Review of the Banach-Stone Theorem

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Abstract: *This is a quick overview of the isomorphism between spaces of continuous functions, or C(X) type spaces, that depend on compact Hausdorff spaces outfitted with the uniform norm. When two compact metric spaces, X and Y, are homeomorphic, Banach assumed the problem in 1932. He came to the conclusion that if C(X) and C(Y) are isometric isomorphic, then X and Y are homeomorphic. Stone then generalized this outcome for a general compact Hausdorff space in 1937. Then it is frequently referred to as the Banach-Stone theorem. There are numerous variations of this classic result. We can derive the topological features of X and Y from Gelfand and Kolmogoroff's algebraic version, which was published in 1939.*

Keywords: *Homeomorphism, Hausdorff , Isomorphism, Riesz Space, Unimodular function.*

1. Introduction

When X and Y are homeomorphic, there is one issue from the time of Banach. This issue was resolved by Banach in 1932 [Banach, 1932, p. 41] for the compact Hausdorff spaces X and Y, and the result is known as the Banach theorem. According to this theorem, the linear structure of C(X) and C(Y) establishes the homeomorphism between X and Y. In other words, X and Y are homeomorphic if and only if C(X) and C(Y) are linear isometric. The distance-preserving function from C(X) and C(Y), which is also linear, is the linear isometry. i.e.

$$T : C(X) \rightarrow C(Y) : T|f1 - f2| = |T(f1) - T(f2)|.$$

The Banach Stone theorem was developed by Stone in 1937 for general compact Hausdorff spaces X and Y. According to the Banach Stone theorem, homeomorphism exists between two compact Hausdorff spaces, X and Y, when C(X) and C(Y) spaces are subjectively isometric. In numerous contexts, this finding has been discovered to have numerous extensions, generalizations, and variants. Here, we review some established findings about algebraic and Riesz isomorphism. Characterizing the topology of X in terms of a particular algebraic structure on C is the main focus (X).

2. Banach-Stone Theorem

The set of all bounded linear functionals on X that have the supremum norm is a normed space itself for every normed space X. Dual space of X is the term for this, and X* serves as the symbol. Consider the following scenario: We are given a set X, a topological space (Y,), and a family F of mappings f: X Y. Create a topology on X such that f F is continuous throughout.

As a result, we must provide the open set collection, OX. This is not difficult; if we select OX = P(X) (power set), we are done; however, this topology is meaningless. Nothing in this topology will converge (apart from the constant sequence). The topology is rather complex. Let (Y, τ) be a topological space and X be a predetermined (non-empty) set. Suppose F is a family of maps with the form *f*: X→ Y. For X, there is a weakest topology that allows all *f* to be continuous. Permit us to

2 attempt to describe this topology. We'll abbreviate it as τ_w. Keep in mind that we must have if $f \in F$ is continuous on X. i.e.

$$\forall\, f \in F \text{ and } \forall\, V \in \tau,\ f^{-1}(V) \in \tau_w.$$

On the other hand, any topology containing the set

$$\psi = \{f^{-1}(V) : V \in \tau, f \in F\}$$

will ensure that with regard to that topology, f is continuous. Therefore, $\psi \subseteq \tau_w$. The topology generated by the set

$$\psi = \{f^{-1}(V) : V \in \tau, f \in F\}$$

on X is called the weak topology on X and is denoted by $\sigma(X, F)$. For any normed space X, $X \subseteq X^{**}$. For $x \in X$, let $g_x: X^* \to K$ be defined by $g_x(f) = f(x)$. Then $g_x \in X^{**}$. Thus, we obtain a canonical mapping $C : X \to X^{**}$, defined by

$$C_x = g_x$$

i.e.

