}}$;
(ii) $p, q \in (0,1)$, such that $p + q = 1$;
$g \in C(I_{\mathbb{T}}, \mathbb{R})$ is symmetric with respect to $pa + qb = \gamma$ on $[a,b]$, for all $a, b \in I_{\mathbb{T}}$, that is,

$$g(\gamma - qt) = g(\gamma + pt), \qquad \text{for all } t \in [0, b-a].$$

Then

$$f(p\phi(x) + q\phi(y)) \leq p\frac{\int_a^\gamma f(\phi(t))g(\phi(t)) \diamond_{\phi_h} t}{\int_a^\gamma g(\phi(t)) \diamond_{\phi_h} t} + q\frac{\int_\gamma^b f(\phi(t))g(\phi(t)) \diamond_{\phi_h} t}{\int_\gamma^b g(\phi(t)) \diamond_{\phi_h} t}$$
$$\leq pf(\phi(x)) + qf(\phi(y)). \tag{1.5}$$

The two-variable time scales delta, nabla and diamond-α calculi were introduced by Albrandth and Morian[1], Bohner and Guseinov [3, 4], Guseinov[9] and Ozkan and Kaymakçalan [11] respectively.

Ozkan and Kaymakçalan [11] gave the following definition of a partial $\diamond_{\alpha_1}$ derivative;

Definition 1.4. [11] Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. We say that f has a partial $\diamond_{\alpha_1}$ derivative $\frac{\partial f(t_1,t_2)}{\diamond_{\alpha_1} t_1}$(with respect to t_1) if for each $\epsilon > 0$, there exists a neighbourhood Ut_1 (open in the relative topology of $\mathbb{T}_1$) of t_1 such that

$$\left|\alpha_1[f(\sigma_1(t_1),t_2)-f(s,t_2,)]\mu t_1 s+(1-\alpha_1)[f(\rho_1(t_1),t_2)-f(s,t_2)]\nu t_1 s-f^{\diamond_{\alpha_1}}(t_1,\, t_2)\mu t_1 s\nu t_1 s\right| < \epsilon|\mu t_1 s\nu t_1 s| \quad (1.6)$$

for all $s \in Ut_1$, where $Ut_1 ms = \sigma_1(t_1) - s, \nu t_1 s = \rho_1(t_1) - s$.

The $\diamond_{\alpha_2}$ partial derivative was respectively defined (see [11]).

Motivated by the recent results of these authors; Fagbemigun and Mogbademu [6], Fagbemigun et al. [7] and Ozkan and Kaymakçalan [11], we discuss the following new concepts of Fagbemigun and Mogbademu [8].

2 Preliminaries

In the sequel, we shall need the following new definitions recently introduced in [8].
Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is a complete metric space with the metric d defined by

$$d((x,y),(x^{'},y^{'})) = ((x-x^{'})^2 + (y-y^{'})^2)^{\frac{1}{2}}, \qquad \forall \quad (x,y),(x^{'},y^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Let σ_i, ρ_i, $(i = 1,\, 2)$ denote respectively the forward jump operator, backward jump operator, and the diamond-ϕ_h dynamic differentiation operator on $\mathbb{T}_i$.

Definition 2.1.[8] Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and ϕ be a given real valued function. f is said to have a partial $\diamond_{(\phi_h)_1}$ derivative $\frac{\partial f(\phi(t_1,t_2))}{\diamond_{(\phi_h)_1} t_1}$(wrt t_1), at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_1 of t_1 such that

$$\Bigg|\left(\tfrac{\lambda}{h(\lambda)}\right)_1^s[f(\sigma_1(t_1),t_2) - f(m,t_2,)]\mu t_1 m$$
$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)_1^s[f(\rho_1(t_1),t_2) - f(m,t_2)]\nu t_1 m - f^{\diamond_{(\phi_h)_1}}(\phi(t_1,\, t_2))\mu t_1 m\nu t_1 m\Bigg|$$
$$< \epsilon|\mu t_1 m\nu t_1 m|, \tag{2.1}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $m \in Ut_1$, where $Ut_1 m = \sigma_1(t_1) - m$, $\nu t_1 m = \rho_1(t_1) - m$.

