Maximal Duality Mappings On Banach Spaces

Biseswar Prashad Bhatt
Lecturer of Mathmatics, Saraswati Multiple Campus, Tribhuvan University
Email: bpbhatt92@gmail.com

Submitted June 4 ; Reviewed June 13; Accepted June 26, Published July 28

Abstract:
The analytical features of a Banach space K are characterized by the duality mappings on a Banach space K. One example of a monotone duality mapping on K is the subdifferential of proper convex functions on K. In this case, we look at various instances of normalized duality mappings as well as the idea of monotone operators on K. The surjectivity of the duality mappings and the notions of hemicontinuity and demicontinuity are crucial. If A and B are two monotone mappings then their sum is always monotone mapping but the sum of maximal monotone mapping may not be maximal in general. Ultimately, the circumstance that results in the sum of two maximal monotone sets becoming a maximal monotone is revealed.

Keywords: Convex set, Duality mapping, Maximal monotone operator, Demicontinuity, Reflexivity.

1. Introduction

We use K for real Banach space and K^* for its dual. The evaluation of $f \in K^*$ at $x \in K$ is denoted by (f, x) or (x, f). The mapping $J : K \to 2^{K^*}$ means J is the multivalued mapping from K with the range as the subset of K^*. We call a space K strictly convex if for all $x, y \in K$ with $x \neq y$, $||x|| - ||y|| = 1$, $||\mu x + (1 - \mu)y|| < 1, \quad \forall \mu \in (0, 1)$. It is equivalent to say that K is strictly convex if and only if there is no any line segments on the boundary of the unit ball. We call the space K uniformly convex if for $0 < \epsilon < 2$, there exists $\delta > 0$: if $||x|| = 1, ||y|| = 1$ and $||x - y|| \geq \epsilon$. If $A : K \to K^*$, we can define A with its subgraph in $X \times X^*$

$$\{(x, f) \in K \times K^*, f = Ax\}.$$

A subset of $K \times K^*$ is called monotone if for each $(x_i, f_i) \in A, i = 1, 2,$

$$(f_1, -f_2, x_1 - x_2) \geq 0.$$

A is maximal monotone if no any other monotone set contains it properly. The sum of two mappings A and B is defined as

$$A + B = \{(x, f + g) : (x, f) \in A \ \ \& \ \ (x, g) \in B\}.$$

If A and B are monotone sets in $K \times K^*$, then $A + B$ are clearly monotone. But it is not confirmed that $A + B$ is maximal monotone if A and B are maximal monotone.

There are different contribution in this area by Lescarrent (1965) , Browder(1965) and Rockfeller (1970). The common assumption is that at least one of A and B has a domain with nonvoid interior. Here we study some useful interior for the case having neither $D(A)$ nor $D(B)$ has non void interior from (Brezis, 1970).
In first section we discuss about duality mappings and the characteristics of the Banach space using the nature of dual map. Then in second section we study some pertubation results on monotone mappings.

2. Duality mappings on Banach space

A continuous and strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be weight function if

$$\phi(0) = 0 \quad \text{and} \quad \lim_{n\to\infty} \phi(n) = \infty.$$

For any Banach space K, the mapping $J : K \to 2^{K^*}$ defined by

$$J_x : \{f \in K^* : (f, x) = ||f||||x||, ||f|| = \phi(||x||)\}$$

is called the duality mapping of weight ϕ. If the weight function ϕ is identity , J is called normalized duality mapping.

Example 1. *For a Banach space K, the mapping $A : K \to 2^{K^*}$ defined by*

$$A(x) = \partial(\frac{1}{2}||x||^2)$$

is the normalized duality mapping on K.

From **(Asplund, 1967)** we get the Banach space K is smooth if and only if each duality mapping J of weight ϕ is single valued: in this case

$$\langle J_x, y\rangle = \lim_{t\to\infty} \frac{\psi(||x + ty||) - \psi||x||}{t}$$

where $\psi(t) = \int_0^t \phi(x)dx$ a convex function on $\mathbb{R}_+$.

Proposition 1. *If K is a Hilbert space then the normalized duality mappings on K are linear and vice versa.*

The normalized duality mapping is the identity operator after a Hilbert space is identified with its dual in view of the Riesz representation theorem. It is important to know whether the identification is done before calculating the duality mapping or afterwards.

