Dominated and Bounded Convergence Results of Sequential Henstock Stieltjes Integral in Real Valued Space

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Abstract: *In this paper, we prove the dominated and bounded convergence results for real-valued Sequential Henstock Stieltjes integral.*

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1 Introduction and Preliminaries

In 1955 and 1957 respectively, R. Henstock and J. Kursweil independently gave a Riemann-type integral called the Henstock integral. It is a kind of non-absolute integral which includes the Riemann, Improper Riemann, Newton and Lebesgue integral. Many authors have studied Henstock integral which is now known as the Kursweil-Henstock integral, since Kursweil defined the same integral though they went different ways in developing and applying the theory. For simplicity, we shall refer the Kursweil-Henstock integral and its general form as the Henstock integral in this paper, see [1]-[6]. It is well known that the Henstock integral is equivalent to the Denjoy integral, Perron integral and Denjoy-Perron integrals. The equivalence of the Henstock integral and Sequential Henstock integral has been discussed in [5]. In this paper, we prove the dominated and bounded convergence theorems for the Sequential Henstock Stieltjes integral.

The symbols used in this paper are as follows: $\mathbb{R}$ and $\mathbb{N}$ for a set of real and natural numbers respectively, $\{\delta_n(x)\}_{n=1}^{\infty}$ as set of gauge functions of $x \in [a, b]$, and P_n as set of partitions of subintervals of a compact interval $[a, b]$ for $n = 1, 2, 3, \cdots$

The following useful definitions of Sequential Henstock integral are needed.

Definition 1.1. **[5]**(Sequential Henstock Integral). A function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable on $[a, b]$ if there exists a number $\alpha \in \mathbb{R}$ and a sequence of gauge functions $\delta_\mu(x) \in \{\delta_n(x)\}_{n=1}^{\infty}$ such that for each $\delta_n(x)$-fine tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ we have

$$U(f, P_n) = \sum_{i=1}^{n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \to \alpha, n \to \infty,$$

i.e. $\alpha = \int_{[a,b]} f$.

Sequel to Definition 1.1, we give a new definition as follows for Sequential Henstock Stieltjes integral.

Definition 1.2.(Sequential Henstock Stieltjes Integral). A function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock Stieltjes integrable on $[a, b]$ with respect to an increasing function $g : [a, b] \to \mathbb{R}$ if there exists a number $\alpha \in \mathbb{R}$ such that for $\varepsilon > 0$ there exists a sequence of positive functions $\delta_\mu(x) \in \{\delta_n(x)\}_{n=1}^{\infty}$ such that $\mu \leq n$ and for every $\delta_n(x)$-fine tagged partitions $P_n = \{(u_{(i-1)_n}), u_{i_n}), t_{i_n}\}$ and $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ we have

$$U(f, g, P_n) = \sum_{i=1}^{n} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))] \to \alpha, \text{ as } n \to \infty.$$

We say that $\alpha \in \mathbb{R}$ is the Sequential Henstock Steiltjes integral of f with respect to g on $[a,b]$ with $\alpha = \int_a^b fdg$

Definition 1.3. **[5]**(Uniform Integrability). Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of functions for $k \in \mathbb{N}$ and a function $g : [a,b] \to \mathbb{R}$. Then f_k is uniformly Sequential Henstock Stieltjes integrable with respect to g on $[a,b]$ if

i. the integral $\int_a^b f_k dg$ exists for each $k \in \mathbb{N}$,

ii. for $\varepsilon > 0$ there exists a sequence of gauges $\delta_\mu(x) = sup_{\mu\in\mathbb{N}}\{\delta_n(x)\}_n^\infty$ and $n \geq \mathbb{N}$ on $[a,b]$ such that the inequality

$$\left| \int_a^b f_k dg - U(f_k, dg, P_n) \right| < \varepsilon, \tag{1.1}$$

holds for each $\delta_n(x)$- fine partition P_n of $[a,b]$ for $n = 1, 2, 3, \cdots$.

Now, we state the following lemma which was proved in [4], and useful in the proof of our main theorems.