$$(C_x)(f) = g_x(f) = f(x).$$

The weakest topology defined on X^* which makes all $g_x : X^* \to K$ continuous is called weak$*$ topology $\sigma(X^*, X)$. Any set G is open in W^* -topology of X^* iff for every $g \in G$ there exists an $\epsilon > 0$ and $x_1, x_2, \ldots, x_n \in X$ such that

$$\{f \in X^* : |(f - g)(x_i)| < \epsilon\} \subset G.$$

The convergence of $\{fn\} \subset X^*$ in $\sigma(X*, X)$ is given as,

Proposition 1. [Fabian M., 2001, p.67] *For a linear normed space X, a sequence $\{f_n\} \in X^*$ converges weakly to f iff $f_n(x)$ converges to $f(x)$ for all $x \in X$.*

A compact Hausdorff space X can be embedded into the dual space $C(X)^*$ of $C(X)$ through the evaluation map

$$\delta : X \to C(X) \text{ defined by } \delta_x = f(x).$$

This can be proved by the help of Urysohn,s lemma.

For all $x_1, x_2 \in K$ with $x_1 \neq x_2$ and for $f \in C(X)$, $f(x_1) \neq f(x_2) \Rightarrow \delta_{x1}(f) \neq \delta_{x2}(f) \Rightarrow \delta(x_1) \neq \delta(x_2)$.

Thus δ is one-one. Now to show δ is continuous, let k_α is a net converging to x in X. By the condition of weak convergence, for every $f \in C(X)$,

$$f(x_\alpha) \to f(x) \Longrightarrow \delta_{x\alpha} \to_{w*} \delta_x.$$

Lastly, since every continuous functions on a compact sets are homeomorphism, δ is homeomorphism onto it's image. Function δ_x lies in the unit ball of $C(X)^*$. From [Arens et al.,(1947), p. 501] every evaluation functional and their negatives are the extreme point of the unit ball of $C(X)^*$.

Extreme point of any convex subset *K* of a linear space *L* is the point of *K* which does not lie in the interior of any line segment of *K*.

Definition 1 (Adjoint operators) *For a bounded linear operator $T : X \to Y$, the* *new operator $T^* : Y^* \to X^*$ defined by*

$$\forall g \in Y^* \text{ and } x \in X,\ (T^* g)(x) = g(Tx)$$

is called the adjoint operator of T.

This adjoint operator satisfies the condition

$$T^* f(x) = g(x) = (g \circ T)(x) = g(T(x)).$$

Theorem 1. [Richard, (2002), p. 25**](Banach-Stone Theorem)** *For a compact Hausdorff spaces X and Y, if T is the surjective isometry between C(X) and C(Y) then there exists a homeomorphism $\tau : Y \to X$ and $|h(y)| = 1$ for all $y \in Y$ such that*

$$T(f)(y) = h(y)f(\tau(y)),\ \forall f \in C(X) \text{ and } y \in Y.$$

We will discover that a linear isometry T: C(L)→ C(K) must be a weighted composition map of the form (T f) = a (f h), where a is a continuous scalar function with |a| = 1 and h is a homeomorphism from K onto L. There is further discussion of the initial Banach proof for compact metric spaces. Furthermore, we incorporate the Mazur-Ulam theorem concerning the linearity of isometries between normed spaces.

3. Algebraic Isomorphism between the C(X) apaces

The algebraic extension of Banach-Stone theorem gives an idea about how the topological properties can be extracted from the algebraic nature of *C(X)* spaces. It was first given by Gelfand and Kolmogorff [Gelfand and Kolmogorff, 1939, p. 13] in 1939 for compact spaces. Here *X* will be a compact Hausdorff space and we shall consider on *C(X)* its algebra structure.

Definition 2 (Algebra Isomorphism). *A map ψ between two algebraic structure A and B is called algebraic isomorphism if for all $a_1, a_2 \in A$,*

(1) $\psi(a_1 + a_2) = \psi(a_1) + \psi(a_2)$;
(2) $\psi(a_1 \cdot a_2) = \psi(a_1) \cdot \psi(a_2)$;
(3) $\psi(1_X) = \psi(1_Y)$ and
(4) ϕ is bijective.

