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Numerical Mean-square and Asymptotic Stability Analysis for the Weak Simpson Method[a)]

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ABSTRACT

A weak Simpson method has order of weak convergence one in general and has order of weak convergence three under certain additional assumptions. The proposed method has the potential to overcome some of the numerical instabilities that are often experienced when using explicit Euler method. This work aims to determine the mean-square stability region of the weak Simpson method for linear stochastic differential equations with multiplicative noises. In this work, a mean-square stability region of the weak Simpson scheme is identified, and step-sizes for the numerical method where errors propagation are under control in a well-defined sense are given. The main results are illustrated with numerical examples.

Keywords: Stochastic differential equation, numerical method, Euler-Maruyama method, mean-square stability, asymptotic stability, weak convergence

1 INTRODUCTION

We developed the Weak Simson Method to construct accurate approximations on fixed time intervals to solutions of the following system of stochastic differential equations (SDEs)

$$X(s) = x + \int_0^s b(X(r))\,\mathrm{d}r + \sum_{k=1}^{M} \int_0^s \sigma_k(X(r))\nu_k\,\mathrm{d}W_k(r), \quad s \geq 0, \tag{0.1}$$

where $x \in \mathbb{R}^d$, $M \in \mathbb{N}$ is a positive integer, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_k : \mathbb{R}^d \to \mathbb{R}$, and for $k = 1, \ldots, M$, $\nu_k \in \mathbb{R}^d$, and $W_k(t)$ are independent one dimensional Brownian motions. Here for each k, ν_k represents the direction along which the random noise W_k enters the system (0.1equation.0.1). Suppose the coefficients b and σ are measurable and are such that a weak solution to (0.1equation.0.1) exists and is unique in probability law. Typically the coefficients b and σ are assumed to satisfy the Lipschitz continuity and the linear growth condition; see, for example, [13] or [7]. SDEs have a wide range of applications in areas such as ecosystem modeling, mathematical finance, and risk management. In most of the practical applications, we can not find explicit solutions for the underlying SDEs, as with most ordinary differential equations (ODEs). In such situations, numerical approximation then becomes the one viable approach.

Most stochastic differential equations can not be solved explicitly. However, a great deal of useful qualitative information can be obtained about the behavior of their solutions. Asymptotic behavior and the impact of small changes in initial values are of particular interest in applications. We know that if a differential equation is well-posed, then a solution exists and is unique; moreover, the solution is continuous with respect to the initial value in

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some sense. The concept of stability is an extension of this idea to an infinite time interval ([9]). In this paper we shall study mean-square stability of our proposed method in relation to a scalar Itô equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x_0 \in \mathbb{R}^d, \tag{0.2}$$

where we assume that (0.2equation.0.2) has a steady solution $X(t) \equiv 0$. In some sense, the stability of a numerical scheme refers to the conditions under which the impact of an error vanishes asymptotically over time, see, for example, Bruti-Liberati and Platen (2008). Generally, the notion of numerical stability is challenging to quantify. Nevertheless various concepts of numerical stability for different schemes have been extensively studied by many authors. Most of the lit- erature in numerical stability use specially designed test equations, see for instance, Kloeden and Platen (1992). We systemically analyze the stability properties of our scheme for the given family of test equations.

2 THE WEAK SIMPSON METHOD

Preliminaries

The error criteria to be used depend on the type of application. If one is interested in just generating $X(T)$ sufficiently accurately (in the distributional sense), an appropriate error criterion may be

$$\sup_{f\in\mathscr{C}} \left|\mathbb{E}[f(X(T))] - \mathbb{E}[f((Y(N))]\right|,$$

for a suitable class $\mathscr{C}$ of smooth functions, where $Y(.)$ is the simulated path. The accuracy of the sample path approximation can be measured by a criterion such as

$$\left[\sup_{\mathbb{t}\in[,\mathbb{T}]} |\mathbb{X}(\mathbb{t}) - \mathbb{Y}(\mathbb{t})|\right]$$

assuming that $X(\cdot)$ and $Y(\cdot)$ can be generated on a common probability space, or, for some suitably chosen p, via an L_p error criterion such as

$$\mathbb{E}\left[\int^T |X(t) - Y(t)|^p\right] dt,$$

$$\mathbb{E}\left[\sum_{0\le n\le T} |X(n) - Y(n)|^p\right]$$

in continuous and discrete time respectively ([**?**]). We recall the following definition from [9].

