

<https://doi.org/10.3126/jme.v6i1.95911>

## An Innovative Activity for Learning Groups of Symmetries in Abstract Algebra

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### Article Info:

Received: March 4, 2025 | Revised: August 11, 2025 | Accepted: January 18, 2026

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**Abstract:** *In many cases, group theory is the first axiomatic mathematics taught, and so is abstract algebra. Thus, the way teachers teach and students learn group theory is considered the foundation with respect to the development of mathematical abilities and attitudes towards mathematics at the higher level. It has been realised that the permutation group ( $S_3$ ) is the most common example throughout group theory, and students need to remember elements and their compositions. Review of different solution initiatives with practical perspectives, such as inclusion of seminar, visual and analytical approach, collaborative approach with Interactive Set Language ( ISETL) and penny moving, etc., and theoretical perspectives such as inquiry-based mathematics education, linking informal knowledge to formal mathematics and constructivist learning became helpful in this innovation. This innovative hands-on activity was developed by following a design thinking approach with a view to bringing changes in the learning of a group of symmetries at the undergraduate level. This hands-on activity has two parts: one on the regular polygon (equilateral triangle and square) and permutation using these polygons, including five worksheets to facilitate learning.*

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**Keywords:** *Group theory, symmetric group, hands-on activity, permutation groups.*

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### Context

In a good lecture, students are busy making notes and trying to understand the ways to remember how those symbols are used. Students can't get a chance to interact with friends or with other tangible objects like triangles and squares. They have to remember the symbols such as  $r$ ,  $r^2$ ,  $r^3 = e$ ,  $f$ ,  $rf$ ,  $r^2f$  and their operations in order to learn the group of symmetries. They don't get time to think about reflectional and rotational symmetries, and as a result, they cannot develop their own mathematical understanding. This paper tries to discuss an innovative way for students to use hands-on activities in the process of learning groups of symmetries and permutation groups, following inquiry-oriented instruction, so that students can develop their own mathematical understanding as mathematicians do.

Undergraduate is considered the first and foundation level of university education. In most cases, abstract algebra is the first mathematical course in an axiomatic approach. Thus, it is considered a foundation course with reference to developing axiomatic reasoning in students. In general, group theory is the first axiomatic concept taught in abstract algebra. Thus, teaching group theory requires several considerations. For example, students' approach of reasoning in mathematics, transition from calculation types of problems to axiomatic reasoning, using symbols of set and operation. The lecture method is not able to address those considerations. As a result, the lecture method becomes 'a bad teaching instrument' (Halmos, 1975, as cited in Freedman, 1983). It can be easily inferred that there is a big gap in the way students learn at the school and university levels. In school, students learn the techniques of solving problems and practice the techniques by solving similar problems. When they reach the level of learning abstract algebra at university, they try to grasp the lecture and try to catch every word as the teacher tells. Relevance and abstraction (Siu, 2001) are two major reasons why abstract algebra is so difficult. It is well accepted that most of the undergraduate students experience cultural shock (Siu, 2004) when they face abstract algebra for the first time. They cannot grasp the content of the lecture in their minds, and as a result, many students try to pass the course without proper understanding. Thus, group theory is the foundational topic to deal with the content and development of attitude towards abstract mathematics (Dubinsky et al., 1994). In this context, the 'theorem-proof-corollary' approach (Schubert et al., 2012) is creating problems not only in understanding the content of mathematics but also in developing an attitude towards abstract mathematics. There are some scholarly works trying to develop alternative solutions to lecture methods.

The study of symmetry algebraically was started by Galois (1811-1832), who used symmetrical transformation in the solvability of a polynomial, which provided a framework for understanding symmetry algebraically (Farooq, 2023). The main objective of teaching abstract algebra is to enable students to reason abstractly (Soto et al., 2024). Algebraic thinking requires axiomatic rules and manipulation of symbols. Though it is abstract in nature, the scholars started to explore possibilities of developing algebraic structures from concrete. Abstract reasoning should be facilitated by reducing complexity. Visualisation is one of the approaches for developing relationships between the objects of thought and the learner (Soto et al., 2024). A transformation in which no property of the shape is changed is called symmetry. Mathematically, let  $T$  be a transformation and  $F$  be a shape. Then  $F$  is a symmetrical shape, if  $T(F) = F$  (Usiskin et al., 2003). This can guide us that a mathematical object as an input, a transformation as a process, and we get the same mathematical object as an output under the symmetry.