To prove the Gelfand and Kolmogoroff theorem we need the following theorem from Stone [Stone M., (1937), p. 79].

Lemma 1. *For a compact Hausdorff space X and for every nonzero multiplicative functional $\psi : C(X) \to \mathbb{R}$ there exists a unique $x \in X$ such that $\psi\delta_x$, that is $\psi(f) = f(x)$, for all $f \in C(X)$.*

Proof. Since ψ is a nonzero multiplicative functional,

$$\psi(1) = \psi(1 \cdot 1) = \psi(1) \cdot \psi(1).$$

This implies that $\psi(1) = 1$. Similarly,

$$\forall\lambda \in R,\ \psi(\lambda) = \lambda.$$

Moreover, $\psi(g) \neq 0$ when $Z(g) = \phi$, since in this case $1/g \in C(X)$ and

$$1 = \psi(1) = \varphi(g) \cdot \psi(1/g).$$

Now we have to show that for every $f \in C(X)$ there exists $x \in X$ such that

4 $$\psi(f) = f(x).$$

On contrary suppose that for some function $f \in C(X)$ we have

$$\forall x \in X, g(x) = f(x) - \psi(f) \neq 0 .$$

Thus $Z(g) = \phi$ and

$$\psi(g) = \psi(f) - \psi(f) = 0,$$

a contradiction.

Since X is compact, the intersection of all the finite family of closed subsets of X is nonempty. So there exists $x \in X$ with

$$x \in \cap Z(f - \psi(f))$$

i.e.

$$\psi = \delta_x.$$

Lastly the uniqueness of x comes from the fact that C(X) seperates the points of X.

Theorem 2 (Gelfand and Kolmogoroff , (1939), p. 13). *Let X and Y be a compact spaces. Then C(X) and C(Y) are isomorphic as algebras if and only if X and Y are homeomorphic. Moreover, every algebra isomorphism $T : C(Y) \to C(X)$ is of the form $T f = f \circ h$ where $h \to Y$ is a homeomorphism.*

Proof. Let $h : X \to Y$ be a homeomerphism between X and Y . Here we have to show that $T : C(Y) \to C(X)$ defined by $T f = f \circ h$ is an algebra isomerphism. For $f_1, f_2 \in C(Y)$ and $x \in X$,

$$\begin{aligned} T(f_1 + f_2)(x) &= (f_1 + f_2) \circ h(x) \\ &= (f_1 + f_2)\, h(x) \\ &= f_1(h(x)) + f_2(h(x)) \\ &= f_1 \circ h(x) + f_2 \circ h(x) \\ &= T\, f_1(x) + T\, f_2(x). \end{aligned}$$

Similarly, we can show that

$$T(f_1 \cdot f_2)(x) = T\, f_1(x) \cdot T\, f_2(x).$$

This shows that T is algebraic homomorphism.

Again, let $1X \in C(Y)$. Then

$$T1_X(y) = 1_x \circ h(y) = 1.$$

Thus $T(1_x) = 1_Y$.

To show T injective, let $f_1, f_2 \in C(Y)$ then

$$\forall x \in X,\ T\, f_1(x) = T\, f_2(x) \Rightarrow f_1 \circ h(x) = f_2 \circ h(x) \Rightarrow f_1 h(x) = f_2 h(x) .$$

This implies that

$$f_1 = f_2.$$

Finally, for every $f \circ h \in C(X)$ there exist $f \in C(Y)$ and $h : X \to Y$ such that

$$T\, f = f \circ h.$$

Thus $T : C(Y) \to C(X)$ a algebra isomorphism. Conversely, let $T : C(Y) \to C(X)$ is an algebra isomorphism. For each $x \in X$, define $\psi : C(Y) \to \mathbb{R}$ by

$$\psi(f) = \delta_x \circ T(f) = T\, f(x).$$

For $f_1, f_2 \in C(Y)$,

$$\begin{aligned} \psi(f_1, f_2) &= \delta_x \circ T(f_1 \cdot f_2) \\ &= \delta_x \circ (T(f_1) \cdot T(f_2)) \end{aligned}$$

$$= \delta_x \circ (T f_1) \cdot \delta_x \circ (T f_2)$$
$$= \psi(f_1) \cdot \psi(f_2).$$