Definition 0.1. *We say that an approximating process Y converges in the strong sense with order $\gamma \in (,\infty]$ if there exists a finite constant K and a positive constant δ_0 such that*

$$\mathbb{E}\big[\,|X(T) - Y(N)|\,\big] \le Kh^\gamma, \quad N = \frac{T}{h},$$

for any time discretization stepsize $0 < h < \delta_0$. The strong order of convergence measures the rate at which the "mean of the error" decays as $h \to 0$.

In fact this definition generalizes the standard convergence criterion for ordinary differential equation, reducing to the usual definition when the diffusion coefficient of (0.1equation.0.1) is zero.

Strong convergence allows an accurate approximation to be computed and involves direct simulation of the sample path and demands the approximation be close to that of the Itô process. The order of convergence of strong approximation is sometimes less in the stochastic case than in the corresponding deterministic case, see, for example, [9]. It is also observed in [2] that strong explicit methods, particularly, the widely used Euler-Maruyama method, sometimes work unreliably and generate large errors for certain step-sizes.

But if the goal is to have a good approximation of the probability distribution of the solution $X(t)$, individual realizations are not of primary interest. Weak approximations are used in simulating functionals of the form $\mathbb{E}[f(X(T))]$, where $T > 0$ and f is some function. For instance, the arbitrage-free price of a European call option is given by $\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S(T)-K)^{+}|\mathscr{F}_t]$, in which $\mathbb{Q}$ is the risk-neutral measure, $r > 0$ is the discounting factor, K is the strike price, and $S(T)$ is the price of the underlying asset at time T. For weak approximation, in leu of Definition 0.1thm.0.1 for strong approximation, a less demanding alternative is to measure the rate of decay of the "error of the means". This leads to the concept of weak convergence order. We recall the following definition for weak convergence from [9].

Definition 0.2. *We say that a time discrete approximation Y converges in the weak sense with order $\beta > 0$ if for any $f \in C^{2(\beta+1)}(\mathbb{R}^d)$ there exists a finite constant K and a positive constant δ_0 such that*

$$|\mathbb{E}[f(X(T))] - \mathbb{E}[f(Y(N))]| \leq Kh^{\beta}, \quad N = \frac{T}{h} \tag{0.3}$$

for any time discretization with maximum step size $h \in (,\delta_0)$.

If the stochastic part of the differential equation is zero and the initial value is deterministic, the definition reduces to the usual deterministic convergence criterion for ordinary differential equation and also agrees with the strong convergence criterion.

We state the following standing assumptions throughout the thesis:

(A1) The coefficients of (0.1equation.0.1) satisfy the Lipschitz and linear growth conditions:

$$\begin{aligned} &|b(x) - b(y)| + \sum_{k=1}^{M} |\sigma_k(x) - \sigma_k(y)| \leq \kappa |x-y|, \\ &|b(x)| + \sum_{k=1}^{M} |\sigma_k(x)| \leq \kappa(1+|x|), \end{aligned} \tag{0.4}$$

for all $k = 1, \ldots, M$ and $x, y \in \mathbb{R}^d$, where κ is a positive constant.

(A2) For each $k = 1, \ldots, M$, we have $\inf_{x\in\mathbb{R}^d}\{\sigma_k(x)\} > 0$. In addition, there exists a positive constant $\lambda \in (0,1]$ such that for any $x, \xi \in \mathbb{R}^d$ we have

$$\lambda |\xi|^2 \leq \xi^T a(x)\xi \leq \lambda^{-1} |\xi|^2, \tag{0.5}$$

where ξ^T denotes the transpose of ξ and $a(x) := \sum_{k=1}^{M} \sigma_k^2(x) v_k v_k^T$.

(A3) For all multi-index α with $|\alpha| \leq 8$, we have

$$|D^{\alpha} b(x)| + \sum_{k=1}^{M} |D^{\alpha}\sigma_k(x)| \leq K(1+|x|^p), \text{ for all } x \in \mathbb{R}^d, \tag{0.6}$$

where K and p are positive numbers.

