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Speeds of Decay of Temperature Distribution in a Laterally Insulated Copper Bar

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Abstract

Employing one-dimensional heat equation, speeds of decay of the temperature distribution in a laterally insulated copper bar are analyzed. Eigenfunctions and their corresponding Eigenvalues of the model equation are derived and analyzed with different period of time. The temperature distribution with given initial temperatures on copper bar in period of time are also analyzed. One-dimensional heat equation of a laterally insulated copper bar is also solved for examining and analyzing its temperature distribution Eigenfunction and corresponding Eigenvalues. Further, finite difference method is used to solve the temperature distribution numerically, and comparing its results with analytical solution.

Keywords: eigenvalues, eigenfunctions, finite difference method, mathematical modelling

Introduction

The temperature distribution of the insulated copper bar is modelled by the partial differential equation. This is one dimensional heat equation in parabolic form. Several research activities are carried out in the past decades. Chabrowski (1984) discussed the problem of nonlocal value for nonlinear fractional q-difference equation. Lin (1994) demonstrated the existence, uniqueness, and continuous dependence of the solution of the one dimensional heat equation, and studied as finite difference methods, backward Euler and Crank-Nicolson scheme. Olmstead and Roberts (1996) derived the solution of the integral equation as an Eigenfunctions. Pokhrel and Lamsal (2020) observed the two dimensional partial differential equation. Pokhrel et al. (2020) used partial differential equation. Pokhrel (2022) analyzed the vibration of the string by discussing the Eigenfunctions and Eigenvalues of the one dimensional wave equation.

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In this paper, the model equation of one dimensional heat equation is presented with insulated copper bar. The temperature distributions are analyzed with different Eigenfunctions and Eigenvalues. Effects of the Eigenfunctions and the corresponding Eigenvalues on the temperature distribution on the insulated copper bar with different density, thermal conductivity and specific heat capacity of the copper bar initially are observed. I also analyze the temperature distribution by solving the model equation numerically. The temperature distribution is obtained by discritization model equation based on finite difference method as Crank Nicolson schemes. I compare the results obtained from the analytical and numerical approaches.

Methods

Formulation of Model Equations

The temperature is transferred from the regions of higher temperature to regions of lower temperature. In a metal rod, heat is transferred from regions of higher temperature to regions of lower temperature. The change of heat energy with respect to time Δ t is

$$s \times (\rho \, . \, A \, \Delta x) \, u(x, t + \Delta t) - s \times (\rho \, . \, A . \, \Delta x) \, u(x, t)$$

$$= s \rho A \, \Delta x \, [u(x, t + \Delta t) - u(x, t)] \qquad \ldots(1)$$

Applying the Fourier's law in heat transfer which states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area, and its change from left to right boundary as:

$$\Delta t \times A\left(-k\frac{\partial u}{\partial x}\right)_x - \Delta t \times A\left(-k\frac{\partial u}{\partial x}\right)_{x+\partial x} = \Delta t \; kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \ldots(2)$$

Equations (1) and (2) give the conservation of heat energy (Mohebbi and Dehghan, 2010) as:

$$s \rho A \; \Delta x) \, [u(x, t + \Delta t) - u(x, t)] = \Delta t \; kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial u}{\partial x}\right)_x\right],$$

$$\therefore \quad \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \frac{k}{s \rho}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial u}{\partial x}\right)_x\right].$$

Here, s is specific heat capacity, k is thermal conductivity, ρ is density and A cross sectional area of the rod of length L. Taking Δ t → 0 on both sides, to get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x^2},$$

where $c^2 = k/s\rho$ is called thermal diffusivity. This is a parabolic partial differential equation, and its solution is a function of space x and time t. So, u (x, t) gives the temperature distribution at x in time t.