Therefore $\psi = \delta_x \circ T : C(Y) \to \mathbb{R}$ is a nonzero multiplicative functional.
So from above lemma there exists unique $y = h(x) \in Y$ such that

$$\delta_x \circ T = \delta_{h(x)} \text{ i.e.}$$

$$\forall f \in C(Y),\ T f(x) = f(h(x)).$$

Thus the map satisfies

$$\forall f \in C(Y),\ T f = f \circ h \in C(X).$$

Now it remains to show that h is continuous. Let $\{x_n\} \subseteq X$ and $x \in X$ such that

$$x_n \to x.$$

Suppose that

$$h(xn) \to h(x).$$

Since f is continuous,

$$f(h(x_n)) \to f(h(x)).$$

This implies that

$$T f(x_n) \to T f(x).$$

This contradicts $x_n \to x$ since $T f \in C(X)$. Thus for $x_n \to x$ in X,

$$h(x_n) \to h(x) \text{ in } Y.$$

Therefore $h : X \to Y$ is continuous map. Since *h* is defined between compact spaces, *h* is homeomerphism between *Y* and *X.*

This result gives the characterization of the topological space *X* from the algebraic characters of the ring *C(X).*

4. Riesz isomorphism between the *C(K)* spaces

4.1. Riesz space and Riesz homomorphism.

Definition 3. *(1) A linear space L is called an ordered vector space if L is partially ordered in such a manner that the partial ordering is compactible with the algebraic structure i.e. if* f, g ∈ L *then* f ≤ g *implies*

$$\forall h \in L, f + h \leq g + h$$

and f ≥ 0 *implies*

$$\forall a \in R,\ a \geq 0,\ a f \geq 0 .$$

(2) A real linear spsce L is called a Riesz space if L is partially ordered in such a manner that (a) L is lattice (b) L is an ordered vector space.

Example 1. *(1) The cartesian space* $\mathbb{R}^2$*, partially ordered by* $(x_1, x_2) \leq (y_1, y_2)$ *if* $x_1 < y_1$ *or if* $x_1 = y_1$*and* $x_2 \leq y_2$ *is Riesz space.*

(2) Let X be a non-empty set and let B(X) be the collection of all bounded real valued functions dened on X. Clearly B(X) is a vector space under the pointwise addition and scalar multiplication which is ordered by the positive cone

$$B(X)_+ = \{f \in B(X) : f(t) \geq 0\ \ \forall t \in X\}.$$

Thus $f \geq g$ *holds iff* $f - g \in B(X)_+$*. Obviously*

$$(f \vee g)(t) = \max \{f(t), g(t)\} \text{ and}$$
$$(f \wedge g)(t) = \min \{f(t), g(t)\}$$

for each $t \in X$ *&* $f, g \in B(X)$*. Thus B(X) is a Riesz space.*

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Definition 4. *The collection of all $u \in L$ satisfying $u \geq 0$ is called the positive cone of L. Elements in the positive cone are called positive elementts. We denote it by*

$$L^+ = \{u \in L : u \geq 0\}.$$

Furthermore for arbitrary $f \in L$, we define $f^+ = f \vee o$, $f- = (-f) \vee 0$, $|f| = (-f) \vee f$.