Definition 2.2.[8] Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$ and $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ an increasing function with the property that $h(t) > 0$ for all $t \geq 0$. f is said to have a "partial $\diamond_{(\phi_h)_2}$ derivative" $\frac{\partial f(\phi(t_1,t_2))}{\diamond_{(\phi_h)_2} t_2}$(wrt t_2), at $(t_1,t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for each $\epsilon > 0$, there exists a neighbourhood Ut_2 of t_2 such that

$$\Bigg|\left(\tfrac{\lambda}{h(\lambda)}\right)_2^s[f(t_1,\, \sigma_2(t_2) - f(t_1,\, m)]\mu t_2 m$$
$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)_2^s[f(t_1,\, \rho_2(t_2) - f(t_1,\, m)]\nu t_2 m - f^{\diamond_{(\phi_h)_2}}(\phi(t_1,\, t_2))\mu t_2 m\nu t_2 m\Bigg|$$
$$< \epsilon|\mu t_2 m\nu t_2 m|, \tag{2.2}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and for all $n \in Ut_2$, where $Ut_2 m = \sigma_2(t_2) - m$,
$\nu t_2 m = \rho_2(t_2) - m$.

These derivatives are also denoted by $f^{\diamond_{(\phi_h)_1}}(\phi(t_1,\, t_2))$ and $f^{\diamond_{(\phi_h)_2}}(\phi(t_1,\, t_2))$ respectively.

Before we define the double diamond-ϕ_h dynamic integral, we shall employ the following remark of [3].

Remark 2.1. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$. If the delta (Δ) and nabla (∇) integrals of f exist on $\mathbb{T}_1 \times \mathbb{T}_2$, then the following types of integrals can be defined:

(i) $\Delta\Delta$-integral over $R^0 = [a,b) \times [c,d)$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta) \times [\gamma,\partial)$;

(ii) $\nabla\nabla$-integral over $R^1 = (a, b] \times (c, d]$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times (\gamma, \partial]$;

(iii) $\Delta\nabla$-integral over $R^2 = [a, b) \times (c, d]$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times (\gamma, \partial]$;

(iv) $\nabla\Delta$-integral over $R^3 = (a, b] \times [c, d)$, which is introduced by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times [\gamma, \partial)$.

Now let $\bar{U}(f)$ and $\bar{L}(f)$ denote the upper and lower Darboux Δ-integral of f from a to b ; $\underline{\mathrm{U}}(f)$ and $\underline{\mathrm{L}}(f)$ denote the upper and lower Darboux ∇-integral of f from a to b respectively. Given the construction of $U(f)$ and $L(f)$, which follows from the properties of supremum and infimum, we give the following definition.

Definition 2.3. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and ϕ be a given real valued function. If f is Δ-integrable on $R^0 = [a, b) \times [c, d)$ and ∇-integrable on $R^1 = (a, b] \times (c, d]$, then it is $\diamond_{\phi_h}$-integrable on $R = [a, b] \times [c, d]$ and

$$\int_R f(\phi(t,\, k)) \diamond_{(\phi_h)_1} \phi(t) \diamond_{(\phi_h)_2} \phi(k) = \left(\frac{\lambda}{h(\lambda)}\right)^s \int\int_{R^0} f(\phi(t,\, k))\Delta_1\phi(t)\Delta_2\phi(k)$$

$$+ \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int\int_{R^1} f(\phi(t,\, k))\nabla_1\phi(t)\nabla_2\phi(k), \tag{2.3}$$

for all $s \in [0, 1]$, $0 \leq \lambda \leq 1$ and $t, k \in J_{\mathbb{T}}$.

Since $\bar{U}(f) \geq \bar{L}(f)$ and $\underline{\mathrm{U}}(f) \geq \underline{\mathrm{L}}(f)$, we obtain the following result.

Theorem 2.1. Let f be a real-valued function on $\mathbb{T}_1 \times \mathbb{T}_2$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ a nonzero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and ϕ be a given real valued function. If f be $\diamond_{\phi_h}$-integrable on $R = [a, b] \times [c, d]$, provided its delta (Δ) and nabla (∇) integrals exist, then

(i) If $\phi_h = 1$, f is $\Delta\Delta$-integrable on $R^0 = [a, b) \times [c, d)$;

(ii) If $\phi_h = 0$, f is $\nabla\nabla$-integrable on $R^1 = (a, b] \times (c, d]$;

(iii) If $\phi_h = \frac{1}{2}$, f is $\Delta\Delta$-integrable and $\nabla\nabla$-integrable on R^0 and R^1

(iv) If $\phi_h = \alpha$, f is double $\diamond_\alpha$-integrable on $R = [a, b] \times [c, d]$.

3 Double integral inequalities of Hermite-Hadamard type for ϕ_h-convex functions

Here, we obtain double integral inequalities of Hermite-Hadamard type in which upper and lower bounds for the quantity

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(\frac{p\phi_1(t) + q\phi_2(t)}{p+q}\right) \diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p,$$

are provided for the generalized class of ϕ_h-convex functions (1.2), defined on linear spaces of time scales.