Analytical Method

Consider a rod of length of length L with temperature u (x, t) in some fixed interval x = 0 to x = L, and for time t from t = 0 to t = ∞. The temperature distribution is not closed as in the case of Laplace and Poisson equations but propagate from initial temperature to open-ended region satisfying initial and boundary conditions as in Fig. (1). So, temperature distribution on rod is governed (Wazwaz, 2009) by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x^2}, \qquad \ldots(3)$$

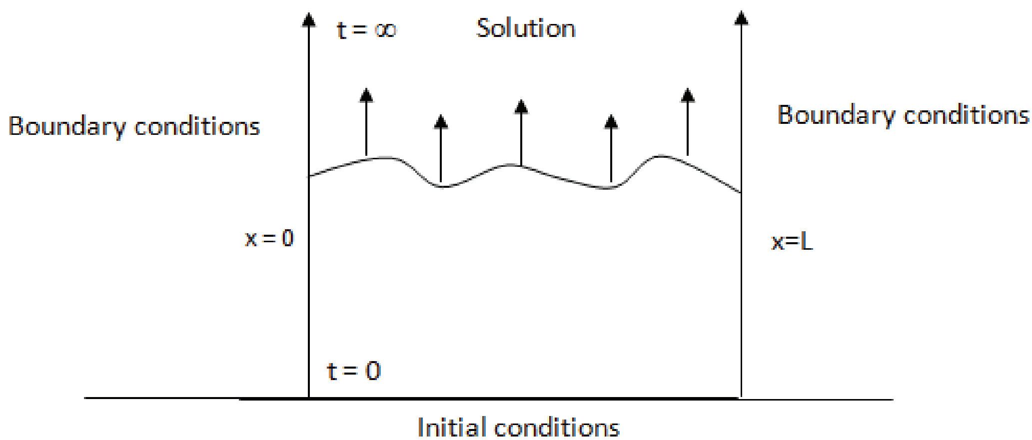

Figure 1 *Initial and boundary condition boundary condition (This figure is modified from Pokhrel (2018)).*

with initial and boundary conditions

u (x, 0) = f (x), u (0, t) = 0, u (L, t) = 0. …(4)

Applying the method of separation of variables (Pokhrel, 2018), a solution for (3) can be constructed. Let u(x, t) = T (t) X (x) be the general solution of (3). As the variables are separated, to obtain

$$\frac{X''}{X} = \frac{T'}{c^2 T} = k, \text{ (constant).}$$

Then, the temperature distribution function can be written as

$$u(x, t) = \begin{cases} (c_1 e^{-px} + c_2 e^{px})\, c_3 e^{c^2 p^2 t}, & \text{if } k = p^2, \\ (c_4 \cos px + c_5 \sin px)\, c_6 e^{-c^2 p^2 t}, & \text{if } k = -p^2, \\ (c_7 x + c_8)\, c_9, & \text{if } k = 0, \end{cases} \quad \ldots(5)$$

where c_1, …, c_9 are constants. The temperature distribution functions given by (5) that do not involve the trigonometric terms are not physically meaningful (Gorguis & Chan, 2008). So, the only feasible solution of the temperature function in (5) is the second trigonometric solution that satisfy the boundary conditions

u(0, t) = 0 °C, u (L, t) = 0. …(6)

Eigenfunctions of (3) is defined (Dabaral et al., 2009) as:

$$u_n(x, t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda n^2 t}, \quad \ldots(8)$$

corresponding Eigenvalues $\lambda_n = c_n \pi / L$, for n = 1, 2,….

The solution of model equation is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda n^2 t}, \quad \ldots(9)$$

where B_n is the constant which will be found by the given initial temperature. Considering a laterally insulated bar of length L whose ends are kept at 0°C, with initial temperature

$$u(x, 0) = \begin{cases} 100\ x, & \text{if } 0 < x < \frac{L}{2}, \\ 100\ (L - x), & \text{if } \frac{L}{2} < x < L \end{cases}$$

The analytical solution (Pokhrel, 2018) is given by

$$u\ (x, t) = \frac{400\ L}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin (n\pi/2)}{n^2} \sin \frac{n\ \pi\ x}{L}\ e^{-\lambda n^2 t}. \quad \text{....(10)}$$

Results and Discussion

Time evolution of Eigenfunctions with variation of Eigenvalues

The density, ρ = 8.92 gm /cm^2, specific heat capacity, s = 0.092 cal /gm°C, and thermal conductivity, k = 0.95 cal/(cm s °C) are the physical data for a copper bar (Lepik, 2011). So, $c^2 = k / s\ \rho$ gives $c^2 = 0.95/(0.092 \times 8.92) = 1.1576$ cm^2/s, L = 4 cm. Hence, the associated squared Eigenvalues $\lambda_n^2 = c^2\ (n^2\ \pi^2)/L^2$ are:

$\lambda_1^2 = (1.1576 \times 1^2\ \pi^2)\ /16 = 0.7141$, $\lambda_3^2 = (1.1576 \times 3^2\ \pi^2)\ /\ 16 = 6.4269$,

$\lambda_5^2 = (1.1576 \times 5^2\ \pi^2)\ /\ 16 = 17.8516$, $\lambda_7^2 = (1.1576 \times 7^2\ \pi^2)\ /\ 16 = 34.9892$.