Definition 5. (Riesz subspace) *A linear subspace V of L is called a Riesz subspace if $f, g \in V$ implies $f \vee g \in V$.($f \wedge g$ also lies in V since $f + g = f \vee g + f \wedge g$.)*

Example 2. *The space C[0, 1] is the Riesz space. The linear subspace V of C[0, 1] consisting of all constant function on [0, 1] is Riesz subspace of C[0, 1].*

Definition 6. (1) *A linear map $\phi : L \rightarrow M$ is called positive (in notation $\phi \geq o$), if $\phi(L^+) \subset M^+$.*
(2) *A linear map $\phi : L \rightarrow M$ is called a Riesz homomorphism if $f, g \in L^+$ and $f \wedge g = 0$ implies*

$$\phi(f) \wedge \phi(g) = 0.$$

Theorem 3. [Junge E., (1977), p. 15] *Let $\phi : L \rightarrow M$ be a linear map then the following statements are equivalents*

(1) ϕ is Riesz homomorphism
(2) $\phi(f \wedge g) = \phi(f) \wedge \phi(g)$ $\forall f, g \in L$
(3) $\phi(f \vee g) = \phi(f) \vee \phi(g)$ $\forall f, g \in L$.

Theorem 4. [Junge E., (1977), p. 78] *For a compact Hausdorff space X and every $a \in X$ define $\phi_a : C(X) \rightarrow R$ by*

$$\phi_a(f) = f(a) \ (f \in C(X)).$$

*Then ϕ_a is Riesz homomorphism and $\phi_a(1) = 1$. Conversely, for every Riesz homomorphism $\phi : C(X) \rightarrow R$ with $\phi(1) = 1$ there exists a unique $a \in X$ such that $\phi = \phi_a$. *

Proof. Let f, g ∈ C(X). Now,

$$\begin{aligned}\phi_a(f \wedge g) &= (f \wedge g)(a)\\ &= f(a) \wedge g(a)\\ &= \phi_a(f) \wedge \phi_a(g) \ \forall f, g \in C(X).\end{aligned}$$

Thus ϕ_a is Riesz homomerphism. And $\phi_a(1) = 1(a) = 1$ being constant function. Conversely, let ϕ be a Riesz homomerphism $C(X) \rightarrow \mathbb{R}$ and $\phi_a(1) = 1$. If possible suppose that for every $a \in X$ there exist an $f_a \in C(X)$ such that $\phi(fa) \neq \phi a(fa)$.
For each a, set

$$g_a = |f_a - \phi(f_a)1|.$$

Then

$$g_a(a) = |\phi_a(fa) - \phi(fa)| > 0$$

while

$$\phi(g_a) = |\phi(f_a) - \phi(f_a)\phi(1)| = 0.$$

Because X is compact and each g_a is continuous, there exists $a_1, a_2, \ldots, a_m \in X$ such that

$$X = \cup_i \{x \in X : g_{ai}(x) > 0\}.$$

Now let $g = g_{a1} \vee \ldots \vee g_{am}$. Then

$$\forall x \in X, g(x) > 0 .$$

As g is continuous and X is compact, there must exist a $\delta > 0$ such that 7

$$\forall x \in X, g(x) > \delta.$$

Then

$$\phi(g) \geq \phi(\delta_1) = \delta\phi(_1) = \delta > 0.$$

On the other hand

$$\phi(g) = \phi(g_{a1}) \vee \ldots \vee \phi(g_{am}) \text{ [since } \phi \text{ is Riesz homomerphism]}$$
$$= 0$$

which is contradiction.
Thus there must exist a a $\in$ X such that

$$\phi = \phi_a.$$

The uniqueness follows from Urysohn's lemma; if x, y $\in$ X and $x \neq y$ then there exist an f $\in$ C(X) such that

$$f(y) = 1 \text{ and } f(x) = 0$$

i.e.

$$\phi_x(f) = 0 \text{ and } \phi_y(f) = 1.$$

If ϕ is a continuous map of a compact Hausdorff space Y into a compact Hausdorff space X, then $\Phi : f \to f \circ \phi$ is a Riesz homomorphism of C(X) into C(Y) with $\Phi(1) = 1$.
The converse is explained in the following.

