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Understanding Osculating Circle and Sphere: Modeling with Mathematica

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Abstract: *This research paper investigates how a digital tool Mathematica supports students' comprehension of mathematical concepts, specifically the osculating circle and osculating sphere in differential geometry. This research is grounded on Technological Pedagogical Content Knowledge (TPACK) framework, to answer a research question "how do students understand the dynamics of the osculating circle and osculating sphere using mathematical simulation?" I employed a qualitative phenomenography methodology, a branch of phenomenology among thirteen students from the Central Department of Education (CDED), Tribhuvan University. I gathered data through semi-structured interviews, direct observations, and recordings of students' interactions with Mathematica-based dynamic and interactive simulations. I analyzed the data through generation of transcriptions, meaning condensation, categorization of structural aspects, and generation of thematic outcomes reflecting on students' understanding of osculating concepts and processes. The study found that Mathematica-coded dynamic visualization enhanced students' conceptualization of osculating process linking formal definitions with mental images. Particularly, it helped in visualizing how the osculating circle and sphere evolve along a curve. The simulations helped bridge the gaps between concept-definition and concept-image. The study suggests that digital pedagogy aligned with the TPACK framework can promote students' conceptual understanding of advanced mathematics.*

Keywords: *Osculating circle, Osculating sphere, Mathematica, Phenomenography, Differential geometry*

Introduction

Osculating circle and sphere is a fundamental concept in differential geometry, which is a local geometric property of curvature and torsion, means at each point in a space curve, it can be classified into various types (Cheng et al., 2025). In this study, we focus on Frenet frame to understand osculating circle and osculating sphere.

The Frenet frame is a matrix equation with three unit vectors that are mutually orthogonal: the tangent vector, which is along the direction of curve; the normal vector, in the direction towards the center of curvature; and the binormal vector, perpendicular to osculating plane (Cheng et al., 2025). For example, if we consider a space curve $C : I \subset R \rightarrow E^3$, which is continuous at least up to its fourth-order derivatives, then Serret–Frenet equation is given as:

$$\begin{bmatrix} \vec{t}' \\ \vec{n}' \\ \vec{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}$$

where κ represents the curvature, and τ represents the torsion of the curve, and $'$ denotes the derivative with respect to the arc parameters (Cheng et al., 2025).

Differential geometry is a higher mathematics, which is abstract by its nature. However, there are number of researches that have demonstrated that mathematical abstraction can be materialized and visualized. For example, in a work by Dhakal (2019), Sulastri et al (2021) mentioned that

> "…mathematical abstraction can be materialized using examples, using graphical images, using analogies, and using metaphors, researchers have tried to materialize mathematical abstraction. There are also researches that have tried conceptual and ideational reasoning, metalinguistic and mathematical reasoning, procedural and conceptual reasoning, syntactic and semantic reasoning, cognitive and metacognitive reasoning to materialize mathematical abstractions." Dhakal (2019)

However, students are still lacking mathematical competency, as expected, which the researcher experienced while tutoring a course "Differential Geometry" in university classes of Masters Level.

The learners of these days are experiencing AI (Artificial Intelligence) and digital technologies. Therefore, students wants to consolidate their comprehension of mathematical concepts through digital tools, which are also discussed in a number of research papers (Assencio et al., 2015; Dhakal, 2023; Dhakal & Sharma, 2016). As a researcher, I also see that there is a space to do pedagogical exercise with digital learning object to materialize mathematical abstraction in 'osculating circle and osculating sphere' while teaching master-level students. Literature discussed that these kinds of digital tools have been integrated in pedagogical activities (Kim & Md-Ali, 2017; Onwu Iji & Abah, 2018). In this context, I conducted this study to support students' understanding of advanced mathematics, to contribute to the literature, and to justify if abstract mathematical concepts can be materialized by using digital tools.