Figure (2) shows that Eigenfunctions of temperature distribution are parabolic with Eigenvalue λ_1 = 0.845. At length of copper bar L = 2 cm, temperature of Eigenfunctions have maximum 162.1139 °C, 121.8346 °C, 91.5633 °C, and 68.8132 °C at time t = 0.0 s, 0.4 s, 0.8 s, and 1.2 s respectively. During the interval of time period at being the same speed, the maximum temperatures of Eigenfunctions are decreasing as in Fig. (2).

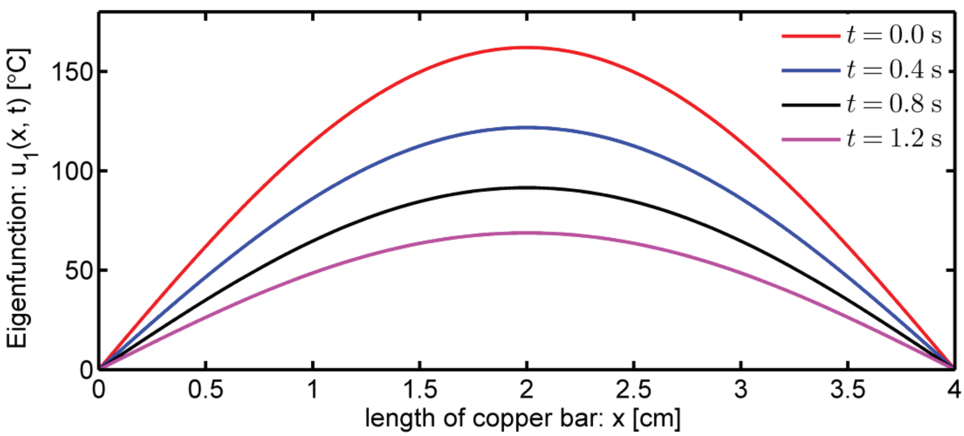

Figure 2 *Eigenfunction with different period of time with speed* $\lambda_1 = c\ \pi\ /L$.

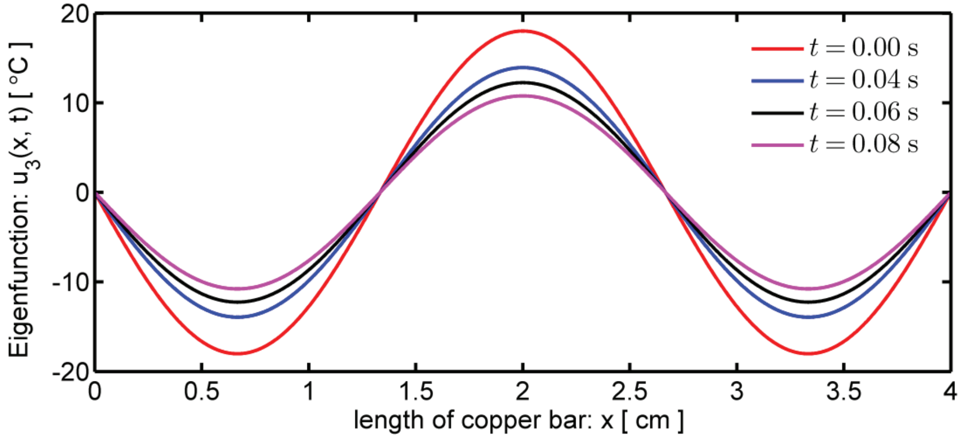

Figure 3 *Eigenfunctions with different period of time with speed* $\lambda_3 = 3\ c\ \pi\ /L$**.**

Figure (3) reveals that Eigenfunctions of temperature distribution are sinusoidal with Eigenvalue λ_3 = 2.5351. At length of copper bar, L = 2 cm, temperature of Eigenfunctions have maximum 18.0127 °C, 13.9294°C, 12.2493°C, and 10.7718 °C at time t = 0.00 s, 0.04 s, 0.06 s, and 0.08 s respectively. At length of copper bar, L = 0.65 cm, 3.35 cm, temperature of Eigenfunctions have minimum -18.0127°C, -13.9294°C, -12.2493°C, and -10.7718°C at time t = 0.0 s, 0.04 s, 0.06 s, and 0.08 s respectively. During the interval of time period at being the same speed, the maximum and minimum temperatures of Eigenfunctions are decreasing as in Fig. (3).

