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An Approximation for General Transit Time Function

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Abstract

People have unpleasant experience of driving through congested road network of urban areas, where transit time is not constant but flow-dependent. Network flow over time with flow-dependent transit times is more realistic than flow over time with fixed transit time on arcs which is an active area of research from last two decades. In traffic assignment problem, transmission time of the vehicles directly depends on the number of vehicles entering on the road. So, general transit time function is non-decreasing, convex and left-continuous. In this paper we discuss how a general transit time function can be approximated by the step function. It relaxes the inflow-dependent transit times to a bow graph. We calculate maximum bow flow by using temporally repeated flow for the given time horizon. Our aim is to show how flow value in step function converges to the flow value of general transit time function by increasing number of points in the domain of the step function.

Key words. Inflow-dependent transit times, bow flow, temporally repeated flow, step function, inflow-preserving flow.

1 Introduction

In developing countries like Nepal, the tendency of people clustering in the cities and metropolitan areas are increasing. The metropolitan areas are facing series of congestion problems due to increasing number of vehicles, which deteriorate the quality of life. Rapid increase in the population of cities, unavoidable travel demand and lack of new transportation facilities worsen the traffic condition. Around the middle of last century with the outstanding work of Ford and Fulkerson[8], a systematic mathematical treatment of network flow theory started. Historically, the study of network flows mainly originated from problems related to transportation of goods and people. In our daily life, network are visible everywhere. A good network of water supply and power supply make our life comfortable and convenient at homes. A good network of cable keeps us connected to the world at every moment. Similarly, a good network of roads keeps mobility of a city normal. However, the flow on network affects the transit time.

In all of these problem domains and in many more, we intend to move some entity from one point to another in an underlying network and to do as efficiently as possible. Traditionally, flow over time in fixed transit time on the arcs, the period it takes to transverse such arc does not depend on the prevalent flow situation on it. Any one caught up in a traffic jam realize that the above assumption fails to encapsulate the practical situations. We would like to take a bit liberty to simplify the idea here. For example, let us suppose that the average transit time to travel from City Center of Kathmandu to Tribhuvan university is

30 minutes technically. However, we hardly find that the distance is covered exactly in 30 minutes every time as it is likely to get affected by the number of vehicles running on the street. Thus, the transit time is not constant, but flow-dependent.

A highly reliable model of flow-dependent transit times on arcs should be considered density, speed and flow rate. However, one can hardly find any algorithmic framework which garners the above mentioned fundamentals and cannot generate reasonable conclusion even for modest network size. Hence we focus on the model with inflow-dependent transit times.

Merchant and Nemhauser[14] were the first to provide the model for flow-dependent transit times, where for every arc there is both a flow-dependent cost function and exit function. It determines the amount of traffic that can leave the arc in dependence of the amount of flow on that particular arc. This model is difficult to be handled due to the characteristics of non-linearity and non-convexity. Carey [4] slightly improved the model of Merchant and Nemhauser [14] by replacing maximum out flow with actual out flow. Carey's model [5] is a convex programming model in which the capacity outflow function is convex in addition to the cost function. Thus the latter model exploits rich convex optimality properties [7]. Carey and Subrahmanian[6] modified the model to convex linear approach for addressing the flow behaviour with link travel times, mainly depending on inflow-outflow rates and some traffic control effects ahead on the links.

In the literature, different approaches of modeling flow-dependent transit times are depicted. We will briefly depict two known models: inflow-dependent and load-dependent transit times. In these models, time is an essential component for the flow of vehicles that travels through a road network. Transmission time directly depends on the flow rate of vehicles on the road.

In inflow-dependent transit times, described by Köhler et al. [10], transit time on arc depends on the current rate of inflow into that arc. It is relevant in many applications like evacuation planning, communication network and production system. The transit time function measures how fast the transit time on an arc increases as flow rate grows. Commonly used transit time functions are monotonically increasing and convex. The major disadvantage of this model is that it does not satify first in first out (FIFO) property.