It is well-known that under Assumption (A1), the stochastic differential equation (0.1equation.0.1) has a unique strong solution; see, for example, [7, 13] or [17]. Moreover, we have the following moment estimate:

Lemma 0.3 ([17])**.** *Assume* (A1)*. Let $T > 0$ be fixed. Then for any positive constant p, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X^x(t)|^p\right] \leq C < \infty, \quad x \in \mathbb{R}^d \times \mathbb{M}, \tag{0.7}$$

where the constant C satisfies $C = C(x,T,p) > 0$ and X^x denotes the solution to (0.1equation.0.1) *with initial condition $x \in \mathbb{R}^d$.*

Remark 0.4. *We note that Assumptions (A1)–(A3) are slightly weaker then those in [1], where it is assumed that $b, \sigma_k, k = 1, \ldots, M$ are bounded with bounded and continuous partial derivatives up to the sixth order and that $\inf_{x\in\mathbb{R}^d}\{\sigma_k(x)\} > 0$ for each k. Also,* (0.5equation.0.5) *plays an important role in a certain Gaussian tail estimate in the proof of Lemma* **??***.*

3 THE ALGORITHM

The weak Simpson method can be summarized as follows. Let $T > 0$ and $\Pi := \{0 = t_0 < t_1 < ... < t_N = T\}$ be a subdivision of $[0,T]$. Let $\{\eta_{1k}^{(i)}, \eta_{2k}^{(i)} : i \in \mathbb{N}, k \in \{1,2,\ldots,M\}\}$ be a collection of mutually independent normal random variables with mean zero and variance 1. Fix $\theta \in (0,1)$ and define

$$\alpha_1 = \frac{5}{12\theta(1-\theta)} \text{ and } \alpha_2 = \alpha_1 - 1 = \frac{5-12\theta+12\theta^2}{12\theta(1-\theta)}. \tag{0.8}$$

In this work, we take constant discretization stepsize $h = T/N$ and so $t_i = ih$ for $i = 0,1,\ldots,N$. Let $Y_0 = X(0) = x_0$ and, for $i = 1,2,\ldots,N$, we repeat the following steps:

Step 1.

$$Y_i^* = Y_{i-1} + b(Y_{i-1})\theta h + \sum_{k=1}^{M} \sigma_k(Y_{i-1})\nu_k \eta_{1k}^{(i)} \sqrt{\theta h}. \tag{0.9}$$

Step 2.

$$\begin{aligned} Y_i =& Y_i^* + (\alpha_1 b(Y_i^*) - \alpha_2 b(Y_{i-1}))(1-\theta)h \\ &+ \sum_{k=1}^{M} \sqrt{[\alpha_1\sigma_k^2(Y_i^*) - \alpha_2\sigma_k^2(Y_{i-1})]^+}\, \nu_k\, \eta_{2k}^{(i)} \sqrt{(1-\theta)h}. \end{aligned} \tag{0.10}$$

We call such an algorithm the *weak Simpson method*.

4 NUMERICAL MEAN SQUARE STABILITY ANALYSIS FOR THE WEAK SIMPSON METHOD

The concept weak convergence given in Definition **??** concerns the accuracy of a numerical method over a finite interval $[0,T]$ for small step sizes Δt. However, in many applications the long-term behavior of an SDE is of interest. The stability of various stochastic processes has been extensively studied by many authors; see for instance [8, 10, 11, 12, 17] and references therein.

In simulations and numerical approximations, roundoff and truncation error, sampling error, random number bias, etc. are common. The utility of a numerical method depends upon its ability to control the propagation of such errors in extended time horizon. Concerning the long-time behavior or stability analysis of numerical schemes, the following two questions are fundamental:

(i) Do the numerical solutions of SDEs preserve stability properties of the original SDEs? And if the answer is yes,

(ii) for what range of step sizes Δt so that the numerical solutions are stable in appropriate senses?

These question have received a lot of attention; some recent developments in this line of research can be found in [3, 4, 5, 14, 15] and references therein.

In this work, we are concerned with mean square and almost surely asymptotic stability analysis for the weak Simpson method (0.9equation.0.9)–(0.10equation.0.10). As in the aforementioned references on numerical stability, we will focus on the linear test equation

$$X(t) = X(0) + \int_0^t \lambda X(t)\,\mathrm{d}t + \int_0^t \mu X(t)\,\mathrm{d}W(t), \quad t \geq 0, \tag{0.11}$$

for real or complex constants λ and μ.

Linear stability analysis for deterministic case

We start with the deterministic case when $\mu = 0$ and hence (0.11equation.0.11) reduces to

$$\begin{aligned} \frac{\mathrm{d}X(t)}{\mathrm{d}t} &= \lambda X(t), \quad t > 0 \\ X(0) &= x \neq 0. \end{aligned} \tag{0.12}$$

Here $\lambda \in \mathbb{C}$ is constant. The solution to (0.12equation.0.12) is $X(t) = xe^{\lambda t}$ and hence $\lim_{t\to+\infty} X(t) = 0$ if and only if $\lambda \in \mathbb{C}^-$, where $\mathbb{C}^-$ denotes the left-half complex plane. This is the stability region for (0.12equation.0.12).