Figure (4) shows that Eigenfunctions of temperature distribution are periodic with period 2, which is sinusoidal function with Eigenvalue λ_5 = 4.2252. At length of copper bar, L = 0.45 cm, L = 2 cm, L = 3.55 cm, the temperature of Eigenfunctions have maximum 6.4846 °C, 5.4244 °C, 4.5375 °C, and 3.7957 °C at time t = 0.00 s, 0.01 s, 0.02 s, and 0.03 s respectively.

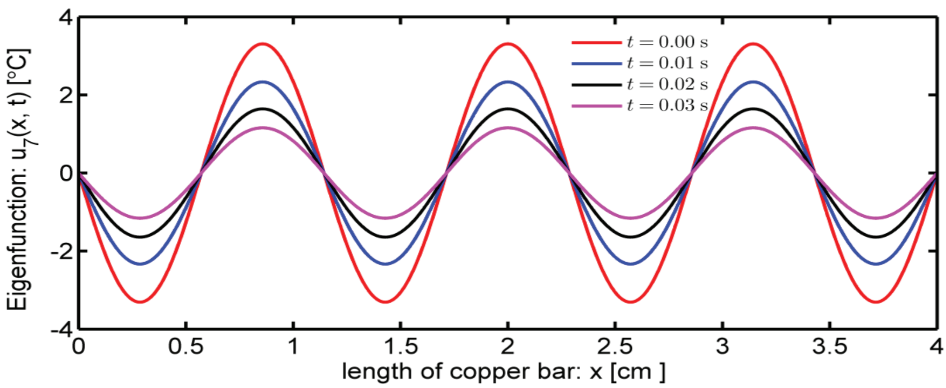

Figure 4 *Eigenfunctions with different period of time with speed* $\lambda_3 = 5\ c\ \pi\ /L$.

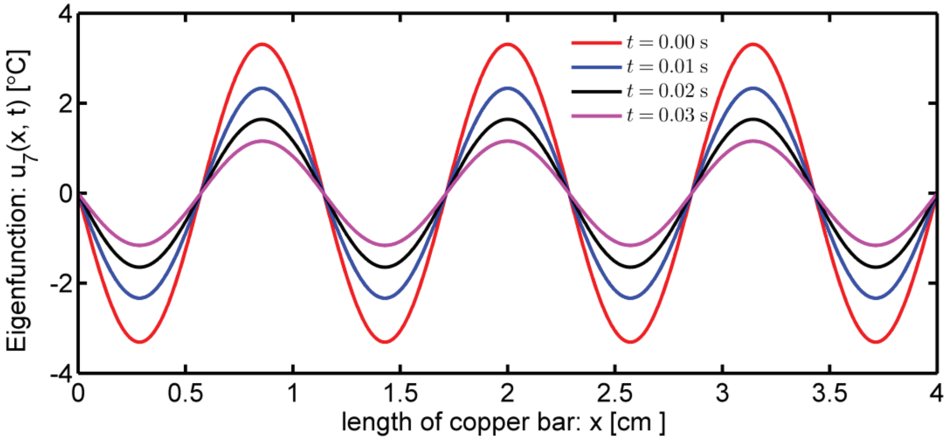

Figure 5 *Eigenfunctions with different period of time with speed $\lambda_7 = 7 c \pi /L$.*