To over come this difficulty, Köhlar and Skutella[11] provide more realistic approach of transit time. They assume that the transit time depends not only on the amount of flow entering an arc but also on the amount of flow being currently in the arc i.e., load. They provide a model in this setting and gave a 2-approximation algorithm for the quickest source-sink flow problem. Baumann and Köhlar[3] have shown that earliest arrival flow do not exist in general for the case of flow-dependent transit times and provide a relaxed version of the problem i.e., α-earliest arrival flow problem. They relax the time instead of flow. The lower and upper bounds of the α-earliest arrival problem are calculated by them. Among these two models our main focus is on the model of inflow-dependent transit times.

In this paper, our aim is to show how the general transit time function is approximated by piece-wise constant, non-decreasing and left(right)-continuous step function. To reflect the inflow-dependent nature of the flow by taking numerical example, we calculate the flow value by using lower as well as upper step functions and justified the result of theoretical approximation. In both cases we have shown that flow value of step function converges to the actual flow value numerically. Together with this, we calculate inflow-preserving flow, which is same as the flow value obtained by step functions in all cases. We have illustrated by an example how to calculate temporally repeated flow for a network having bow graph.

The plan of the paper is as follows. In Section 2, we start with notations and definitions. In Section 3, we include model and formulation of inflow-dependent transit times. In Section 4, we provide the numerical approximation of transit time function by taking lower and upper step functions and also calculate inflow-preserving flow in all cases. This paper concludes in Section 5.

2 Preliminaries

We start with some notations for static flow problems, then we move on to the flow over time setting with constant transit time, as introduced by Ford and Fulkerson[8]. Together with this we briefly discuss about flow overtime with flow-dependent transit times.

Static flows. Given a network $\mathcal{N} = (V, A, u_a, s, t)$, the set of nodes V is partitioned into source node, intermediate nodes and sink node. Flow originates from the source node, travels through the arcs of the network via intermediate nodes to the sink, where it finally disappear. More precisely, a function $x : A \to \mathbf{R}^+$ is called the static flow function if the capacity constraints are satisfied, i.e., it holds $x_a \leq u_a$ for all $a \in A$. A static flow is said to observe the flow conservation in node v if

$$\sum_{a \in A^-(v)} x_a - \sum_{a \in A^+(v)} x_a = 0 \tag{1}$$

holds. Here $A^+(v)$ and $A^-(v)$ denote the set of outgoing arcs of node v and incoming arcs into node v respectively [1].

Flow over time with constant transit time. In static flow problem we are given a network $\mathcal{N}$ with source, sink and capacity on the arc. In addition to this, flow over time contains constant transit time τ_a on the arc. We denote the flow rate on arc a at time θ by $\phi_a(\theta)$ and the flow over time function as $\phi : A \times T \to \mathbf{R}^+$. In flow over time with fixed transit time the flow on arc a progresses at constant speed. Actually, the transit time τ_a of an arc a is the amount of time it takes to travel from the tail(a) to the head(a) of that arc. In general a flow entering arc a at time θ arrives at head(a) at time $\theta + \tau_a$. We say that any flow over time has time horizon T if no flow is entering an arc a after the time $T - \tau_a$.

Temporally repeated flow. [8] Let x be feasible static $s-t$ flow in $\mathcal{N}$ with path decomposition $(x_p)_{p \in P}$ where P is a set of $s-t$ paths. If the transit time $\tau_p = \sum_{a \in P} \tau_a$ of every path $p \in P$ is bounded above by T, the static $s-t$ flow x can be turned into a temporally repeated $s-t$ flow ϕ as follows.

Lemma 1. *The value of a flow over time ϕ that is computed as a temporally repeated flow of a static flow x equals*

$$|\phi| = \sum_{p\in\mathbf{P}}(T-\tau_p)x_p = T|x| - \sum_{a\in A}\tau_a x_a. \qquad (2)$$

Observe that the flow value is independent of the path decomposition P. One natural goal is to find the temporally repeated flow having maximum flow value. The existence of such a kind of maximum $s-t$ flow over time is given by Ford and Fulkerson [8].

Theorem 1. *[8] There always exists a temporally repeated flow that is a maximum $s-t$ flow over time such a temporally repeated can be computed using a static min cost $s-t$ flow.*

Flow over time with flow-dependent transit times. Although flow over time are much more suitable than static flows to model real-world situations. Often the constant transit times do not model reality in a sufficiently precise way. A more accurate method for describing this correlation is provided by the use of flow-dependent transit times instead of constant transit time.