The weak Simpson method (0.9equation.0.9)–(0.10equation.0.10) applied to (0.12equation.0.12) produces the recurrence

$$Y_n = \left(1 + \lambda h + \frac{5}{12}\lambda^2 h^2\right) Y_{n-1}. \tag{0.13}$$

Then it follows from (0.13equation.0.13) that

$$\lim_{n\to+\infty} Y_n = 0 \text{ if and only if } \left|1 + \lambda h + \frac{5}{12}\lambda^2 h^2\right| < 1 \tag{0.14}$$

Therefore for a given step size $h > 0$, the stability region for the weak Simpson method is

$$S_w = \{\lambda \in \mathbb{C} : \left|1 + \lambda h + \frac{5}{12}\lambda^2 h^2\right| < 1\} \tag{0.15}$$

It is more common to speak of the region of absolute stability as a region in the complex λh-plane. Setting $z = \lambda h = x + iy$ in (0.15equation.0.15) and detailed calculation gives

$$S_w = \{(x,y) \in \mathbb{R}^2 : 25x^4 + 25y^4 + 50x^2y^2 + 120x^3 + 120xy^2 + 264x^2 + 24y^2 + 288x < 0\}. \tag{0.16}$$

The stability domain (0.16equation.0.16) of weak Simpson method for deterministic linear test equation is shown in Figure 1stability domain for weak-Simpson method (blue shaded region)figure.1.

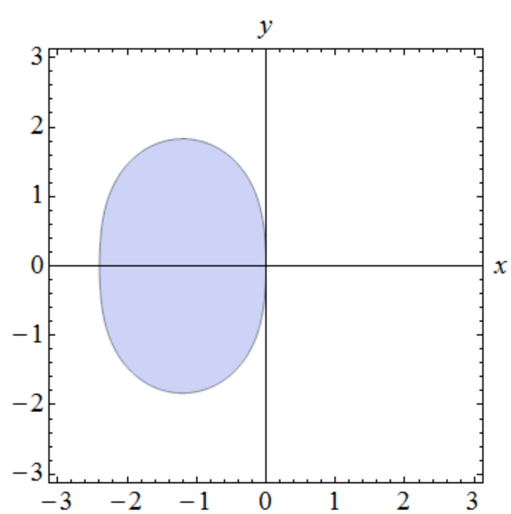

FIGURE 1. stability domain for weak-Simpson method (blue shaded region)

Mean square stability analysis

Returning to the SDE (0.11equation.0.11), where we assume that λ and μ are real constants and that $X(0) \neq 0$, since the solution is $X(t) = X(0)e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$, we have

$$\lim_{t\to+\infty} \mathbb{E}[|X(t)|^2] = 0 \text{ if and only if } 2\lambda + \mu^2 < 0, \tag{0.17}$$

and

$$\lim_{t\to+\infty} |X(t)| = 0 \text{ with probability 1 if and only if } \lambda - \frac{1}{2}\mu^2 < 0. \tag{0.18}$$

It is clear from (0.17equation.0.17) and (0.18equation.0.18) that mean square stability implies asymptotic stability but not vice versa. Note that if $W = \{W(t) : 0 \le t < \infty\}$ is a Brownian motion, so is the process $-W = \{-W(t) : 0 \le t < \infty\}$. Thus we can and will assume without loss of generality in the rest of the subsection that $\mu > 0$. For ease of later presentation, we denote by

$$S_P = \{(\lambda, \mu) \in \mathbb{R} : 2\lambda + \mu^2 < 0\}$$

the set of ordered pairs of real parameters (λ, μ) so that the trivial solution of (0.11equation.0.11) is mean square stable.