In this study, Worlfram Mathematica was selected as the primary dynamic visualization and computation tool because it has both the capacity to dynamically represent complex mathematical equations with precision, and real-time interactivity to visualize it. There are number of softwares related to mathematics like MatLab, Maple, SageMath, Python, and GeoGebra, to list a few. However, I (as a researcher) am more friendly with Mathematica language to manipulate. The department, where I teach, have the education version of Mathematica licence. This licence helps us integrate both symbolic computation and numerical analysis with dynamic visualization in a single environment. Therefore, I used mathematics as primary coding tool because it also supports custom coding. Also, using Wolfram Cloud online account/environment, Mathematica allowed students to observe and interact with the custom built applets with free user account. Then students can experience the instantaneous simulation of osculating circle and sphere as they evolve along a curve. Therefore, I chose Mathematica as an appropriate tool in this research within the TPACK and Connectivism learning framework.

Problem Statement

I am a faculty member of mathematics education teaching at University Campus, Central Department of Mathematics Education for 16 years. I have experienced that students frequently struggle with the abstract nature of the dynamics of osculating circle and osculating sphere during my teaching. I see that student often prefer to memorize formulas rather than grasping their underlying meaning. As I see, there is a foundational gap in the students' ability to bridge the concept-image and concept-definition, these students often find it challenging to the dynamics of osculating circle and osculating sphere and their inter relationships. For example, while teaching at master-level students at the university, I have realized that students incorrectly assumed that great circles of osculating sphere are the osculating circle, which it is not, in general, true. Seeing this learning gap, and to address these learning issues, I thought that it is pedagogically good to integrate digital tools to support my instructional strategies that can influence and shape students' cognitive structures to make sense of abstract concepts such as osculating circles and spheres. These enhanaced congnitive structures can bridge the concept-image and concept-definition of osculating circle and osculating sphere.

Research Question

The study investigates evolution and refinement of students' understanding of mathematical concept-images and defintions as they interact with dynamic visualizations. The central question to guide this study is:

> *How do students understand the dynamics of osculating circle and osculating sphere using mathematical simulation?*

Theoretical Framework

In this study, I used 'Technological Pedagogical Content Knowledge (TPACK)' as a theoretical guiding lens (Herring et al., 2014; Koehler et al., 2014; Mishra et al., 2007) to uncover students' understanding of dynamics of osculating circle and osculating sphere. This framework helps to understand how the integration of technological knowledge, pedagogical knowledge and content knowledge helps uncover conceptual meaning of osculating circle and osculating sphere. This research work is also guided by connectivism as learning theory, which is also called the learning theory of digital age (Herring et al., 2014; Mukhlis et al., 2024). Therefore, TPACK and Connectivism are considered as the theoretical framework for this study.

Method

The purpose of this research is to provide a comprehensive information on how Mathematica-coded dynamic applets influence students' understanding of mathematical concepts. So, a qualitative phenomenography (a part of phenomenology (Creswell et al., 2018; Denzin & Lincoln, 2018)) study methodology was used to understand how university students comprehend mathematical concepts: the dynamics of osculating circle and osculating sphere utilizing Mathematica coded visualizations.

Phenomenography is a research approach, which is also discussed in a literature (Creswell et al., 2018; Denzin & Lincoln, 2018; Steffe & Nesher, 2009). It was originated around 1970s with an interest to study 'how student as learner experience the action of doings, for example reading texts, solving problems, and listening to lectures'. In this study context, this user/learner experience is manifested using digital tools. So, I deployed this phenomenography research approach to consider human experience as 'subject of understanding'(Steffe & Nesher, 2009).

Phenomenography, a part of phenomenology, adopts a non-dualistic ontological perspective on human experience. Within this framework, the object and the subject are not viewed as distinct entities. Instead, experience is understood as a relational phenomenon—what the subject perceives is inherently tied to the object being perceived (Creswell et al., 2018; Denzin & Lincoln, 2018; Steffe & Nesher, 2009). In the context of learning, phenomenology means that a learners may encounter with a problem is not with a fixed, objective "problem" but rather with a problem as experienced during the learning process. For the phenomenography, there is no such thing as a problem in a fixed and isolation state, but it is always interpreted through the lens of the learner's experience.

A common assumption adopted within this phenomenography is that student may go through the same text, same problem, same lecture, however, they are equipped with the learning experience differently, the internalized experience can be different. This insight identified the driving force behind this study methodology "phenomenographic research" to better understand the nature of how learners (students) experience and understand the content 'osculating circle and osculating sphere' during my teaching.