At length of copper bar, L = 1.25 cm, L = 2.75 cm, the temperature of Eigenfunctions have minimum values - 6.4846 °C, -5.4244°C, - 4.5375 °C, and - 3.7957 °C at time t = 0.00 s, 0.01 s, 0.02 s, and 0.03 s respectively. During the interval of time period at being the same speed, the maximum and minimum temperatures of Eigenfunctions are decreasing as in Fig.4. Figure 5 reveals that Eigenfunctions of temperature distribution are sinusoidal with Eigenvalue $\lambda_7 = 5.9153$. At length of copper bar, L = 0.75 cm, L = 2 cm, and L = 3.25 cm, temperature of Eigenfunctions have maximum 3.3084 °C, 2.3317°C, 1.6432 °C, and 1.1581 °C at time t = 0.00 s, 0.01 s, 0.02 s, and 0.03 s respectively. At length of copper bar, L = 0.30 cm and L = 1.30 cm, L = 2.60 cm, L = 3.60 cm, temperature of Eigenfunctions have minimum -3.3084 °C, -2.3317 °C, - 1.6432 °C, and - 1.1581 °C at time t = 0.00 s, 0.01 s, 0.02 s, and 0.03 s respectively. During the interval of time period at being the same speed, the maximum and minimum temperatures of Eigenfunctions are decreasing as in Fig. 5.

Numerical Methods

Consider the rectangle $R = \{(x, t) : x_1 \leq x \leq x_n, t_1 \leq t < \infty\}$, and it is subdivided into sub-rectangles with sides $\Delta x = h$ and $\Delta t = k$ as shown in Fig.6. The finite difference scheme for $\partial^2 u / \partial x^2$ at time steps t_j and t_{j+1} in Fig.7 are:

$$. \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \quad \text{and} \quad \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}.$$

Taking the average of spatial second derivative of u at the time steps t_j and t_{j+1}

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}],$$

with forward finite difference of time derivative of u

$$\frac{\partial u}{\partial t} = \frac{1}{k}[u_{i,j+1} - u_{i,j}],$$

the model equation (3) reduces to finite difference numerical scheme as:

$$-r\,u_{i-1,j+1} + (2r+2)\,u_{i,j+1} - r\,u_{i+1,j+1} = r\,u_{i-1,j} + (2-2r)\,u_{i,j} + r\,u_{i+1,j}, \quad \ldots(11)$$

where $r = k\,c^2/h^2$. This method is Crank-Nicolson numerical scheme formula (Dehghan, 2000). The left hand side of the equation involves the unknown values of u,