In this, we consider problems in which transit times may vary according to the current situation of flow in arcs. If we consider the speed, density and flow rate along the arc, then it is known as flow-dependent transit times. Inflow-dependent transit times is the relaxation of flow-dependent transit times. The rate of inflow is measured when it enters in the arc and moves with constant speed throughout the arc. Distinct from this, another relaxation is load-dependent transit times in which all amount of flow in the arc is considered as one unit and travels with same speed [7]. Transit time differs when any unit of flow enters or leaves the arc continuously. Our main concern is approximation of transit time function with inflow-dependent transit times.

Time-expanded network. The discrete flow over time problems are dealt in [8] introducing static time-expanded network $\mathcal{N}^{\mathcal{T}} = (V^T, A^M \cup A^H)$, where $V^T = \{v_\theta : v \in V, \theta = 0, 1, \ldots, T\}$, $A^M = \{(v_\theta, w_{\theta+\tau_e}) : a = (v, w) \in A, \theta = 0, 1, \ldots, T-\tau_a\}$ and $A^H = \{(v_\theta, v_{\theta+1}) : v \in V, \theta = 0, 1, \ldots, T-1\}$. A moment arc $a \in A^M$ has the constant capacity u_a, whereas a holdover arc in A^H has infinite capacity that allows storage of flow at intermediate nodes. The flow over time in $\mathcal{N}$ is equivalent to the static flow in $\mathcal{N}^{\mathcal{T}}$. To realize it, at a point of time θ, one can take a flow on $a_\theta \in A^M$ as the flow amount into $a \in A$, and conversely, the average flow on $a \in A$ of flow over time ϕ in T as the flow value into $a_\theta \in A^M$.

3 Inflow-dependent transit times

The main objective of this section is to study about flow over time with inflow-dependent transit times, which is an extension of the flow over time with fixed transit time. In inflow-dependent transit times [10], transit time experienced by an immeasurably small unit of flow on an arc is determined when entering the arc and only depends on the inflow rate

at that moment of time. In this, flow entering arc a at time θ arrives at head(a) at time $\theta + \tau_a(\phi_a(\theta))$. Particularly, the transit time of an arc only depends on the current flow rate. In time-dependent flow, all arc must be empty from time T on, so for all arcs $a \in A$ and $\theta \in \mathbf{R}^+$ we have $\theta + \tau_a(\phi_a(\theta)) < T$ whenever $\phi_a(\theta) > 0$.
Flow conservation, in this case, is modeled as

$$\sum_{a \in A^+(v)} \int_{0 \le \theta < \zeta} \phi_a(\theta) d\theta - \sum_{a \in A^-(v)} \int_{\theta \ge 0 : \theta + \tau_a(\phi_a(\theta)) \le \zeta} \phi_a(\theta) d\theta \le 0, \tag{3}$$

for all $\zeta \in [0, T)$ and $v \in V \setminus \{s, t\}$ and equality holds for $v \in V \setminus \{s, t\}$ at time $\zeta = T$. The flow over time ϕ satisfies the supply and demands if

$$\sum_{a \in A^+(v)} \int_{0 \le \theta < \zeta} \phi_a(\theta) d\theta - \sum_{a \in A^-(v)} \int_{\theta \ge 0 : \theta + \tau_a(\phi_a(\theta)) \le \zeta} \phi_a(\theta) d\theta = D, \tag{4}$$

for $v \in \{s, t\}$. The value of $s - t$ flow over time ϕ is given by

$$|\phi| := \sum_{a \in A^+(s)} \int_0^T \phi_a(\theta) d\theta - \sum_{a \in A^-(s)} \int_{\tau_a}^T \phi_a(\theta) d\theta. \tag{5}$$