The study is based on thirteen students (5 femailes and 8 males), studying a course "Differential Geometry" during the academic year 2024, in the Central Department of Education, Tribhuvan University. In this study, I have, as a researcher, used multiple data sources, which is good for data triangulation (Creswell et al., 2018; Denzin & Lincoln, 2018). I have used interview guideline, observation checklist, and students inform interactions as data collection tools. These tools are well-suited to answer the 'how part' of research question, which is how do student understand the dynamics of osculating circle and osculating sphere.

Results

In this study, the data were collected from 13 students of M.Ed. in mathematics education during 2024 cohort in Central Department of Education, Tribhuvan University. During the content delivery, initially, it was found that many students' concept images of osculating circles often remained abstract or solely based on algebraic formulas. As mentioned in the textbook by Pundir et all and Koirala and Dhakal and others (Koirala & Dhakal, 2024; Liu et al., 2023; Pundir et al., 2021; Saglam et al., 2004), if a curve C: $\vec{r} = \vec{r}(s)$ be defined with neighboring points P, Q, R on C, then osculating circle at P is defined as limiting position of circle through P, Q, R as Q, R approach to P.

In mathematical notation, it is given by

$$\text{Osculating circle at P} = \lim_{Q,R \to P} \text{circle PQR}$$

In this representation, the center of osculating circle is given by

$$c = \vec{r} + \rho\vec{n}$$

In the preliminary study, it was experienced that, students were not confident about the fact that 'osculating circle lies in the osculating plane', and another fact 'osculating circle has, in general, three points of contact with the curve', and also 'the tangent of osculating circle and space curve are the same' at the point of contact. From this initial experience, it was found that students were struggling with the cognitive visualization of the dynamics of 'osculating circle'.

To visualize the concept-image of osculating circle, a regular space curve is chosen, which is

$$C: \vec{r} = (t\cos t,\ t\sin t,\ t) \qquad -3\pi < t < 3\pi$$

In the textbook by Pundir et al., and Koirala and Dhakal (Koirala & Dhakal, 2024; Pundir et al., 2021), unit tangent vector of the space curve is defined by

$$\vec{t} = \vec{r}'$$

It is also mentioned that, if parameter t is used instead of the arc length s, the unit tangent vector is defined as

$$\vec{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

The students were expected to have the foundation knowledge to understand the dynamics of osculating circle and osculating sphere. One of these foundational knowledge are considered as fundamental vectors. The fundamental vector of space curve, which have three mutually orthogonal unit vectors are defined by

$$\vec{t}\ , \vec{n}\ , \text{and } \vec{b}$$

which exist at each point on three-dimensional space curve. These vectors move along the curve, so are called moving trihedrons. These vectors are mutually perpendicular, so are also called orthogonal triads. As mentioned in textbook by Pundir et all and Koirala and Dhakal (Koirala & Dhakal, 2024; Pundir et al., 2021),

> "*…the three fundamental unit vectors $\vec{t}$, $\vec{n}$, and $\vec{b}$ move in a space curve along the positive directions and satisfy the following relations*
>
> 1. $\vec{t}.\vec{t} = 1,\ \vec{n}.\vec{n} = 1,\ \vec{b}.\vec{b} = 1$
> 2. $\vec{t}.\vec{n} = 0,\ \vec{n}.\vec{b} = 0,\ \vec{b}.\vec{t} = 0$
> 3. $\vec{t} \times \vec{t} = 0,\ \vec{n} \times \vec{n} = 0,\ \vec{b} \times \vec{b} = 0$
> 4. $\vec{t} \times \vec{n} = \vec{b},\ \vec{n} \times \vec{b} = \vec{t},\ \vec{b} \times \vec{t} = \vec{n}$"

Before students compute the equation of osculating circle and osculating sphere, they are expected to have sound understanding of computational formula of curvature and torsion. As mentioned in the textbook by Pundir et al. and Koirala and Dhakal (Koirala & Dhakal, 2024; Pundir et al., 2021), the computational formula of curvature and torsion of space curve are described by two theorems as below.