i.e., $u_{i-1,j+1}$, $u_{i,j+1}$, $u_{i+1,j+1}$,

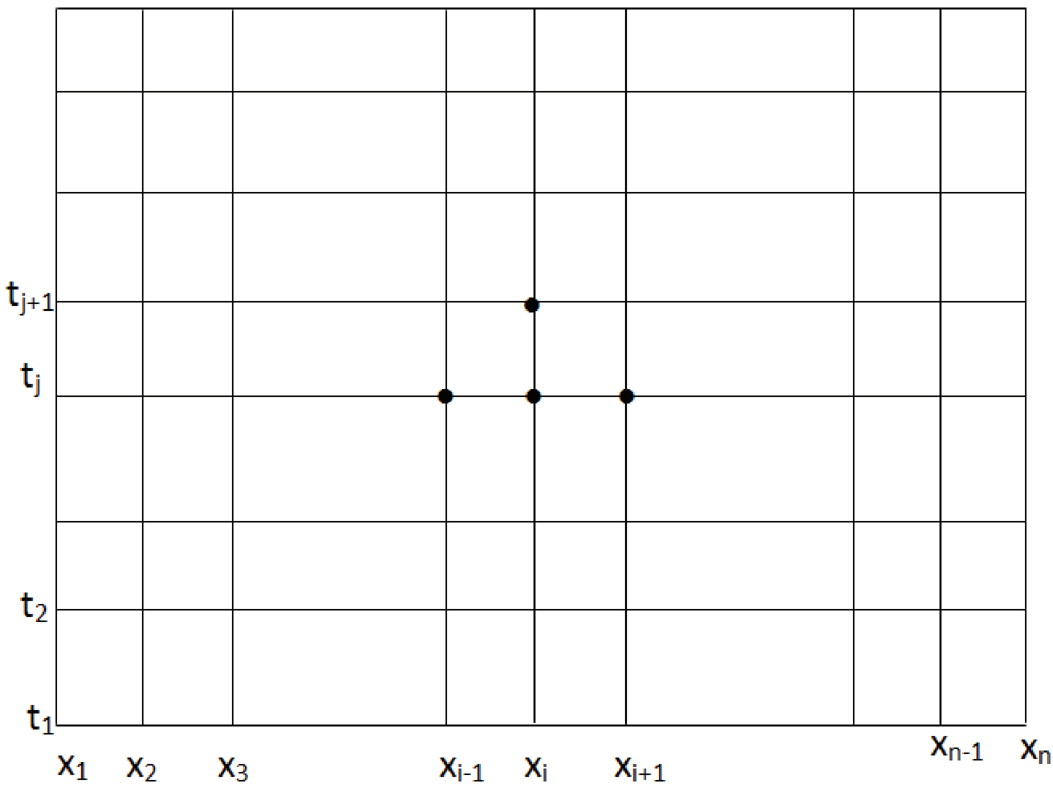

Figure 6 *Mesh-grid for* $t_1 \le t < \infty$, *and* $x_1 \le x \le x_n$ *(This figure is modified from Pokhrel (2018)).*

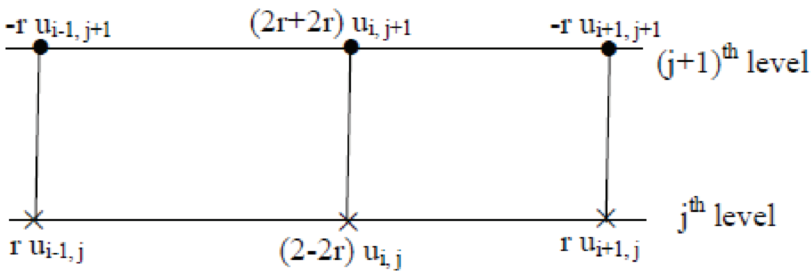

Figure 7 *Crank-Nicolson stencil is that crosses represent the known values of u in j^{th} level, and solids represent the unknown values of u in $(j + 1)^{th}$ level (This figure is modified from Pokhrel (2018)).*

and right hand side of the equation involves the known values of u, i.e., $u_{i-1,j}$, $u_{i,j}$, $u_{i+1,j}$ as shown in Fig. 7. Thus, there are n equations and n unknown variables, the system AX = B is closed. Denoting first three terms involving $u_{i-1,j}$, $u_{i,j}$, and $u_{i+1,j}$ of right side on (11) by

$b_i = r\, u_{i-1,j} + (2 - 2r)\, u_{i,j} + r\, u_{i+1,j}$,

the equation (11) becomes

$$u_{i,j+1} = \frac{r}{2+2r}\left[u_{i-1,j+1} + u_{i+1,j+1}\right] + \frac{b_i}{2+2r},$$

and by releasing the $(j+1)^{th}$ level, Gauss-Seidal iteration formula is written as:

$$u_i^{(n+1)} = \frac{r}{2+2r}\left[u_{i-1}^{(n)} + u_{i+1}^{(n+1)}\right] + \frac{b_i}{2+2r}.$$

This is numerical scheme that contains three unknowns u_i, u_{i-1} and u_{i+1}.

Numerical Results

In a laterally insulated bar of length L = 4 cm, whose ends are kept 0 °C. The length of each sub-intervals is h = 0.50, and c^2 = 1.1576 cm^2/ s, and time interval k = 0.4 s, so $r = k\,c^2/h^2 = (0.4 \times 1.1576)/(\pi^2/16) = 0.7507$.

Table 1 *Temperature distribution on the copper bar with length L = 4 cm.*

	0	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
0.0	0	50	100	150	200	250	300	350	0

0.2	0	49.1386	96.4167	135.9563	145.1666	135.9563	96.4167	49.1386	0
0.4	0	45.9195	86.7475	114.4319	125.9606	114.4319	86.7475	45.9195	0
0.6	0	40.7999	75.6350	99.6146	107.7501	99.6146	75.6350	40.7999	0