### Solution Initiatives

Scholars started to work on bringing alternative methods to the lecture method in teaching abstract algebra during the 1980s. The inclusion of a seminar (Freedman, 1983) in the regular lecture was one of the initiatives. This inclusion of the seminar brought an alternative thought of that time and was one of the innovative ideas. But it was not able to bring changes in the learning process as expected. This can be said from the conclusion of the initiative as "No special magic attached to abstract algebra" (Freedman, 1983, p. 644). Next, there was a discussion about the need for connecting the visual and analytical approach in teaching abstract algebra using an example of the dihedral group  $D_4$  (Zazkis et al., 1996). They suggested visualisation and analysis as mutually dependent rather than a dichotomy. This pedagogical approach is limited only to the visual and analytical thinking part, which has a limited area in abstract algebra to apply. Moreover, Smith (1997) researched a collaborative approach to learning with special computer activities, ISETL (Interactive Set Language) and noticed that students had developed a deeper understanding of some of the concepts and learned to do proofs much better. But still, the problem lies in the coverage of all concepts and collaboration. Another initiative is to use concrete examples, namely, a penny moving in a  $2 \times 2$  checkerboard to introduce the Klein-Vier group (Edwards and Brenton, 1999, as cited in Schubert et al., 2012). Scholars also tried to address these

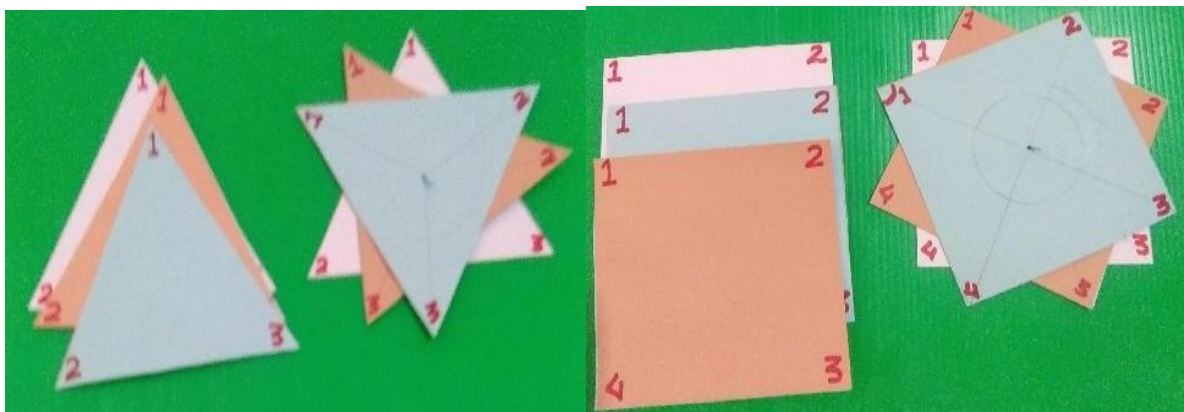
problems by providing theoretical lenses to engage students to think as Mathematicians do. For example, Inquiry-Based Mathematics Education (IBME) in undergraduate mathematics learning.