3.1 Bow graph

As described in [10, 9], the bow graph denoted by $\mathcal{N}^B = (V^B, A^B, u_e, \tau_e, s, t)$, arises from the original graph by expanding arc $a \in A$ according to its transit time function. In bow graph $\mathcal{N}^B$, every arc $e \in A^B$ has capacity u_e and a constant transit time $\tau_e \in \mathbf{R}^+$. To construct the bow graph, let us take an arc $a \in A$ having capacity u_a with breakpoints $0 = u_0 < u_1, ... < u_m = u_a$ and corresponding transit times $\tau_1 < \tau_2 < ... < \tau_m$, where $\tau_a^s(x) := \tau_i$ for $x \in (u_{i-1}, u_i]$. This means that flow entering the arc a at the rate $x \in (u_{i-1}, u_i]$ needs τ_i time unit to traverse arc. The arc a is replaced by two types of arcs: bow arcs $b_1, b_2, ..., b_m$ and regulating arcs $r_1, r_2, ..., r_m$. The bow arcs are uncapaciated and they represent all possible transit times of arc a. The transit times of bow arc b_i is given by $\tau_i,\ i = 1, 2, ..., m$. The regulating arcs have zero transit time and they limit the amount of flow entering the bow arcs. Their capacities are chosen according to the breakpoints of transit time function $\tau_a^s(x_a)$, that is, the capacity of arc r_i is set to $u_i,\ i = 1, 2, ..., m$. The set of regulating arcs and bow arcs of an arc $a \in A$ is denoted by A_a^B. For every arc $e \in A_a^B$, we denote $a(e)$ as the corresponding arc a in A. The size of A_a^B is linear in the number of breakpoints of τ_a^s. In bow graph nodes in $\mathcal{N}^B$ corresponding to nodes in $\mathcal{N}$ are original nodes and the remaining nodes are artificial.

It is to be note that flow entering in an original arc $a \in A$ at any time θ might experience different transit times in bow arcs of $\mathcal{N}^B$. So to accumulate the flow in to a single bow arc for each time θ, we use to push the flow from fast flow carrying arcs to slower flow carrying arc, called inflow-preserving flow.

Inflow-preserving flow. Let ϕ^B be a flow over time in bow graph $\mathcal{N}^B$, then flow ϕ^B is inflow-preserving if for every original arc $a \in A$ and at every point in time θ, the flow ϕ^B sends flow into at most one bow arc [12].

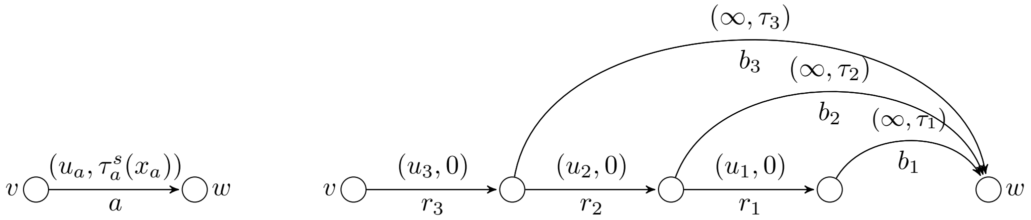

Figure 1: Expansion of a single arc $a = (v, w)$ in the bow network with travel times τ_1, τ_2 and τ_3, for at most u_1, u_2 and u_3 flow units, respectively.

3.2 Temporally repeated flow for flow dependent transit times

Ford and Fulkersion[8] already defined the notion of temporally repeated flow in case of fixed transit time. Köhlar et al.[10] extended this idea in the case of flow-dependent transit times.

Lemma 2. *The value of a temporally repeated flow ϕ with flow-dependent transit times $(\tau_a)_{a\in A}$ and underlying path decomposition $(x_p)_{p\in\mathbf{P}}$ is given by*

$$|\phi| = \sum_{p\in\mathbf{P}} (T - \tau_p(x))x_p = T|x| - \sum_{a\in A} \tau_a(x_a)x_a. \tag{6}$$

Example 1. Figure 2(a) represents a network with bow graph. Flow value by using temporally repeated flow with time horizon $T = 5$ is 13, where there exists no augmenting path that yields the flow more than 13. There are three paths $P_1 = s{-}v{-}w{-}t$, $P_2 = s{-}w{-}t$ and $P_3 = s{-}v{-}t$. Arc (s, v) has transit time 0, flow value 3 units and capacity is also 3 units. Similarly transit time, flow value and capacity of other arcs as shown in the Figure 2(a). Arc (v, t) contains two types of arcs: regulating arcs and bow arcs, with inflow-dependent transit times. Regulating arcs r_1, r_2 having capacities 2 and 1 with transit time zero and bow arcs b_1, b_2 having infinite capacity with transit times 2 and 3, respectively. In path P_1 and P_2 transit time is constant with flow value 4 each. In path P_3 transit times are inflow-dependent with flow value 5. Total flow value for $T = 5$ is calculated as follows.