1. $\kappa = |\vec{r}' \times \vec{r}''|$ and $\tau = \frac{[\vec{r}', \vec{r}'', \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2}$ with $[\vec{r}', \vec{r}'', \vec{r}'''] = \kappa^2\tau$ for parameter s, as arc length.
2. $\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$ and $= \frac{[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$ with $[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = \kappa^2\tau\dot{s}^6$, for arbitrary parameter t.

During the classroom teaching, students were exposed to the mathematical equation of center of osculating circle, which is

$$c = \vec{r} + \rho\vec{n}$$

where rho (ρ) is the radius of osculating circle, whose reciprocal is the curvature of the space curve, related by the formula

$$\kappa = \frac{1}{\rho}, \rho = \frac{1}{\kappa}, \kappa\rho = 1$$

To help students understand these concepts better, the computation of fundamental vectors, curvature, the radius and center of osculating circle of space curve are drawn through a custom equation as below.

$$\vec{r} = (t \cos t, t \sin t, t)$$

The computation of fundamental vectors, curvature, the radius, and center of osculating circle are coded in Mathematica as below.

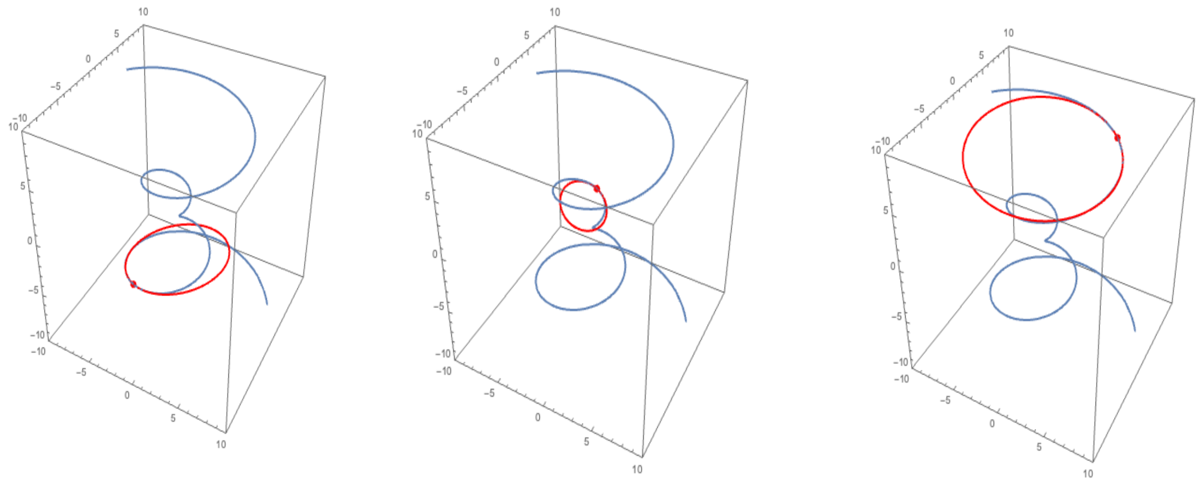

Figure 1: osculating circle and curvature

WOLFRAM CLOUD

(unnamed)

```
In[65]:= r[t_] := {t Cos[t], t Sin[t], t}
uT[r_][t_] := r'[t] / Sqrt[r'[t] . r'[t]]
uN[r_][t_] := uT[r]'[t] / Sqrt[uT[r]'[t] . uT[r]'[t]]
uB[r_][t_] := Cross[uT[r][t], uN[r][t]]
a[r_][t_] := (Sqrt[r'[t] . r'[t]])^3 / Sqrt[Cross[r'[t], r''[t]] . Cross[r'[t], r''[t]]]
c[r_][t_] := r[t] + a[r][t] * uN[r][t]
Manipulate[
 Module[{ce = c[r][t], ra = a[r][t], osc},
  osc = ParametricPlot3D[
   ce + ra * Cos[u] * uT[r][t] + ra * Sin[u] * uN[r][t], {u, 0, 2 Pi},
   PlotStyle → Directive[Red, Thick]];
  Show[
   ParametricPlot3D[r[s], {s, -9, 9}, PlotStyle → DarkBlue],
   Graphics3D[{Red, PointSize[Large], Point[r[t]]}],
   osc,
   PlotRange → {{-25, 25}, {-25, 25}, {-10, 10}},
   Boxed → True
  ]
 ],
 {{t, 2, "parameter t"}, -8, 8}
]
```