Applying the weak Simpson method to (0.11equation.0.11) produces the following iterative sequence:

$$Y_n = \left[A + B\eta_1^{(n)}\right] Y_{n-1} + \sqrt{(1-\theta)h\left[C + D\eta_1^{(n)} + E\left(\eta_1^{(n)}\right)^2\right]^+}\,\eta_2^{(n)}\,|Y_{n-1}|,\ n = 1, 2, \ldots, \tag{0.19}$$

where $\{\eta_1^{(n)}, \eta_2^{(n)}, n = 1, 2, \ldots\}$ are mutually independent Gaussian random variables with mean zero and variance one, and

$$\begin{aligned}
A &:= 1 + \lambda h + \frac{5}{12}\lambda^2 h^2, \\
B &:= \mu\sqrt{\theta h}\left(1 + \frac{5}{12\theta}\lambda h\right), \\
C &:= \mu^2\left(1 + \alpha_1\lambda^2\theta^2 h^2 + 2\alpha_1\lambda\theta h\right) \\
D &:= 2\alpha_1\mu^3(\lambda\theta h + 1)\sqrt{\theta h}, \\
E &:= \alpha_1\mu^4\theta h > 0.
\end{aligned} \tag{0.20}$$

Apparently C is positive when $\lambda \ge 0$. When $\lambda < 0$, it is easy to see that C is positive when $0 < h < \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right)$.

The sequence Y_n of (0.19equation.0.19) is mean-square stable if $\lim_{n\to\infty} \mathbb{E}[|Y_n|^2] = 0$ ([4]). By the construction, $Y_{n-1}, \eta_1^{(n)}$ and $\eta_2^{(n)}$ are mutually independent. Thus it follows from (0.19equation.0.19) that

$$\mathbb{E}[|Y_n|^2] = \mathbb{E}[|Y_{n-1}|^2]\left(A^2 + B^2 + (1-\theta)h\mathbb{E}\left[\left[C + D\eta_1^{(n)} + E\left(\eta_1^{(n)}\right)^2\right]^+\right]\right). \tag{0.21}$$

Lemma 0.5. *Assume either one of the following is true:*

$$\lambda \ge 0, \textit{ and } h > 0; \tag{0.22}$$

$$\lambda < 0, \textit{ and } 0 < h < \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right). \tag{0.23}$$

Then we have

$$\mathbb{E}\left[[C + D\eta_1^{(n)} + E(\eta_1^{(n)})^2]^+\right] \le C + E + o(h^2). \tag{0.24}$$

Proof. Consider the function $g(x) := C + Dx + Ex^2$, $x \in \mathbb{R}$. As we observed before, both C and E are positive. By straightforward computations, $g(x) > 0$ for $x \in (-\infty, x_1) \cup (x_2, \infty)$, where

$$x_1 = -\frac{1 + \sqrt{\frac{\alpha_2}{\alpha_1}} + \lambda\theta h}{\mu\sqrt{\theta h}} < x_2 = \frac{-1 + \sqrt{\frac{\alpha_2}{\alpha_1}} - \lambda\theta h}{\mu\sqrt{\theta h}}.$$

Note from (0.8equation.0.8) that $0 < \frac{\alpha_2}{\alpha_1} < 1$. Thus $-1 + \sqrt{\frac{\alpha_2}{\alpha_1}} < 0$ and $x_2 < 0$. Moreover, $x_2 \to -\infty$ as $h \downarrow 0$. Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$ denote the probability density function of a standard normal random variable. Since $\eta_1^{(n)}$ is normally distributed with mean 0 and variance 1, we compute

$$\begin{aligned}
\mathbb{E}\Big[[C + D\eta_1^{(n)} + E(\eta_1^{(n)})^2]^+\Big] &= \int_{-\infty}^{x_1} (C + Dx + Ex^2)\varphi(x)\,dx + \int_{x_2}^{\infty} (C + Dx + Ex^2)\varphi(x)\,dx \\
&\le C + E\int_{-\infty}^{\infty} x^2\varphi(x)\,dx + D\int_{-\infty}^{x_1} x\varphi(x)\,dx + D\int_{x_2}^{\infty} x\varphi(x)\,dx \\
&= C + E - D\int_{x_1}^{x_2} x\varphi(x)\,dx \\
&= C + E + \frac{D}{\sqrt{2\pi}}\left[\exp\left(-\frac{x_2^2}{2}\right) - \exp\left(-\frac{x_1^2}{2}\right)\right].
\end{aligned}$$

The proof will be complete if we can show that

$$\frac{D}{\sqrt{2\pi}}\left[\exp\left(-\frac{x_2^2}{2}\right) - \exp\left(-\frac{x_1^2}{2}\right)\right] = o(h^2) \text{ as } h \downarrow 0.$$