Lemma 1.1. **[4]** Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of functions and $g : [a,b] \to \mathbb{R}$ be an increasing function satisfying the following conditions:

i. The integral $\int_a^b f_k dg$ exists for each $k \in \mathbb{N}$.

ii. $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in [a,b]$,

iii. There exist $\beta, \gamma \in \mathbb{R}$ such that the inequalities

$$\beta \leq \sum_{i=1}^{n} \int_{u_{(i-1)n}}^{u_{i_n}} f_{s_i} dg \leq \gamma,$$

holds for all partitions P_n of $[a,b]$ and all $s_1, s_2, \cdots, s_n \in \mathbb{N}$. Then, f_k is uniformly Sequential Henstock Stieltjes integrable with respect to g. Then, the integral $\int_a^b fdg$ exists and

$$\lim_{k\to\infty} \int_a^b f_k dg = \int_a^b fdg.$$

Moreover, we have

$$\lim_{k\to\infty} \left(\sup_{t\in[a,b]} \left| \int_a^t f_k dg - \int_a^t fdg \right| \right) = 0.$$

2 Main Results

We state and give the proof of theorems in our main results.

Theorem 2.1. (Dominated Convergence). Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of functions which is Sequential Henstock Stieltjes integrable with respect to an increasing function $g : [a,b] \to \mathbb{R}$ and is satisfying the following conditions:

i. The integral $\int_a^b f_k dg$ exists for each $k \in \mathbb{N}$,

ii. $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for all $x \in [a,b]$,

iii. There exist Sequential Henstock Stieltjes integrable functions $h_i, h_2 : [a,b] \to \mathbb{R}$ such that $\int_a^b h_1 dg$ and $\int_a^b h_2 dg$ exist, where $h_1 \leq f_k \leq h_2$ on $[a,b]$ for each $k \in \mathbb{N}$.

Then, f_k is uniformly Sequential Henstock Stieltjes integrable with respect to g. Then, the integral $\int_a^b fdg$ exists and

$$\lim_{k\to\infty}\int_a^b f_k dg = \int_a^b fdg.$$

Moreover, we have

$$\lim_{k\to\infty}\left(\sup_{t\in[a,b]}\left|\int_a^t f_k dg - \int_a^t fdg\right|\right) = 0.$$

Proof. From Lemma 1.1, following from condition (iii). Let $\beta = U(h_1, dg, P_n)$ and $\gamma = U(h_2, dg, P_n)$.
If P_n is a sequence of divisions on $[a,b]$ and $s_1, s_2, \cdots, s_n \in \mathbb{N}$. i.e.
$P_n = (u_{(i-1)_n}, u_{i_n}) \in [a,b]$ for $n = 1, 2, 3, \cdots$, then

$$\beta = U(h_1, dg, P_n) \le \sum_{i=1}^{n}\int_{u_{(i-1)_n}}^{u_{i_n}} f_{s_i} dg \le U(h_2, dg, P_n) = \gamma.$$

This shows that the assumption of Lemma 1.1 is satisfied and the proof is complete.

Theorem 2.2.(Bounded Convergence). Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of function and $g : [a,b] \to \mathbb{R}$ be an increasing function satisfying the following conditions:

i. The integral $\int_a^b f_k dg$ exists for each $k \in \mathbb{N}$

ii. $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in [a,b]$,

iii. There exist a constant $M \ge 0$ such that $|f_k(x)| \le M$ for all $k \in \mathbb{N}$ and $x \in [a,b]$. Then, f_k is uniformly Sequential Henstock Stieltjes integrable with respect to g, the integral $\int_a^b fdg$ exists and

$$\lim_{k\to\infty}\int_a^b f_k dg = \int_a^b fdg.$$

Moreover, we have

$$\lim_{k\to\infty}\sup_{t\in[a,b]}\left(\left|\int_a^t f_n dg - \int_a^t fdg\right|\right) = 0.$$

Proof. If P_n is a sequence of partitions on $[a,b]$ and $s_1, s_2, \cdots, s_n \in \mathbb{N}$. where $P_n = (u_{(i-1)_n}, u_{i_n}) \in [a,b]$ for $n = 1, 2, 3, \cdots$, then

$$\left|\sum_{i=1}^{n}\int_{u_{(i-1)_n}}^{u_{i_n}} f_{s_i} dg\right| \le \sum_{i=1}^{n}\left|\int_{u_{(i-1)_n}}^{u_{i_n}} f_{s_i} dg\right| \le \sum_{i=1}^{n} Mvar_{u(i-1)_n}^{u_{i_n}} g = Mvar_a^b g.$$

by the assumptions of Lemma 1.1, which is also satisfied with

$$-Mvar_a^b g \le \left|\sum_{i=1}^{n}\int_{u_{(i-1)_n}}^{u_{i_n}} f_k dg\right| \le Mvar_a^b g.$$

This completes the proof.

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