Definition 1. ***(Ciaoranescu, (1990),p.153)*** *A mapping $A : K \to 2^{K^*}$ is said to be monotone if for any $x, y \in D(A)$ and $u \in A_x, v \in A_y$,*

$$\langle u - v, x - y\rangle \geq 0$$

and a maximal monotone if it is monotone and

$$\langle u - v, x - y\rangle \geq 0 \implies (x, u) \in G(A) \ \ \textit{for} \ \ (y, v) \in G(A).$$

Example 2. *Let $\Omega \in \mathbb{R}^n$ be a bounded domain, $q \geq 2$. Define the Sobolev space*

$$W^{1,q}(\Omega) = \{g : D_i g \in L^q(\Omega), 0 \leq i \leq n\}$$

under the norm

$$||g||_{1,q} = \Big(\sum_{i=0}^{n} ||D_i g||_q^q\Big)^{1/q}$$

Let $W_0^{1,q}(\Omega)$ denote the closure of the test function space $\mathcal{C}_0^\infty(\Omega)$ where the norm is defined as

$$||g||_{1,q}^0 = \Big(\sum_{i=0}^{n} ||D_i g||_q^q\Big)^{1/q}.$$

The pseudo-Laplacian operator

$$A : W_0^{1,q}(\Omega) \to \Big(W_0^{1,q}(\Omega)\Big)^*$$

is defined by

$$Ag = -\sum_{i=1}^{n} D_i\Big(|D_i g|^{q-2} D_i g\Big), \ g \in W_0^{1,q}(\Omega).$$

Now for $h, g \in W_0^{1,q}(\Omega)$,

$$\begin{aligned}
\langle Ag, h\rangle &= \int_\Omega \Big(-\sum_{i=1}^{n} D_i\Big(|D_i g|^{q-2} D_i g\Big) h\Big) \\
&= -\sum_{i=1}^{n} \int_\Omega D_i\Big(|D_i g|^{q-2} D_i g\Big) h \\
&= -\sum_{i=1}^{n} \Big(|D_i g|^{q-2} D_i g\Big) h \ |_{\partial\Omega} + \sum_{i=1}^{n} \int_\Omega |D_i g|^{q-2} D_i g D_i h \\
&= 0 + \sum_{i=1}^{n} \int_\Omega |D_i g|^{q-2} D_i g D_i h
\end{aligned}$$

And so $|\langle Ag, h\rangle| = |\sum_{i=1}^{n} \int_\Omega |D_i g|^{q-2} D_i g D_i h|$.

Since

$$|D_i g|^{q-2} D_i g \in L^p(\Omega)$$

and

$$D_i h \in L^q(\Omega).$$

from Holders inequality

$$
\begin{aligned}
|\langle Ag, h\rangle| &\leq \sum_{i=1}^{n} |\int_{\Omega} |D_i g|^{q-2} D_i g D_i h| \\
&\leq \sum_{i=1}^{n} |||D_i g|^{q-1}||_p ||D_i h||_q \\
&\leq \Big(\sum_{i=1}^{n} |||D_i g|^{q-1}||_p^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} ||D_i f||_q^q\Big)^{\frac{1}{q}}
\end{aligned}
$$

Using

$$
\begin{aligned}
|||D_i g|^{q-1}||_p^p &= \int_{\Omega} |D_i g|^{(q-1)p} \\
&= \int_{\Omega} |D_i g|^q \\
&= ||D_i g||_q^q
\end{aligned}
$$

We find

$$
\begin{aligned}
|\langle Ag, h\rangle| &\leq \Big(\sum_{i=1}^{n} ||D_i g||_q^q\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} ||D_i h||_q^q\Big)^{\frac{1}{q}} \\
&= \Big(||g||_{1,q}^0\Big)^{\frac{q}{p}} ||h||_{1,q}^0.
\end{aligned}
$$

This shows that A is well defined and bounded.
Moreover,

$$\langle Ag - Ah, g - h\rangle \geq 0.$$

Hence operator A is strictly monotone.

Theorem 1. ***(Petryshyn, 1970)*** *A Banach space K is strictly convex if and only if the duality mapping J of weight ϕ is strictly monotone.*

As a consequence result, we say that all duality mappings on a strictly convex Banach spaces are strictly monotone.