$$\begin{aligned}
|\phi| &= \sum_{p\in\mathbf{P}} (T - \tau_p(x))x_p \\
&= (T - \tau_{P_1}(x))x_{P_1} + (T - \tau_{P_2}(x))x_{P_2} + (T - \tau_{P_3}(x))x_{P_3} \\
&= (5 - 1) * 1 + (5 - 3) * 2 + [(5 - 2) * 1 + (5 - 3) * 1] \\
&= 4 + 4 + 5 = 13.
\end{aligned}$$

The network with bow graph and its Time-expanded graph (Fan graph) of Example 1 is shown in Figure 2.

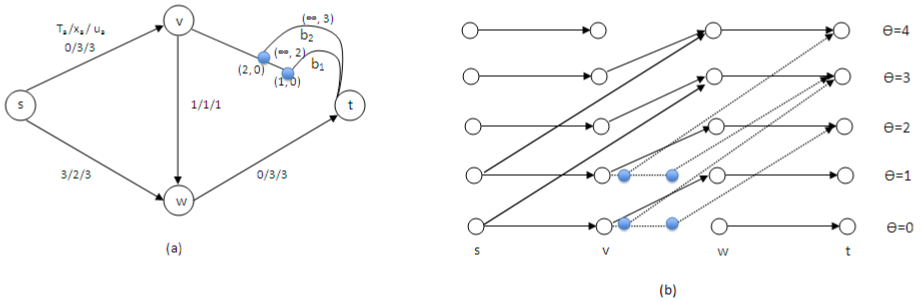

Figure 2: (b) Represents Time-expanded graph of network (a) with bow.

4 Approximation to general transit time function

In general, transit time functions are monotonic, non-decreasing and convex. For better reflection of inflow-dependent flow we use bow graph or fan graph. The idea is to approximate transit time function by step function. For this we need the technical requirement that the transit time function must be left-continuous in case of lower step function and right-continuous in case of upper step function.

4.1 Theoretical approximation

Köhlar et al.[10] use the approximation of general transit time function by using step function theoretically. The theorem given below fulfills the requirements of constructing bow graph $\mathcal{N}^B$ according to step functions.

Theorem 2. *If $\tau : [0, u] \rightarrow \mathbf{R}^+$, be non-negative, non-decreasing and left continuous function, then for $\beta, \gamma > 0$, there exist a step function $\tau^s : [0, u] \rightarrow \mathbf{R}^+$ with*

(i) $\tau^s(x) \leq \tau(x) \leq (1+\gamma)\tau^s(x) + \beta$ *for every* $x \in [0, u]$,

(ii) the number of break points of τ^s is bounded by $\lceil log_{1+\gamma}(\tau(u)/\beta)\rceil + 1$.

4.2 Numerical approximation

We use the approximation of general transit time function by using step function. To justify the approximation with the help of an example, we use lower as well as upper step function.

Example 2. Consider the one-arc network, as shown in the Figure 3, together with the simple linear transit time function defined by $\tau(x) = 2x$ for $0 \leq x \leq 2$ and a capacity two. We consider a flow model with inflow-dependent transit times. Let $T = 6$ be the considered time horizon. If we send flows from s to t at a flow rate of 2 in time interval $[0, 2)$ and flow rate linearly decreasing from 2 to 0 in the time interval $[2, 6)$ then flow of 8 units has been reached to the sink t by time $T = 6$. In fact, this is the maximum amount of flow that can be sent from s to t in this time horizon.

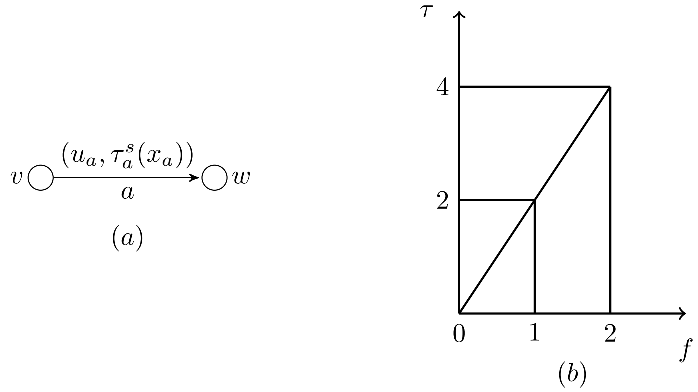

Figure 3: (b) Represents inflow-dependent transit times of single arc (a).