With the start of the new millennium, there is an effort to link informal mathematical ideas to formal mathematics at the undergraduate level. Scholars focused on using students' informal knowledge and strategy to build formal mathematics (Larsen, 2002, 2004) and learning of group theory by reinventing concepts (Weber & Larsen, 2008) are the milestones of the development of Inquiry-based learning (IBL) in undergraduate mathematics. A research done in the United States shows that the students in IBL had better gains in an undergraduate mathematics course than those in a non-IBL course. Understanding and thinking, confidence, persistence and positive attitude about mathematics, working with others, seeking help, and appreciating differences were respectively cognitive, affective and collaborative gains (Laursen et al., 2014 as cited in Kuster et al., 2017). This implies that there is growing research attention in undergraduate mathematics learning. But a little work has been explored directly on teaching axiomatic mathematics (Melhuish & Strickland, 2019). There is a research gap in combining all these initiatives in undergraduate abstract algebra, especially in group theory.

In this context, classroom research on using hands-on activities to help students learn group theory, especially focusing on groups of symmetries, can be an innovative idea to address the problems. Hands-on learning is a form of inquiry-based learning in which learners manipulate and use materials in order to gain a better understanding (Haury & Rilliero, 1994, as cited in Sulik, 2014). Inquiry Oriented Instruction, a particular form of Inquiry Based Learning, which moves on generating students' ways of reasoning, building on students' contributions, developing a shared understanding and connecting to standard mathematical language and notation (Kuster et.al., 2017). The main objective of the innovation is to facilitate students to reinvent groups of symmetries and distinguish them from groups of permutations.

### Development of Innovative Activity

In this research, two types of symmetries, reflectional symmetry and rotational symmetry of a regular polygon, are utilised as concrete hands-on toolkits. The implementation sequence and students' learning from the hands-on innovative toolkits by following inquiry-oriented instruction are described herewith. This innovative hands-on activity was developed with a closed constructive discussion with the authors. One of the authors has been teaching group theory for more than a decade, and another follows his lessons in inquiry-based learning. The discussions helped develop and refine the product (the outcome). After the initial preparation of the product, three of the master students with mathematics education major were asked to use the process and go through the process. As per the suggestions obtained from them, the product was refined. The product was again shared with a group of scholars working in innovation, and their feedback was considered important for the improvement of the process.



**Figure 1** Hands-on Activity Toolkit of Equilateral Triangle and Square

### Learning Trajectory with Hands-on Tool

The product consists of tool kits for symmetries of equilateral triangles and those of squares. Once students work with teachers in triangles, they can explore the same in a square using the toolkit. Apart from this toolkit, there are five worksheets prepared with a view to guiding students' learning.

The following synopsis of learning activities is a tentative guideline for using the innovative hands-on activity.

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a) Review: Students are asked to review Group axioms such as closure, associative, identity, and inverse of a set and an operation. Then students explore the reflectional and rotational symmetry of different types of triangles and quadrilaterals. Finally, they are focused on equilateral triangle and square.

b) Symmetries: Students explore lines of symmetries of an equilateral triangle 123 by using cyclic notation as well as rotational symmetries. For example, (12) is the reflectional symmetry about the median of triangle passing through vertex 3. Similarly, (123) is the rotational symmetry of rotation of triangle with center at the circum-center of triangle and around  $123^\circ$  clockwise. Students are expected to list all the symmetries as (1), (123), (132), (12), (13), and (23) as the end of the activity.

c) Combination: Students are asked to combine various combinations asking them to pick randomly any two symmetries. They can use the hands-on toolkits for combining two symmetries. Later, they can be directed to develop a table to show all possible combinations and get the result as given in Table 1.

**Table 1**

*Combinations of Symmetries of an Equilateral Triangle*

$\circ$	(1)	(12)	(13)	(23)	(123)	(132)
(1)	(1)	(12)	(13)	(23)	(123)	(132)
(12)	(12)	(1)	(132)	(123)	(23)	(13)
(13)	(13)	(123)	(1)	(132)	(12)	(23)
(23)	(23)	(132)	(123)	(1)	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	(1)
(132)	(132)	(23)	(12)	(13)	(1)	(123)

d) Test for Group: Students are asked to test whether the set  $G = \{(1), (12), (13), (23), (123), (132)\}$  forms a group under operation 'composition' or combination of symmetries. It is expected that they come with the conclusion that the set along with operation form a group.