To this end, we note from the expression for D in (0.20equation.0.20) and the fact that $x_1 < x_2 < 0$ that

$$\left|\frac{D}{\sqrt{2\pi}h^2}\left[\exp\left(-\frac{x_2^2}{2}\right) - \exp\left(-\frac{x_1^2}{2}\right)\right]\right| \le K_1 \frac{e^{-\frac{x_2^2}{2}}}{h^{\frac{3}{2}}} \le K_2 \frac{h^{-\frac{3}{2}}}{e^{\frac{K_3}{h}}} \to 0,$$

as $h \downarrow 0$, where K_1, K_2, and K_3 are positive constants independent of h. This completes the proof of the lemma. ☐

Putting (0.24equation.0.24) into (0.21equation.0.21) and using the expressions for $A, \ldots, E$ in (0.20equation.0.20), detailed computations reveal that

$$\begin{aligned}
\mathbb{E}[|Y_n|^2] &< \mathbb{E}[|Y_{n-1}|^2]\left(A^2 + B^2 + (1-\theta)h(C + E + o(h^2))\right) \\
&= \mathbb{E}[|Y_{n-1}|^2]\left[1 + (2\lambda + \mu^2)h + \frac{1}{12}[22\lambda^2 + 20\lambda\mu^2 + 5\mu^4]h^2 + o(h^3)\right].
\end{aligned} \tag{0.25}$$

Furthermore, for the expression inside the brackets of the right-hand side of (0.25equation.0.25), we notice that

$$22\lambda^2 + 20\lambda\mu^2 + 5\mu^4 = 22\Big(\lambda + \frac{5}{11}\mu^2\Big)^2 + \frac{5}{11}\mu^4 > 0. \tag{0.26}$$

Next we compute the discriminant

$$\Delta = (2\lambda + \mu^2)^2 - 4\frac{1}{12}(22\lambda^2 + 20\lambda\mu^2 + 5\mu^4) = -\frac{10}{3}\left[\Big(\lambda + \frac{2}{5}\mu^2\Big)^2 + \frac{1}{25}\mu^4\right] < 0.$$

Therefore it follows that for any $h > 0$,

$$1 + (2\lambda + \mu^2)h + \frac{1}{12}[22\lambda^2 + 20\lambda\mu^2 + 5\mu^4]h^2 > 0.$$

Putting this observation into (0.25equation.0.25), we obtain a condition for mean square stability of the weak Simpson method (0.9equation.0.9)–(0.10equation.0.10) for (0.11equation.0.11)

$$1 + (2\lambda + \mu^2)h + \frac{1}{12}[22\lambda^2 + 20\lambda\mu^2 + 5\mu^4]h^2 < 1. \tag{0.27}$$

Since $h > 0$, and noting (0.26equation.0.26), we can rewrite equation (0.27equation.0.27) as

$$0 < h < \frac{-12(2\lambda + \mu^2)}{22\lambda^2 + 20\lambda\mu^2 + 5\mu^4}. \tag{0.28}$$

Note that when $\lambda \geq 0$, the set of h that satisfies (0.28equation.0.28) is an empty set. On the other hand, when $\lambda < 0$, $2\lambda + \mu^2 < 0$, and $h > 0$ satisfies (0.23equation.0.23) and (0.28equation.0.28), then the weak Simpson method (0.9equation.0.9)–(0.10equation.0.10) is mean-square stable for (0.11equation.0.11). In other words, given $(\lambda, \mu) \in S_P$, the combination of (0.23equation.0.23) and (0.28equation.0.28):

$$0 < h < \min\left\{ \frac{-12(2\lambda + \mu^2)}{22\lambda^2 + 20\lambda\mu^2 + 5\mu^4}, \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right) \right\}. \tag{0.29}$$

gives a sufficient condition for mean-square stability of the weak Simpson method (0.9equation.0.9)–(0.10equation.0.10) for (0.11equation.0.11).

Conversely, suppose the weak Simpson method with discretization stepsize $h > 0$ is mean-square stable for (0.11equation.0.11). Note from (0.21equation.0.21) that

$$\mathbb{E}[|Y_n|^2] \geq \mathbb{E}[|Y_{n-1}|^2](A^2 + B^2) = \mathbb{E}[|Y_{n-1}|^2](1 + (2\lambda + \mu^2\theta)h + O(h^2)).$$

Thus for the weak Simpson method to be mean-square stable, λ, μ, and θ necessarily satisfy $2\lambda + \mu^2\theta \leq 0$.