4.2.1 Approximation by lower step function

In this subsection we construct lower step function of given general transit time function by taking number of steps 2, 4, 8 and 16 then we calculate maximum bow flow and inflow-preserving flow for given time horizon $T = 6$.

Approximation by taking 2 steps. For the linear transit time function of Example 2, we define the step function $\tau^s(x)$ by

$$\tau^s(x) = \begin{cases} 0 & \text{for } x \in [0, 1) \\ 2 & \text{for } x \in [1, 2). \end{cases}$$

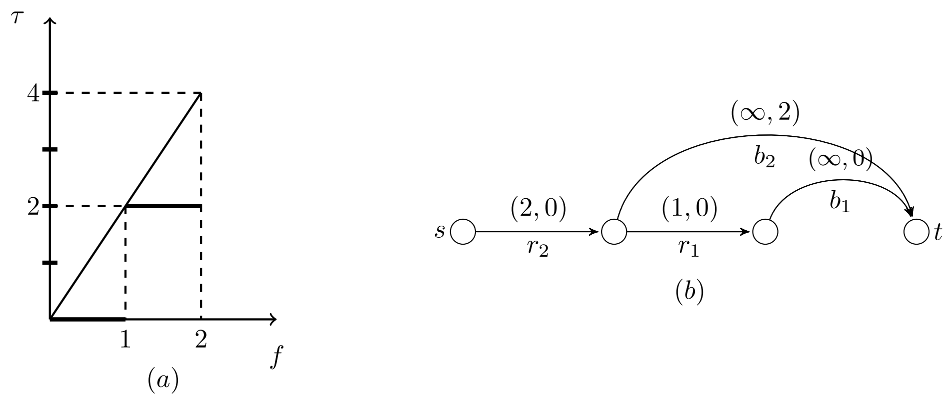

Figure 4: (b) Represents bow graph according to transit time function with 2 steps in (a).

There exist a flow over time in bow graph $\mathcal{N}^B$ which sends $D = 10$ units of flow from s to t within given time $T = 6$. We can calculate this by using temporally repeated flow (TRF). We send flow at rate 1 into the $s - t$ path containing lower bow arc b_1 during time interval $[0, 6)$ and flow rate 1 into the $s - t$ path containing upper bow arc b_2 during time interval $[0, 4)$. The flow value is obtained as follows

$$TRF = \sum_{e \in A^B} (T - \tau_e) x_e^B = (6 - 0) * 1 + (6 - 2) * 1 = 10.$$

We also calculate an inflow-preserving flow over time ϕ in $\mathcal{N}^B$ which satisfies demand $D = 10$ within time $T = 6$. For this, we send flow at rate 2 into bow arc b_2 during time $[0, 4)$, then it sends flow at the rate 1 into bow arc b_1 until time $[4, 6)$.

Approximation by taking 4 steps. For the linear transit time function of above Example 2, we define the step function $\tau^s(x)$ by

$$\tau^s(x) = \begin{cases} 0 & \text{for } x \in [0, 0.5) \\ 1 & \text{for } x \in [0.5, 1) \\ 2 & \text{for } x \in [1, 1.5) \\ 3 & \text{for } x \in [1.5, 2). \end{cases}$$

There exists a flow over time in bow graph $\mathcal{N}^B$ which sends $D = 9$ units of flow from s to t within given time $T = 6$. We calculated by using temporally repeated flow which sends flow at rate 0.5 into the $s - t$ path containing lower bow arc b_1 during time interval $[0, 6)$ and flow rate 0.5 into the $s - t$ path containing upper bow arc b_2 during time interval $[0, 5)$. Similarly, flow rate 0.5 into the $s - t$ path containing upper bow arc b_3 during time interval $[0, 4)$ and flow rate 0.5 into the $s - t$ path containing upper bow arc b_4 during time interval $[0, 3)$.