3. The role of duality mappings in surjectivity

Suppose K be a Banach space. A mapping $A : K \to K^*$ is said to be coerceive if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ with

$$\lim_{t\to\infty} \psi(m) = +\infty :$$

$$(u, x) \geq \psi(||x||) \quad \forall (x, u) \in G(A).$$

The following result is the basic tool to solve various functional equations for monotone operators.

Proposition 2. ***(Browder, 1968)***
Consider a Banach space K which is reflexive. If B a closed convex subset of K and if $A : K \to 2^{K^*}$ a monotone mapping where $D(A) \subseteq B$ and $\phi : B \to K^*$ a monotone, bounded, coercive and demicontinuous operator, then there exists $x_0 \in B$ with

$$(u + \phi x_0, x - x_0) \geq 0, \quad \forall (x, u) \in G(A).$$

Theorem 2. *For any real reflexive Banach space K and a closed convex subset B of K, if $A : K \to 2^{K^*}$ a maximal monotone mapping with $D(A) \subseteq B$ and $\phi : B \to K^*$ a monotone bounded, coercive and demicontinuous operator, then $A + \phi$ is surjective.*

Proof. Let $f \in K^*$, define $A'x = Ax - f$. Then A' and ϕ has the above all properties. So from above proposition there exists $y \in B$:

$$(g' + \phi y, x - y) \geq 0. \ \ \forall (x, g') \in G(A)$$

i.e.

$$(g - (f - \phi y), x - y) \geq 0, \quad \forall (x, g) \in G(A).$$

As A is maximal monotone,

$$(y, f - \phi y) \in G(A).$$

This gives

$$f \in Ay + \phi y.$$

i.e. $\forall f \in K^*$ there exists $y \in B : A(y) = gy + \phi y$.

□

Now we present the surjectivity results for maximal monotone mappings involving such a duality mappings.

Theorem 3. ***(Browder, 1966)*** *Let K be a reflexive Banach space. A maximal monotone and coercive mapping $A : K \to 2^{K^*}$ is surjective.*

Theorem 4. *A monotone mapping $A : K \to 2^{K^*}$ is maximal iff $A + J$ is surjective on any reflexive Banach space K.*

Proof. From Theorem 2, the neccessity condition is completed. Suppose $R(A + J) = K^*$ and A is not maximal monotone then there exists $(x_0, u_0) \in K \times X$:

$$(x_0, u_0) \notin G(A)$$

but

$$(u - u_0, x - x_0) \geq 0, \quad \forall (x, G) \in G(A). \tag{3.1}$$

From hypothesis $(x_1, u_1) \in G(A)$ so that

(3.2) $$u_0 + Jx_0 = u_1 + Jx_1$$

Using $x = x_1$ and $u = u_1$ in (3.1), we get

$$(u_1 - u_0, x_1 - x_0) \geq 0.$$

Hence

$$(Jx_1 - Jx_0, x_1 - x_0) = 0.$$

Since J is strictly monotone, it follows that

$$x_0 = x_1 \in D(A).$$

Thus by (3.2)

$$u_0 = u_1 \in Ax_1 \in G(A),$$

i.e. $(x_0, u_0) \in G(A)$ and this is contradiction.
This completes the proof. □

Corollary 1. ***(Rockfeller, 1970)*** *If $A : K \to 2^{K^*}$ is maximal monotone and $\phi : K \to K^*$ is a monotone bounded and hemicontinuous operator with $D(\phi) = K$, then $A + \phi$ is maximal monotone.*

Proof. As $D(\phi) = K$, $B\phi$ is demicontinuous and hence $B\phi + J$ is demicontinuous. Also it is monotone, bounded and coercive then by Theorem 2, $A + \phi + J$ is surjective. Then from above theorem $A + \phi$ is maximal.

□

4. Pertubation results

The main problem in the pertubation theory of maximal monotone sets is the determination of the condition under which the sum of two maximal sets A and B is maximal. First look at the following example.

Example 3. *Let $K = L^2(\mathbb{R}_+), Ag = -g''$ with $D(A) = \{g \in H^2(\mathbb{R}_+), g(0) = 0\}$ and $Bg = -g''$ with $D(B) = \{g \in H^2(\mathbb{R}_+), g'(0) = 0\}$. Here A and B are maximal monotone but $A + B$ is not maximal monotone.*

The following theorem gives the condition for the sum of maximal monotone mappings to be again a maximal monotone.