Using this code in web version of Mathematica through Wolfram Cloud demonstrated that intersection dynamics of osculating circle and osculating sphere. However, the Wolfram Cloud is licenced software, so to make it available in open source in cross platform, anytime anywhere, any platform, these code were re-deployed in the researcher webpage https://www.bedprasaddhakal.com.np/2024/07/osculating-circle-and-osculating-sphere.html using Javascript for smooth interaction through any device and even without Mathematica licence (or user account). This code allows users to slide parameter, dynamically updating and plotting the osculating circle, its center and visualization of the processes, the equivalent snapshot of the visualization are given in Figure 1.

The dynamic visualization allowed students to move a point along the curve and observe the osculating circle "kissing" the curve and continuously adapting its curvature, this dynamic visualization helped students develop an intuitive "concept-image" of best approximation and local curvature of osculating circle. This is justified by a quotation of two students as follows:

> *"...before, I thought curvature was just one number for a whole curve, now I see it's different at every point, and the circle shows how much it bends right there, based on this visualization, I realized that osculating circles are of different sizes, when they are defined along the curve".*

> *"... I realized that, osculating circle is the curve's most precise local circular approximation, capturing curve bends at a single point. It lies in the osculating plane, making it the circle that 'kisses' the curve. By interacting with dynamic visualization, I see the instantaneous curvature both geometrically and algebrically..."*

In the classroom observation, it was found the students were struggling to visualize the concept of osculating sphere. This concept of osculating sphere, in a textbook by Pundir et all and Koirala & Dhakal (Koirala & Dhakal, 2024; Pundir et al., 2021), is defined as, if C: $\vec{r} = \vec{r}(s)$ be a space curve with P,Q,R,S as the four neighboring points on C, then the osculating sphere at P is defined as limiting position of sphere through P,Q,R,S as Q,R,S approach P, which is given as

Osculating sphere at P = $\lim_{Q,R,S \to P}$ sphere PQR

In addition, students were struggling to understand that fact "osculating sphere at P has four points of contact with the curve at P". Even if student compute the equation of the center of osculating sphere, which is

$$c = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

and the radius of osculating sphere be

$$a = \sqrt{\rho^2 + (\sigma\rho')^2}$$

where sigma (σ) and tau (τ) are reciprocal to each other, related by the formula

$$\sigma = \frac{1}{\tau}, \tau = \frac{1}{\sigma}, \sigma\tau = 1$$

It was found that visualization was essential to maintain the synchronization between the concept-definition and concept-image. Therefore, the computation of fundamental vectors, curvature, torsion, the radius, and the center of osculating sphere were computed through a custom curve, given as

$\vec{r}$=(t cos t, t sin t, t)

The computational code of fundamental vectors, curvature, torsion, the radius and center of osculating sphere, which were coded in Mathematica are given below.