We also calculate an inflow-preserving flow over time ϕ in $\mathcal{N}^B$ which satisfies demand $D = 9$ within time $T = 6$. For this, we send flow at rate 2 into bow arc b_4 during time $[0, 3)$, then it sends flow at the rate 1.5 into bow arc b_3 until time $[3, 4)$, it sends flow at the rate 1 into bow arc b_2 until time $[4, 5)$, and flow at the rate 0.5 into bow arc b_1 until time $[5, 6)$.

Approximation by taking 8 steps. For the linear transit time function of above Example 2, we define the step function $\tau^s(x)$ by

$$\tau^s(x) = \begin{cases} 0 & \text{for } x \in [0, 0.25) \\ 0.5 & \text{for } x \in [0.25, 0.5) \\ 1 & \text{for } x \in [0.5, 0.75) \\ 1.5 & \text{for } x \in [0.75, 1) \\ 2 & \text{for } x \in [1, 1.25) \\ 2.5 & \text{for } x \in [1.25, 1.5) \\ 3 & \text{for } x \in [1.5, 1.75) \\ 3.5 & \text{for } x \in [1.75, 2). \end{cases}$$

There exist a flow over time in bow graph $\mathcal{N}^B$ which sends $D = 8.5$ units of flow from s to t within given time $T = 6$. Similarly, we calculate an inflow-preserving flow over time ϕ in $\mathcal{N}^B$ which satisfies demand $D = 8.5$ within time $T = 6$.

Approximation by taking 16 steps. As defined above, if we define step function $\tau^s(x)$ by taking intervals of length 0.125 in flow x then flow over time in bow graph $\mathcal{N}^B$ which sends $D = 8.25$ units of flow from s to t within given time $T = 6$. Similarly, we calculated an inflow-preserving flow over time ϕ in $\mathcal{N}^B$ which satisfies demand $D = 8.25$ within time $T = 6$.

4.2.2 Approximation by upper step function

In Example 2, all step functions taken are lower bound for general transit time function. Next we are going to take step functions that are right-continuous and upper bound for the general transit time function.

Approximation by taking 2 steps. For the linear transit time function of above Example 2, we define the step function $\tau^s(x)$ by

$$\tau^s(x) = \begin{cases} 2 & \text{for } x \in (0,1] \\ 4 & \text{for } x \in (1,2]. \end{cases}$$

There exist a flow over time in bow graph $\mathcal{N}^B$ which sends $D = 6$ units of flow from s to t within given time $T = 6$. By using temporally repeated flow it sends flow at rate 1 into the $s-t$ path containing lower bow arc b_1 during time interval $(0, 4]$ and flow rate 1 into the $s-t$ path containing upper bow arc b_2 during time interval $(0, 2]$. This can also be calculated by

$$TRF = \sum_{e \in A^B} (T - \tau_e) x_e^B = (6-2) * 1 + (6-4) * 1 = 6.$$

We calculate an inflow-preserving flow over time ϕ in $\mathcal{N}^B$ which satisfies demand $D = 6$ within time $T = 6$. For this, we send flow at rate 2 into bow arc b_2 during time $[0, 2)$, then it sends flow at the rate 1 into bow arc b_1 until time $[2, 4)$.

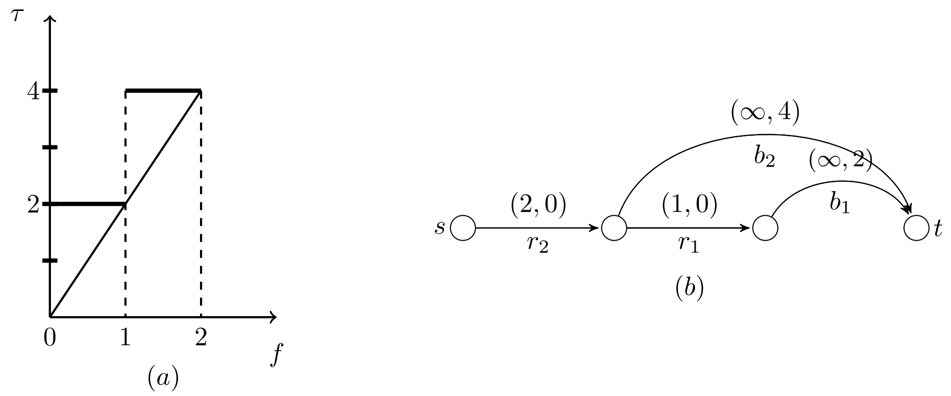

Figure 5: (b) Represents bow graph according to transit time function with 2 steps in (a).