Let $X_{\mathbb{T}}$ be a vector space over the time-scaled field K and let $\phi(x), \phi(y)$ be monotonically increasing functions in $X_{\mathbb{T}}$, $\phi(x) \neq \phi(y)$. Let the segment generated by $\phi(x), \phi(y)$ be defined by

$[a, b] : \{(1-\lambda)\phi(x) + \lambda\phi(y), \quad \lambda \in [0, 1]\}$.

We consider the function $f : [x, y]_{I_{\mathbb{T}}} \subset \mathbb{T} \to \mathbb{R}$ and the attached function $g(\phi(x), \phi(y)) : [0, 1] \subset \mathbb{T} \to \mathbb{R}$ defined by

$g(\phi(x), \phi(y))(\phi(\lambda)) := f[(1-\lambda)\phi(x) + \lambda\phi(y)], \quad \lambda \in [0, 1]$.

Note that f is ϕ_h-convex on $[x, y]$ if and only if $g(\phi(x), \phi(y))$ is ϕ_h-convex on [0, 1].

The concept of ϕ_h-convexity in Definition 1.1 can be extended for functions defined on closed ϕ_h-convex subsets of the linear spaces in the same way as on classical intervals by replacing the interval $I_{\mathbb{T}}$ by the corresponding closed ϕ_h-convex subset E of the linear space $X_{\mathbb{T}}$.

It is well-known that if $(X, ||\cdot||)$ is a normed linear space, then the function $f(x) = ||x||^p, p \geq 1$ is convex on X.

Using the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in [0,1]$, we have for the function $g(\phi(x)) = ||\phi(x)||$ that

$$\begin{aligned}
g(\lambda\phi(x) + (1-\lambda)\phi(y)) \quad &= \left(||\frac{\lambda}{h(\lambda)}\phi(x) + \frac{(1-\lambda)}{h(1-\lambda)}\phi(y)||\right)^s \\
&\leq \left(\frac{\lambda}{h(\lambda)}||\phi(x)|| + \frac{(1-\lambda)}{h(1-\lambda)}||\phi(y)||\right)^s \\
&\leq \left(\frac{\lambda}{h(\lambda)}||\phi(x)||\right)^s + \left(\frac{(1-\lambda)}{h(1-\lambda)}||\phi(y)||\right)^s \\
&\leq \left(\frac{\lambda}{h(\lambda)}\right)^s (||\phi(x)||)^s + \left(\frac{(1-\lambda)}{h(1-\lambda)}\right)^s (||\phi(y)||)^s \\
&= \left(\frac{\lambda}{h(\lambda)}\right)^s g(\phi(x)) + \left(\frac{(1-\lambda)}{h(1-\lambda)}\right)^s g(\phi(y)),
\end{aligned}$$

for any $x, y \in X_{\mathbb{T}}$, $\lambda \in [0,1]$ and h a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and $\phi(t)$ be a monotonically increasing function on $I_{\mathbb{T}}$, which shows that g is ϕ_h-convex on $I_{\mathbb{T}}$. With this concept, an equivalent definition to definition 1.1 is given as follows.

Definition 3.1. Let h be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and let ϕ be a monotonically increasing function. Then inequality (1.2) can be re-written as

$$f\left(\frac{p\phi(x) + q\phi(y)}{p+q}\right) \leq \frac{(\frac{p}{h(p)})^s f\phi(x) + (\frac{q}{h(q)})^s f\phi(y)}{p+q}, \tag{3.1}$$

for all $p,\ q \geq 0$ with $p + q > 0$, $s \in [0,1]$ and $x,\ y \in I_{\mathbb{T}}$.

We can now establish double Hermite-Hadamard-type integral inequalities for the class of ϕ_h-convex functions on time scales linear spaces.

Theorem 3.1. Let h be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$ and let ϕ be a monotonically increasing function. Let E ϕ_h-convex set in a linear space of a time scale interval $X_{\mathbb{T}} \subset \mathbb{T}$ and $f : E \subseteq X_{\mathbb{T}} \to \mathbb{R}$ be an integrable ϕ_h-convex function with respect to the function ϕ_h defined on the set E with ϕ_h Lebesgue integrable on $[a,b]_{I_{\mathbb{T}}} \times [c,d]_{I_{\mathbb{T}}}$. Then, for any $a, b, c, d \geq 0$, with $b > a, d > c$ and for any $a, b, c, d \in E$, $s \in [0,1]$, we have:

$$\begin{aligned}
&f\left(\frac{I(a,b;c,d)}{(b-a)(d-c)}\phi(x) + \frac{I(c,d;a,b)}{(b-a)(d-c)}\phi(y)\right) \\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right) \diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p \\
&\leq \frac{I_{h(p)}(a,b;c,d)}{(b-a)(d-c)} f(\phi(x)) + \frac{I_{h(q)}(c,d;a,b)}{(b-a)(d-c)} f(\phi(y)),
\end{aligned} \tag{3.2}$$

where

$$I(a,b;c,d) = \int_a^b \left(\int_c^d \left(\frac{p}{p+q}\right) \diamond_{(\phi_h)_1} q\right) \diamond_{(\phi_h)_2} p,$$

$$I(c,d;a,b) = \int_a^b \left(\int_c^d \left(\frac{q}{p+q}\right) \diamond_{(\phi_h)_1} q\right) \diamond_{(\phi_h)_2} p,$$

$$I_{h(p)}(a,b;c,d)=\int_a^b\int_c^d\frac{\left(\frac{p}{h(p)}\right)^s}{p+q}\diamond_{(\phi_h)_1}\phi(q)\diamond_{(\phi_h)_2}\phi(p)$$

and

$$I_{h(q)}(c,d;a,b)=\int_a^b\int_c^d\frac{\left(\frac{q}{h(q)}\right)^s}{p+q}\diamond_{(\phi_h)_1}\phi(q)\diamond_{(\phi_h)_2}\phi(p),$$

for all p, $q\geq 0$ with $p+q>0$ and $x,y\in X_{\mathbb{T}}$.

Proof. Consider the function $g_{\phi(x),\phi(y)}:[0,1]\subset\mathbb{T}\to R$ defined by $g_{\phi(x),\phi(y)}(\lambda)=f(\lambda\phi(x)+(1-\lambda)\phi(y))$. This function is ϕ_h-convex on $[0,1]\subset\mathbb{T}$ and by Jensen's double integral inequality of real functions on time scales, we have

$$g_{\phi(x),\phi(y)}\left(\int_a^b\int_c^d\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p\right)$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g_{\phi(x),\phi(y)}\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p,\tag{3.3}$$

which is equivalent to

$$f\left[\frac{\int_a^b\int_c^d\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\phi(x)+\left(1-\frac{\int_a^b\int_c^d\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\right)\phi(y)\right]$$
$$\leq f\left(\frac{\int_a^b\int_c^d\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\phi(x)+\frac{\int_a^b\int_c^d\left(\frac{q}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\phi(y)\right).$$

By definition 3.1 and using a simple calculation, we have

$$f\left(\frac{\int_a^b\int_c^d\left(\frac{p}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\phi(x)+\frac{\int_a^b\int_c^d\left(\frac{q}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p}{(b-a)(d-c)}\phi(y)\right)$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{p}{p+q}\phi(x)+\frac{q}{p+q}\phi(y)\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\left(\frac{p}{h(p)}\right)^s}{p+q}\phi(x)+\frac{\left(\frac{q}{h(q)}\right)^s}{p+q}\phi(y)\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p,$$

which proves the first part of Theorem 3.1.

Under the same assumption of definition 3.1, f is ϕ_h-convex. Thus, integrating inequality (3.1) on the rectangle $[a,\,b]_{I_{\mathbb{T}}}\times[c,\,d]_{I_{\mathbb{T}}}$ over $\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p$ gives

$$\int_a^b\int_c^d f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p$$
$$\leq f(\phi(x))\int_a^b\int_c^d\frac{\left(\frac{p}{h(p)}\right)^s}{p+q}\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p$$
$$+f(\phi(y))\int_a^b\int_c^d\frac{\left(\frac{q}{h(q)}\right)^s}{p+q}\diamond_{(\phi_h)_1}q\diamond_{(\phi_h)_2}p,$$

and the second part of inequality (3.2) is satisfied.

Theorem 3.2. Let h be a non zero non negative function with the property that $h(t)>0$ for all $t\geq 0$ and ϕ a monotonically increasing function. Let E be a ϕ_h-convex set in a linear space of a time scale interval $X_{\mathbb{T}} \subset \mathbb{T}$ and $f : E \subseteq X_{\mathbb{T}} \to \mathbb{R}$ be an integrable ϕ_h-convex function with respect to the function ϕ_h defined on the set E with the mapping $[0,1] : \phi(\lambda) \to f((1-\lambda)\phi(x) + \lambda\phi(y))$ Lebesgue integrable on $[a,b]_{I_{\mathbb{T}}} \times [c,d]_{I_{\mathbb{T}}}$. Then for all $p,\ q \geq 0$, $\phi(x), \phi(y) \in E$ with $p+q>0$, $s \in [0,1]$ and $x, y \in X_{\mathbb{T}}$,

$$2h(\frac{1}{2})f\left(\frac{\phi(x)+\phi(y)}{2}\right)$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left(f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)+f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)\right)\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$$
$$\leq \frac{f(\phi(x)+f(\phi(y))}{(b-a)(d-c)}\int_a^b\int_c^d \diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p.$$