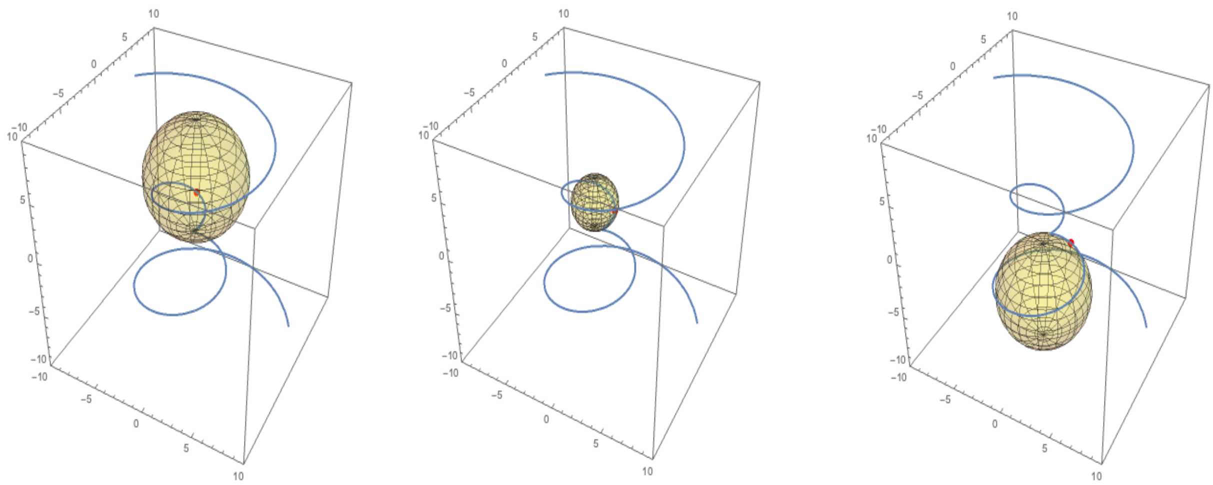

Figure 2: osculating sphere visualization

WOLFRAM CLOUD

(unnamed)

```
In[72]:= r[t_] := {t Cos[t], t Sin[t], t}
uT[r_][t_] := r'[t] / Sqrt[r'[t] . r'[t]]
uN[r_][t_] := uT[r]'[t] / Sqrt[uT[r]'[t] . uT[r]'[t]]
uB[r_][t_] := Cross[uT[r][t], uN[r][t]]
rho[r_][t_] := (Sqrt[r'[t] . r'[t]])^3 / Sqrt[Cross[r'[t], r''[t]] . Cross[r'[t], r''[t]]]
tau[r_][t_] := (Cross[r'[t], r''[t]] . r'''[t]) / (Cross[r'[t], r''[t]] . Cross[r'[t], r''[t]])
c1[r_][t_] := r[t] + rho[r][t] * uN[r][t] + (Derivative[1][rho[r]][t] / tau[r][t]) * uB[r][t]
a1[r_][t_] := Sqrt[(rho[r][t])^2 + (Derivative[1][rho[r]][t] / tau[r][t])^2]
Manipulate[
 Module[{ce1 = c1[r][t], ra1 = a1[r][t], oss},
  oss = ParametricPlot3D[
   ce1 + ra1 {Sin[u] Cos[v], Sin[u] Sin[v], Cos[u]}, {u, 0, Pi}, {v, 0, 2 Pi},
   PlotStyle → Directive[Blue, Opacity[0.2]], Mesh → None, PlotPoints → 15];
  Show[ ParametricPlot3D[r[s], {s, -9, 9}, PlotStyle → DarkGray],
   Graphics3D[{Red, PointSize[Large], Point[r[t]]}],
   oss,
   PlotRange → {{-20, 20}, {-20, 20}, {-10, 10}},
   Boxed → True
  ]
 ],
 {{t, 2, "parameter t"}, -2, 2}
]
```

Using these codes in web version of Mathematica through Wolfram Cloud demonstrated the underlying structures of osculating sphere. Also, these codes were re-deployed in the researcher webpage https://www.bedprasaddhakal.com.np/2024/07/osculating-circle-and-osculating-sphere.html using Javascript. These codes allow users to slide parameter, dynamically updating and plotting the osculating sphere, its center and visualization of the processes, the equivalent snapshot of the visualization are given in Figure 2.

The concept of the osculating sphere, through visualization in Figure 2, enabled students to interactively visualize the sphere's contact with the space curve, its dynamically changing center, and its radius within a three-dimensional environment. Students could rotate, zoom, and pan the 3D models, allowing them to explore the curve and its approximating sphere from multiple perspectives. This is justified by a quotation by two students:

> *"….Through the dynamic visualization, I see that, osculating sphere generalizes the idea closest spherical approximation to a space curve at a given point. I see that its size are different based on the curvature and torsion, so its radius are also different in different point."*

> *"….ahh, when I drag the slider in this visualization, I see that the locus of center of osculating sphere is a different curve, which is traced by the center of osculating circle. Now I understand that the equation of center of osculating circle."*

From the quotation, it is seen that the dynamic visualization allowed students to understand the order of four point contact and locus of center of osculating sphere as:

$$c = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

Also, the dynamics of osculating circle and osculating sphere, both are coded together and visualized as below.