e) Test for Abelian Group: Students then are expected to test themselves whether the group is abelian or not. As seen in the table or combination of reflection and rotation, student can conclude the group is NOT an abelian as  $(123) \circ (23)$  and  $(23) \circ (123)$  are not same.

f) Self-Exploration in a square: As done in the case of equilateral triangle, students can explore the symmetries and their combinations in case of square.

g) Exploration of One-to-One Functions: Students can explore these symmetries by drawing one-to-one functions from a set  $A = \{1, 2, 3\}$  onto itself. They can check the result with the symmetries obtained in the equilateral triangle. Further they can be directed to find one-to-one function from a set  $B = \{1, 2, 3, 4\}$  and compare the symmetries obtained in case of the square.

Students can find reflection symmetries and rotational symmetries themselves. On exploring those symmetries, the six symmetries of an equilateral triangle are expected from students to reinvent. They use those six symmetries to transform the triangle and name the transformations. Then they can test whether these symmetries form a group or not under a combination of symmetries. The toolkit is used to combine those symmetries. There should be a guided instruction in the case of an equilateral triangle, and an independent exploration is expected in the case of a square with the toolkit. It is expected that students can develop such a toolkit, in other regular polygons, so that they can engage in the learning process themselves. In the second part, they use the same regular polygons, for example, an equilateral triangle, to define permutations. It is expected to use the same name and find the reasons why they are the same. Later, there is a difference in the use of the toolkit in the case of a square. This case may be helpful for them to distinguish the case of a triangle from a square. Furthermore, explorations with the other regular polygons are expected to be more abstract based on the reasoning with those toolkits.

### **Key Expectations**

The use of innovative toolkits is highly emphasised in order to develop mathematical skills. The main tenets developed using the innovative toolkits in the following inquiry-oriented instructions are presented here. The class quiz can be administered as an assessment of the learning of groups of symmetries and permutations at the end of the session. Students' performance in the class quiz can be helpful to mention that the use of innovative toolkits is able to simplify the abstract concept of a group of symmetries and a group of permutations. This active learning can be helpful in increasing the performance of students (Freeman, et al., 2014).

The use of toolkits in exploring symmetries and their combinations can be helpful for students to provide a structured approach to investigating patterns and rules, allowing students to transition from concrete to more abstract mathematical concepts. By using the toolkits, students can utilise a computational framework to analyse symmetries and combinations. This process involves thinking, observing data, and analysing patterns to generate rules or patterns. While the discovery of these patterns may not always align with a predetermined plan, it is through this exploration and analysis that students think and work to reinvent the groups.

In addition to the toolkits, worksheets play a vital role in the learning process. Worksheets provide a systematic representation of patterns and offer a platform for testing and verifying their guesses and doubts. Overall, the use of toolkits and worksheets encourages students to engage in active learning, fostering critical thinking, pattern recognition, and generalisation from concrete to abstract forms of groups of symmetries and groups of permutations. This innovative toolkit makes it easy to appropriately scaffold students' learning. This process can be associated with the Goldilocks discourse: Too little scaffolding results in unsystematic exploration, too much scaffolding results in constraining opportunities to think, and the right amount of scaffolding results in productive struggle (Dale & Scherrer, 2015).

The diversity in notation and representation can be associated with individual interpretations and approaches taken by the students, showcasing the different ways in which they understand and communicate the symmetries of the polygons. This innovative hands-on toolkit, as a subject-specific instructional practice, helps discover their potential in the classroom (Bicer, 2021). Working with toolkits, students can develop their level of confidence, which is helpful in developing a positive attitude towards mathematics (Orozco-Guzmán et al., 2020). Overall, the combination of toolkits, worksheets, group work, and challenges created the Goldilocks discourse (Dale & Scherrer, 2015) that fostered excitement, independent thinking, and continuous improvement.