Theorem 5. [[Junge E., (1977), p. 80**]** *Let X and Y be a compact Hausdorff space and Φ a Riesz homomerphism of C(X) into C(Y) such that Φ(1) = 1. Then there exist a unique continuous map ϕ : Y → X such that*

$$\Phi f = f \circ \phi \ (f \in C(X)).$$

Proof. Let $\Phi : C(X) \to C(Y)$ a Riesz homomorphism with $\Phi(1) = 1$. Then there exist unique element $\phi(y) \in C(X)$ such that

$$(\Phi f)(y) = \Phi\phi(y)\,(f)$$
$$= f(\phi(y))\ \forall f \in C(X).$$

Thus we obtain a $\phi : Y \to X$ with the property

$$\Phi f = f \circ \phi \ (f \in C(X)).$$

Now it remains to show that ϕ is continuous. Let $U \subset X$ be open and let $b \in \phi^{-1}(U) \subset Y$. By Urysohn's lemma there exists an f $\in$ C(X) such that

$$f(\phi(b)) = 1$$

while f vanishes at X\U. Setting $g = \Phi f$ we have

$$g \in C(Y),\ g(b) = f(\phi(b)) = 1$$

and g vanishes at Y \ϕ −1 (U).
Now

$$\{y \in Y : g(y) \geq 0\}$$

is open in Y and $b \in \{y \in Y : g(y) > 0\} \subset \phi^{-1}(U)$.
Hence $\phi^{-1}(U)$ is open in X and therefore ϕ is continuous.
The Banach-tone theorem for Riesz isomorphism is the following.

Theorem 6. (Banach-Stone)[[Junge E., (1977), p. 80**]** *Let X and Y be compact Hausdorff space. If C(X) and C(Y) are Riesz isomorphic, then X and Y are homeomorphic.*

Proof. Let Φ be a Riesz isomorphism of C(X) onto C(Y). Let $u = \Phi 1$ and take $v \in C(X)$: $\Phi(v) = 1$. Then there exist a positive number c for which $v < c1$. Then

$$1 = \Phi v \leq c\Phi 1 = cu,$$

So C(Y) > 0 for every y $\in$ Y .

8 The formula,

$(\varphi f)(y) = (\Phi f)(y) / u(y) \quad (f \in C(X),\ y \in Y)$ can be used to define isomerphism φ of $C(X)$ ont $C(Y)$ such taht

$$\begin{aligned}(\varphi 1)(y) &= (\Phi 1)(y) / u(y)\\ &= u(y)\, u(y)\\ &= 1.\end{aligned}$$

Now there exists continuous $\phi_1 : Y \to X$ and $\phi_2 : X \to Y$ with the properties

$$\varphi f = f \circ \phi 1 \ (f \in C(X))$$

and

$$\varphi^{-1} g = g \circ \phi_2 \quad (g \in C(Y)).$$

If $x \in X$, then for every $f \in C(X)$,

$$f(x) = (\varphi -1 \varphi(f)(x) = (f \circ (\phi 1 \circ \phi 2))(x),$$

since X is compact, f is homeomerphic to R so taking $_{Y}$inverse of f on bothsides from left we get

$$x = (\phi 1 \circ \phi 2)(x).$$

Simillarly

$$\forall\, y \in Y\ y = (\phi_2 \circ \phi_1)(y).$$

Thus, ϕ_1 & ϕ_2 are inverses of each others. Thus X and Y are homeomerphic.

The classical Banach-Stone theorem states that if X and Y are compact spaces and B^X and B^Y are equivalent then X and Y are homeomerphic when B is the space of real numbers. Meyer Jerison [Jorsin M., (1950), p. 14] extended this result for the space B^X for strictly convex Banach space B. According to this if X and Y are compact spaces, B a strictly Banach space and ψ an equivalence of B^Y onto B^X, then there exists a homeomerphism h of X onto Y and a continuous map ψ_x from X into the space Φ of rotation of B such that

$$(\psi B)(x) = \varphi_x[\beta(h(x))],\ \beta \in B\ Y.$$

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