Theorem 5. ***(Rockfeller,1970)***

Let K be a reflexive and A and B be two maximal monotone mappings: if $intD(A) \cap D(B) \neq \phi$, then $A + B$ is maximal monotone.

Theorem 6. *Let K be a reflexive Banach space and A, B are two maximal monotone sets in $K \times K^*$ such that*

(1) $D(A) \subset D(B)$
(2) $|B(x)| \leq k(||x||)|Ax| + C(||x||)$,

where $k(r)$ and $C(r)$ are non-decreasing functions of r and $k(r) < 1 \ \ \forall r$. Then $A+B$ is maximal monotone in $K \times K^*$.

Proof. With no loss of generality suppose that $0 \in D(A), 0 \in A_0$ and $0 \in B_0$. We can get it by the shifting domain and range of A and B.
Suppose $\{|| \cdot ||_p\}$ be the family of equivalent norm on K. Then $A+B$ is maximal if for every $f^* \in K^*$ and $u \in K$ there exists p:

$$f^* + J_p(u) \in R(J_p + A + B).$$

We take an equation

$$J_p(x_\lambda) + x_\lambda^* + B_\lambda^t x_\lambda = f^* + J_p(u), [x_\lambda, x_\lambda^*] \in A. \tag{4.1}$$

For every $f^* \in K^*, u \in K$ and a fixed t, this equation has a unique solution x_λ.
Then

$$f^* + J_p(u) \in R(J_p + A + B)$$

is bounded as $||B_\lambda^t x_\lambda||_t$ is bounded as $\lambda \to 0$. Multiplying (4.1) by x_λ, we get

$$||x_\lambda||_t \leq ||f^*||_t + ||u||_t \leq t(||f^*|| + ||u||),$$

as $B_\lambda^t 0 = 0$.
Take $R = 2(||f^*|| + ||u||)$ and t such that $1 < t < 2$ and $k(R)t^2 < 1$. Using (4.1) , we get

$$\begin{aligned} t^{-1}|Ax_\lambda| &\leq ||Ax_\lambda||_t \\ &\leq ||x_\lambda^*||_t \\ &\leq ||f^*||_t + ||B_\lambda^t x_\lambda||_t p + ||u||_t + ||x_\lambda||_t \\ &\leq 2t(||f^*|| + ||u||) + |Bx_\lambda|_t \\ &\leq 2t(||f^*|| + ||u||) + t|Bx_\lambda| \\ &\leq 2t(||f^*|| + ||u||) + tk(R)|A(x)| + tC(R) \end{aligned}$$

This shows that $|Ax_\lambda|$ is bounded then $|Bx_\lambda|t$ is bounded. Hence

$$||B_\lambda^t x_\lambda||_t$$

is bounded. This completes the proof. □

Remark: We can use for $x \in \overline{D(A)}$ there exists a neighbourhood u_x of x, $tk_x < 1$ and a constant C_x :

$$|B_y| \leq kx|Ay| + Cx$$

for all $y \in D(A) \cap u_x$ if K and K^* are uniformly convex.

References:

Asplund, E. (1967) Positivity of duality mappings. *Bull. Amer. Math. Soc.*,73(2):200-203.

Brezis H., Grandall M.G. (1970) Pertubations of nonlinear maximal monotone sets in Banach space. *communications on pure and applied mathematics* 23(1): 123-144.

Browder,F.E. (1968) Nonlinear maximal monotone operators in Banach spaces, *Math. Ann.,* 175(1): 89-113.

Browder F. (1966) *Problems non-lineaires.* University of Montreal Press.

Cioranescu I. (1990) *Geometry of Banach spaces, duality mappings and nonlinear problems*, Kluwer Academic Publishers.

Lescarret, G. (1965) Cas d'addition des applications monotones maximales dans un espace de Hilbert. *C.R. Acad. Sci. Paris*, 261(1): 1160-1163.

Petryshyn, W.V. (1970) Invariance domain theorem for locally A-proper mappings and its implications. *J, Funct, Anal,*5(1): 137-159.

Rockfeller, R.T.(1970) On the maximality of sums of nonlinear monotone operators,*Transactions of American mathematical society.* 149(1), 75-88.