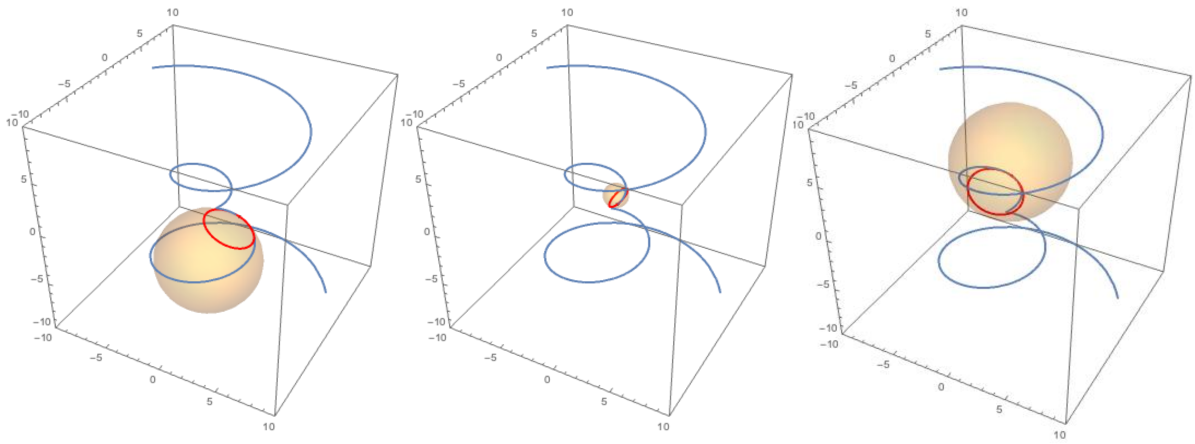

Figure 3: osculating sphere visualization

WOLFRAM CLOUD

(unnamed)

```
In[40]:= r[t_] := {t Cos[t], t Sin[t], t}
uT[r_][t_] := r'[t] / Sqrt[r'[t] . r'[t]]
uN[r_][t_] := uT[r]'[t] / Sqrt[uT[r]'[t] . uT[r]'[t]]
uB[r_][t_] := Cross[uT[r][t], uN[r][t]]
rho[r_][t_] := (Sqrt[r'[t] . r'[t]])^3 / Sqrt[Cross[r'[t], r''[t]] . Cross[r'[t], r''[t]]]
a[r_][t_] := rho[r][t]
c[r_][t_] := r[t] + a[r][t] * uN[r][t]
tau[r_][t_] := (Cross[r'[t], r''[t]] . r'''[t]) / (Cross[r'[t], r''[t]] . Cross[r'[t], r''[t]])
cl[r_][t_] := r[t] + rho[r][t] * uN[r][t] + (Derivative[1][rho[r]][t] / tau[r][t]) * uB[r][t]
al[r_][t_] := Sqrt[(rho[r][t])^2 + (Derivative[1][rho[r]][t] / tau[r][t])^2]
Manipulate[
 Module[{cel = cl[r][t], ral = al[r][t], oss, ce = c[r][t], ra = a[r][t], osc},
  osc = ParametricPlot3D[ce + ra * Cos[u] * uT[r][t] + ra * Sin[u] * uN[r][t], {u, 0, 2 Pi}, PlotStyle → Directive[Red, Thick]];
  oss = ParametricPlot3D[cel + ral {Sin[u] Cos[v], Sin[u] Sin[v], Cos[u]}, {u, 0, Pi}, {v, 0, 2 Pi}, PlotStyle → Directive[Blue, Opacity[0.2]], Mesh → None, PlotPoints → 15];
  Show[
   ParametricPlot3D[r[s], {s, -9, 9}, PlotStyle → DarkGray],
   Graphics3D[{Black, PointSize[Large], Point[r[t]]}],
   osc,
   oss,
   PlotRange → {{-15, 15}, {-15, 15}, {-10, 10}},
   Boxed → True
  ]
 ],
 {{t, 2, "parameter t"}, -2, 2}
]
```