Approximation by taking 4 steps. For the linear transit time function of above Example 2, we define the step function $\tau^s(x)$ by

$$\tau^s(x) = \begin{cases} 1 & \text{for } x \in (0, 0.5] \\ 2 & \text{for } x \in (0.5, 1] \\ 3 & \text{for } x \in (1, 1.5] \\ 4 & \text{for } x \in (1.5, 2]. \end{cases}$$

There exist a flow over time in bow graph $\mathcal{N}^B$ which sends $D = 7$ units of flow from s to t within given time $T = 6$ (using TRF) and inflow-preserving flow also satisfy same demand within same time.

Similarly, if we take 8 steps then demand $D = 7.5$ can sent within time $T = 6$ and if we take 16 steps then demand $D = 7.75$ can sent within time $T = 6$.
Numerical results obtained by approximating transit time function using step function is summarized in the table given below:

Table 1: Flows values for different step functions in Example 1 with $T = 6$

No. of steps	2	4	8	16	32	64	——	Flow for general transit time function
Flow in lower step function	10	9	8.5	8.25	8.125	8.0625	——	8
Flow in upper step function	6	7	7.5	7.75	7.875	7.9375	——	8

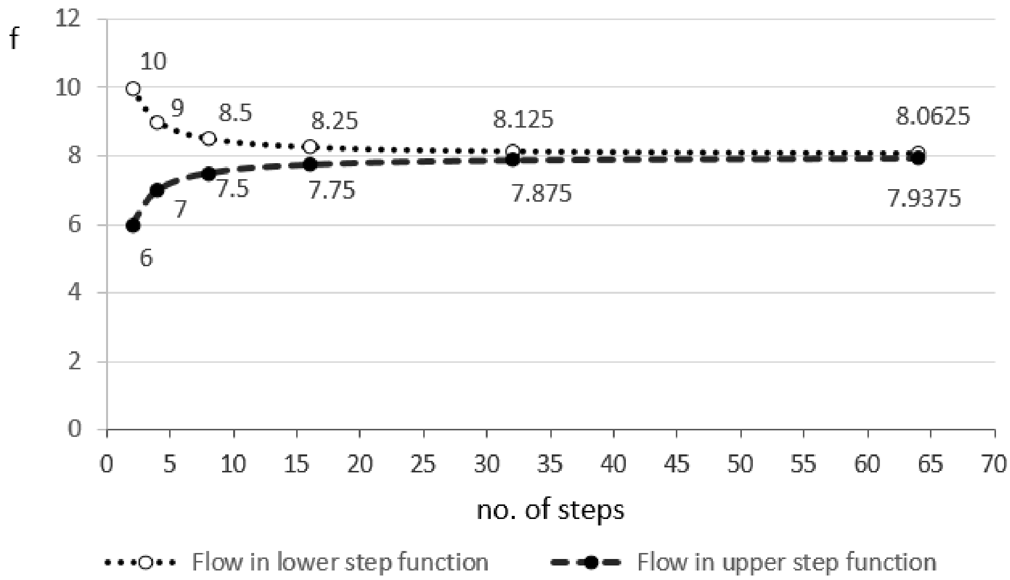

Figure 6: Graphical representation of Table 1.

The convergence of the flow of step function to the flow of general transit time function, as presented in Table 1, is shown graphically in Figure 6.

From the above example, we have following conclusions:

- For an single arc instance, the value of inflow-preserving flow over time in $\mathcal{N}^B$ contains maximum flow over time.
- For a network instance, maximum flow over time exists in case of inflow-dependent transit times.
- If the number of break points (steps) are increased, then flow value is better approximated by step function and it converges to maximum flow of general transit time function.

5 Conclusion

In this paper, we revisited the flow over time with inflow-dependent transit times and the bow graph. We introduced bow graph on general network and applied the idea of temporally repeated flow to calculate the flow value. We approximated the general transit time function by using step function. The approximation by using lower step function and upper step function shows that flow value calculated by the step function converges to the actual flow value of general transit time function by increasing the number of partition points. In all cases we also calculated inflow-preserving flows and showed that they are equal to the flow obtained by using step function. Numerical solution provides the reliability of approximation technique clearly.

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