### Conclusion

Symmetry is considered a pervasive and captivating phenomenon, and the mathematical underpinnings of which reveal a profound underlying structure (Farooq, 2023). Once students grasp the core concepts of reflectional and rotational symmetries, they can extend their understanding to other symmetries. Use of hands-on toolkits help students explore composite symmetries with better understanding of properties and functions associated with these symmetries, making even the testing of groups relatively simple. Overall, the combination of toolkits, hands-on exploration, worksheets, and collaborative group work may create an engaging and dynamic learning environment. The students' excitement, motivation, and confidence are nurtured as they delve into symmetries, derive groups, and encounter both abelian and non-abelian structures.

### Note

This paper is an extension of a paper presented at the 14<sup>th</sup> International Congress on Mathematical Education, Shanghai, 12<sup>th</sup>–19<sup>th</sup> July 2020.

### References

- Arnon, I., Cottinill, J., Dubinsky, E., Oktac, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in Mathematics Education*. Springer Science+Business Media. doi:10.1007/978-1-4614-7966-6
- Bicer, A. (2021). A systematic literature review: Discipline-specific and general instructional practices fostering the mathematical creativity of students. *International Journal of Education in Mathematics, Science and Technology*, 9(2), 252-281. <https://doi.org/10.46328/ijemst.1254>
- Dale, R., & Scherrer, J. (2015). Goldilocks discourse — math scaffolding that's just right. *Phi Delta Kappan*, 97(2), 58 — 61. <https://doi.org/10.1177/0031721715610093>
- Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994, October). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27, 267-305. Retrieved from <https://link.springer.com/content/pdf/10.1007%2F01273732.pdf>.
- Farooq, V. (2023). The beauty of symmetry: Understanding group theory. *Bulletin of Engineering Science, Technology and Industry*, 1(3), 233-237. <https://bestijournal.org>
- Freedman, H. (1983). A way of teaching abstract algebra. *The American Mathematical Monthly*, 90(9), 641-644. Retrieved from <https://www.jstor.org/stable/2323285>.
- Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 111, pp. 8410–8415. <https://www.pnas.org/doi/abs/10.1073/pnas.1319030111>
- Melhuish, K., & Strickland, S. K. (2019). Abstract algebra instructors' noticing of students' mathematical thinking. Retrieved from <http://sigmaa.maa.org/rume/crume2019/Papers/45.pdf>.
- Orozco-Guzmán, M., Villanueva-Cantillo, J., Acuña, F. M., Castro, S. O., & Malo, E. S. (2020). Factors that promote positive attitudes towards mathematics in higher education students. *In Journal of Physics: Conference Series*, 1514(1), 12-27. <https://iopscience.iop.org/article/10.1088/1742-6596/1514/1/012027/meta>
- Schubert, C., Gfeller, M., & Donohue, C. (2012). Using group explorer in teaching abstract algebra. *International Journal of Mathematical Education in Science and Technology*. <https://doi.org/10.1080/0020739X.2012.729680>

- Siu, M. K. (2001). Why is it difficult to teach abstract algebra? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Ed.), *Proceedings of the 12th ICMI Study Conference*. 1, pp. 541-547. Melbourne University. <https://hkumath.hku.hk/~mks/12THICMI.pdf>.
- Siu, M. K. (2004). On learning and teaching tertiary algebra.
- Smith, R. S. (1997). A collaborative learning constructivist approach to abstract algebra using ISETL. <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.42.5029&rep=rep1&type=pdf>
- Soto, H., Lajos, J., & Romero, A. (2024). Teaching Abstract Algebra Concretely via Embodiment. *PRIMUS Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 34(4), 376-391. <https://doi.org/10.1080/10511970.2024.2310235>
- Sulik, E. (2014, Dec 5). Hands-on minds-on learning: Creating engaging learning activities. <https://www.youtube.com/watch?v=hpR0zmIkkU>.
- Usiskin, Z., Peressini, A., Marchisotto, E. A., & Stanley, D. (2003). *Mathematics for High School Teachers: An Advanced Perspective*. <https://doi.org/10.6082/uchicago.5793>
- Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group  $D_4$ . *Journal for Research in Mathematics Education*, 27(4), 435-457. <http://www.jstor.org/stable/749876>.