This conceptual code illustrated above help students to understand how the osculating sphere, with its center and radius were dynamically calculated based on the curve's properties. With this visualization, one student articulated that:

> *"…, oh, I see, I finally get that the osculating circle isn't just great circle, it can be any circle within the sphere, like the section of the osculating sphere by a plane, therefore, I understand that the center of osculating circle and osculating sphere can be different.".*

These statements indicated that the students had a good understanding of the dynamics between osculating circle and osculating sphere, which was difficult to convey without such 3D dynamic interaction. Therefore, the dynamic and interactive applet as demonstrated in Figure 3, the dynamic and interactive nature of osculating circle and osculating sphere, "any circle on sphere can be osculating circle, not only great circle", which demonstrated an improved conceptual understanding.

Findings and Discussion

The findings of this qualitative phenomenography indicate that dynamic visualization, facilitated by Mathematica's dynamic visualization, enhances university students' understanding of differential geometry concepts on the dynamics of osculating circle and osculating sphere. It is realized that, this technology enhanced pedagogy helped students understand abstract concepts beyond superficial rote memorization, leading to a visualized, and clear conceptual comprehension, such kind of findings are consistent with the literature (Dhakal, 2023; Herring et al., 2014; Kim & Md-Ali, 2017). It was found that visualization facilitated the development of accurate, coherent, and dynamics of inter-relationship of osculating circle and osculating sphere through "concept-images", similar findings were also discussed in the literature (Sulastri et al., 2021). It was found that student realized that, "any section of the osculation sphere can be osculating circle, not only a section of great circle", which is also the core value of digital pedagogy as suggested in the TPACK framework (Herring et al., 2014; Koehler et al., 2014; Mishra et al., 2007) and Connectivism (Mukhlis et al., 2024).

In this research, the concept-image of osculating circle and osculating sphere were formed by students' active participation with digital tools, as mentioned by TPACK and connectivism learning theory (Herring et al., 2014; Mukhlis et al., 2024), which focused that active participation with digital tools is essential in meaningful learning, which is a kind of hallmark of TPACK and connectivist learning theory (Herring et al., 2014; Mukhlis et al., 2024). With this conceptualization, the study found that students build knowledge through direct interaction with the digital simulations, which is also an example of active participation.

A considerable number of research consistently indicated that dynamic visual and computational tools like Computer Algebra Software (CAS) , the Mathematica coded applets of osculating circle and osculating sphere positively impact students' understanding and therefore their mathematics achievement, especially in the area of higher-level thinking skills and deeper conceptual understanding (Assencio et al., 2015; Dhakal & Sharma, 2016; Kim & Md-Ali, 2017; Onwu Iji & Abah, 2018). In a similar manner, in this research, the Mathematica coded dynamic and interactive applets of osculating circle and osculating sphere have provided students a cognitive ability to compute, manipulate and observe the changes in the dynamics of osculating circle and osculating sphere dynamically to understand the concept of osculating circle and osculating sphere and their inter-relationships. This is found that, the dynamic simulation fostered student with a comprehensive understanding and a deeper analytical approach, that can be instrumental in problem-solving, this kind of pedagogical thoughtfulness are also the part of TPACK framework (Herring et al., 2014; Koehler et al., 2014; Mishra et al., 2007) and Connectivism (Mukhlis et al., 2024). Therefore, it was found that students understand the dynamics of osculating circle and osculating sphere through concept-image while using Mathematica coded dynamic and interactive mathematical simulations.

Conclusion

Based on the data analysis and interpretation of results comparing with literature, it is concluded that this study contributes valuable empirical insights to the growing body of literature on the pedagogical value of dynamic and interactive visualization tools in advanced mathematics education. The findings support the transformative and potential of thoughtful integration of dynamic and interactive visualization tools in mathematics, such as those developed in Mathematica. Such technological integration with content and pedagogy can foster deeper conceptual and computational understanding in mathematics while learning complex ideas through the formation of "concrete type of concept-image" with coding and simulations of abstract concepts to visualize the processes.

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