We summarize the above discussion into the following theorem:

Theorem 0.6. *The following assertions are true:*

(a) *Given $(\lambda, \mu) \in S_P$, the weak Simpson method is mean-square stable if the discretization stepsize h satisfies* (0.29equation.0.29). *Therefore the mean square stability of the process* (0.11equation.0.11) *implies the mean square stability of the weak Simpson method if the discretization stepsize h satisfies* (0.29equation.0.29).

(b) *Conversely, if the weak Simpson method with discretization stepsize $h > 0$ is mean-square stable for* (0.11equation.0.11), *then the parameters λ and μ of* (0.11equation.0.11) *satisfies* $2\lambda + \mu^2\theta \leq 0$.

We can visualize the stability region when

$$\frac{-12(2\lambda + \mu^2)}{22\lambda^2 + 20\lambda\mu^2 + 5\mu^4} < \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right). \tag{0.30}$$

It is common in the literature to visualize the region of stability in the xy-plane, in which $x = \lambda h$ and $y = \mu^2 h > 0$. Then we using (0.27equation.0.27), we have

$$S_M := \{(x,y) \in \mathbb{R}^2 : 22x^2 + 20xy + 5y^2 + 24x + 12y < 0\}. \tag{0.31}$$

Note that since $22x^2 + 20xy + 5y^2 = 22(x + \frac{5}{11}y)^2 + \frac{5}{11}y^2 > 0$, for any $(x,y) \in S_M$, we necessarily have $24x + 12y < 0$ or $y < -2x$. See Figure 2Real mean-square stability domain for weak Simpson method (crossed hashing)figure.2 for the plot of the mean-square stability domain for the weak Simpson method S_M.

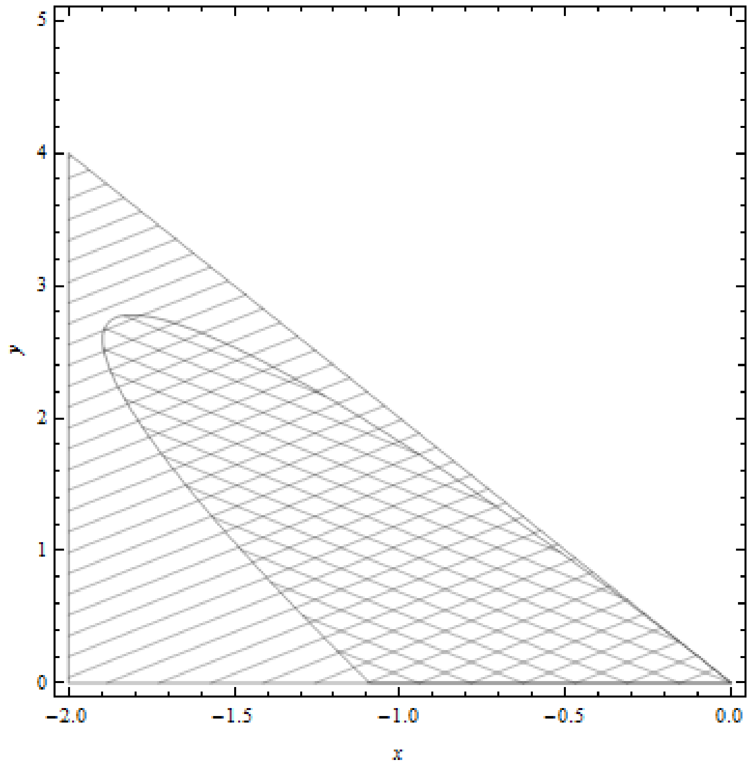

FIGURE 2. Real mean-square stability domain for weak Simpson method (crossed hashing)

Example 0.7. *Again, we test the mean-square stability over* $[0,30]$ *with non-random initial value* $X_0 = 1$*. We take* $\lambda = -2$ *and* $\mu = \sqrt{2}$ *in* (0.11equation.0.11)*. We have* $2\lambda + \mu^2 < 0$ *and hence thanks to* (0.17equation.0.17)*, the trivial solution of* (0.11equation.0.11) *is mean-square stable. The left-hand side of* (0.30equation.0.30) *is equal to* $\frac{6}{7}$ *and the right-hand side of* (0.30equation.0.30) *is equal to* 0.37*. We apply weak Simpson method to simulate 45000 discrete sample paths of* (0.11equation.0.11) *for stepsizes* $h = 1, \frac{1}{2}, \frac{1}{4}$*. The stepsize* $h = \frac{1}{4}$ *satisfy* (0.30equation.0.30) *but not the stepsizes* $h = 1$ *and* $h = \frac{1}{2}$*. Therefore the weak Simpson method is mean-square stable for* (0.11equation.0.11) *when* $h = \frac{1}{4}$*. We plot the sample average of* Y_n^2 *against* $t_n := nh$ *with logarithmically scaled y-axis in Figure 3Mean-square stability testfigure.3. The numerical experiments indicates that weak Simpson method is mean-square stable for* $h = \frac{1}{2}$ *or* $\frac{1}{4}$*, and unstable for* $h = 1$*. These observations are consistent with Theorem 0.6thm.0.6.*