Proof. By ϕ_h-convexity of f,

$$f(\lambda\phi(x)+(1-\lambda)\phi(y)) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(\phi(x)) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(\phi(y)) \tag{3.4}$$

and

$$f((1-\lambda)\phi(x)+\lambda\phi(y)) \leq \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(\phi(x)) + \left(\frac{\lambda}{h(\lambda)}\right)^s f(\phi(y)) \tag{3.5}$$

for any $0 \leq \lambda \leq 1$, and $s \in [0,1]$.
Adding (3.4) and (3.5) gives

$$f(\lambda\phi(x)+(1-\lambda)\phi(y)) + f((1-\lambda)\phi(x)+\lambda\phi(y))$$
$$\leq \left(\left(\frac{\lambda}{h(\lambda)}\right)^s + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\right)[f(\phi(x))+f(\phi(y))]. \tag{3.6}$$

By choosing $\lambda = \frac{p}{p+q}$ and $1-\lambda = \frac{q}{p+q}$ in (3.6), we obtain

$$f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)+f\left(\frac{q\phi(x)+p\phi(y)}{p+q}\right)$$
$$\leq \left(\frac{\frac{p}{p+q}}{h\left(\frac{p}{p+q}\right)}\right)^s + \left(\frac{\frac{q}{p+q}}{h\left(\frac{q}{p+q}\right)}\right)^s [f(\phi(x))+f(\phi(y))] \tag{3.7}$$

for any $p, q > 0$ with $p+q>0$. Then the double integrals

$$\int_a^b\int_c^d f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$$

and

$$\int_a^b\int_c^d f\left(\frac{q\phi(x)+p\phi(y)}{p+q}\right)\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$$

exists since the mapping $[0,1] : \phi(\lambda) \to f((1-\lambda)\phi(x)+\lambda\phi(y))$ is ϕ_h Lebesgue integrable on $[a,b]_{I_{\mathbb{T}}} \times [c,d]_{I_{\mathbb{T}}}$. Hence, integrating inequality (3.7) on the rectangle $[a,b]_{I_{\mathbb{T}}} \times [c,d]_{I_{\mathbb{T}}}$ over $\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$ gives

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left(f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)+f\left(\frac{p\phi(x)+q\phi(y)}{p+q}\right)\right)\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$$
$$\leq \frac{f(\phi(x)+f(\phi(y))}{(b-a)(d-c)}\int_a^b\int_c^d \left(\frac{\frac{p}{p+q}}{h\left(\frac{p}{p+q}\right)}\right)^s + \left(\frac{\frac{q}{p+q}}{h\left(\frac{q}{p+q}\right)}\right)^s \diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p,$$

which is the second inequality of Theorem 3.2.
For the first part, we have, from the ϕ_h-convexity of f that for any $\phi(z_1), \phi(z_2) \in E,\ \lambda = \frac{1}{2}, s = 1,\ h(\frac{1}{2}) \leq 1$. Thus,

$$f\left(\frac{\phi(z_1) + \phi(z_2)}{2}\right) \leq \frac{1}{2h(\frac{1}{2})}(f(\phi(z_1)) + f(\phi(z_2))). \tag{3.8}$$

Choosing $\phi(z_1) = \frac{p\phi(x) + q\phi(y)}{p+q}$ and $\phi(z_2) = \frac{q\phi(x) + p\phi(y)}{p+q}$ in (3.8), then for any $p, q \geq 0,\ p + q > 0$, we get

$$f\left(\frac{\phi(x) + \phi(y)}{2}\right) \leq \frac{1}{2h(\frac{1}{2})}\left(f\left(\frac{p\phi(x) + q\phi(y)}{p+q}\right) + f\left(\frac{q\phi(x) + p\phi(y)}{p+q}\right)\right). \tag{3.9}$$

Integrating (3.9) on the rectangle $[a, b]_{I_{\mathbb{T}}} \times [c, d]_{I_{\mathbb{T}}}$ over $\diamond_{(\phi_h)_1} q \diamond_{(\phi_h)_2} p$ gives the first part of Theorem 3.2.

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