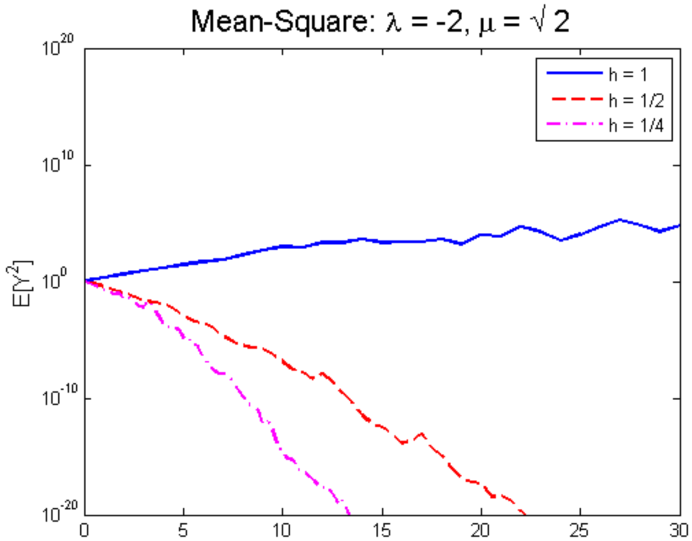

FIGURE 3. Mean-square stability test

REFERENCES

1. Anderson, D. F. and Mattingly, J. C. (2011). A weak trapezoidal method for a class of stochastic differential equations. *Commun. Math. Sci.*, 9(1):301–318.
2. Bruti-Liberati, N. and Platen, E. (2008). Strong predictor-corrector Euler methods for stochastic differential equations. *Stoch. Dyn.*, 8(3):561–581.
3. Higham, D. J. (2000a). *A*-stability and stochastic mean-square stability. *BIT*, 40(2):404–409.
4. Higham, D. J. (2000b). Mean-square and asymptotic stability of the stochastic theta method. *SIAM J. Numer. Anal.*, 38(3):753–769 (electronic).
5. Higham, D. J. (2001). An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.*, 43(3):525–546 (electronic).
6. Hüsler, J., Liu, R. Y., and Singh, K. (2002). A formula for the tail probability of a multivariate normal distribution and its applications. *Journal of multivariate analysis*, 82(2):422–430.
7. Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.
8. Khasminskii, R. (2012). *Stochastic stability of differential equations*, volume 66 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2nd edition.
9. Kloeden, P. E. and Platen, E. (1992). *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin.
10. Kushner, H. J. (1967). *Stochastic stability and control*, volume 33 of *Mathematics in Science and Engineering*. Academic Press, New York.
11. Mao, X. and Yuan, C. (2006). *Stochastic differential equations with Markovian switching*. Imperial College Press, London.
12. Meyn, S. P. and Tweedie, R. L. (2009). *Markov chains and stochastic stability*. Cambridge University Press, Cambridge.
13. Øksendal, B. (2003). *Stochastic differential equations, An introduction with applications*. Universitext. Springer-Verlag, Berlin, sixth edition.
14. Saito, Y. and Mitsui, T. (1996). Stability analysis of numerical schemes for stochastic differential equations. *SIAM J. Numer. Anal.*, 33(6):2254–2267.
15. Saito, Y. and Mitsui, T. (2002). Mean-square stability of numerical schemes for stochastic differential systems. *Vietnam J. Math.*, 30(suppl.):551–560.
16. Walsh, J. B. (1986). *An introduction to stochastic partial differential equations*. Springer, Berlin.
17. Yin, G. G. and Zhu, C. (2010). *Hybrid Switching Diffusions: Properties and Applications*, volume 63 of *Stochastic Modelling and Applied Probability*. Springer, New York.