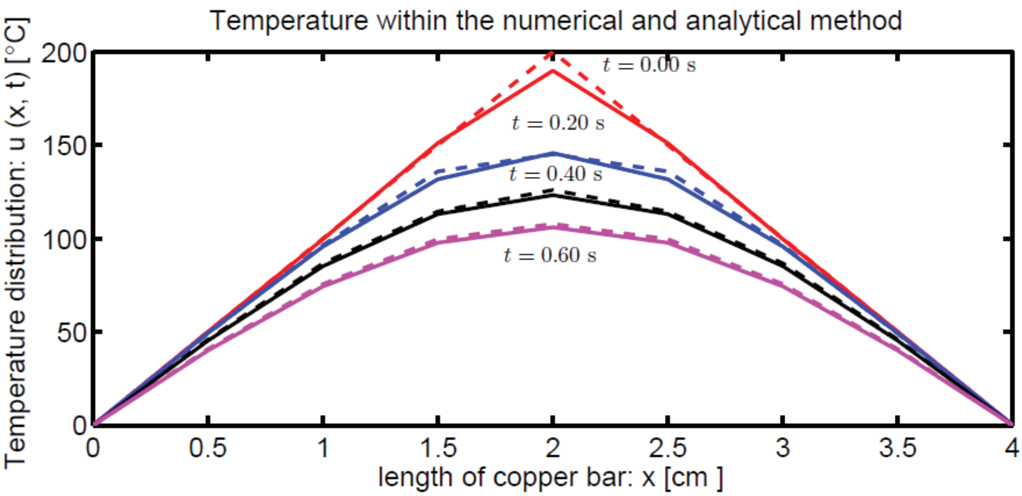

Figure 8 *Temperature distribution on copper bar within Crank-Nicolson and analytical method.*

The initial temperature is u(x, 0) = 100 x for $0 \leq x \leq L/2$ and u(x,0) = 100 (L - x) for $L/2 \leq x \leq L$. Thus $u_{0,0} = 0$, $u_{1,0} = 100 \times 0.50 = 50$, $u_{2,0} = 100 \times 1.0 = 100$, $u_{3,0} = 100 \times 1.5 = 150$, $u_{4,0} = 100 \times 2.0 = 200$, $u_{5,0} = 100 (4 - 2.5) = 150$, $u_{6,0} = 100 (4.0 - 3.0) = 100$, $u_{7,0} = 100 (4 - 3.5) = 50$, $u_{8,0} = 100 (4 - 4) = 0$.

The numerical results of temperature distributions are shown in Table 1. Figure shows that temperatures on copper bar have maximum at x = 2 cm at time t = 0.00 s with higher error, but the temperatures at time t = 0.60 s within the numerical and analytical method are almost same as in Fig.8.

Conclusion

Eigenfunctions and corresponding Eigenvalues of one-dimensional heat equation were derived by transferring the heat with initial temperature on copper bar. A copper bar has density, specific heat capacity, and thermal conductivity. The results were analyzed with Eigenfunctions and Eigenvalues at different period of time. Analytical and numerical methods were used to find temperature distribution on the copper bar.

The results were analyzed by comparing analytic with numerical solution. The results showed that Eigenvalues are the speeds of temperature distribution which are decaying. We examined the temperature distribution obtained by the Crank-Nicolson method and analytical method during the same time interval. We concluded that Eigenvalues is important role to distribute the temperatures on copper bar.

References

Chabrowski, J. (1984). On nonlocal problems for parabolic equations, *Nagoya Math. J.*, *93*, 109-131.

Dehghan, M. (2000). A finite difference method for a non-local boundary value problem for two-dimensional heat equation. *Applied Mathematics and Computation*, *112,* 133-142.

Gorguis, A., Chan, W. K. B. (2008). Heat equation and its comparative solutions. *Computers and Mathematics with Applications*, *55*, 2973-2980.

Lepik, U. (2011). Solving PDEs with the aid of two-dimensional haar wavelets. *Computers and Mathematics with Applications*, *61*, 1873-1879.

Lin, Y. (1994). Analytical and numerical solutions for a class of nonlocal nonlinear parabolic differential equations, *SIAM J. Math. Analysis*, *25*, 1577-1594.

Mohebbi, A., & Dehghan, M. (2010). High-order compact solution of the one-dimensional heat and advection-diffusion equations. *Applied Mathematical Modelling*, *34*, 3071-3084.

Olmstead, W. E., and Roberts, C.A. (1997). The one-dimensional heat equation with a nonlocal initial condition. *Appl. Math. Lett.*, *10* (3), 89 - 94.

Pokhrel, P. R. (2018). Numerical methods. *Numerical solution of partial differential equation*. Vidharty Pustak Bhandar, 315-356.

Pokhrel, P. R. (2022). Variations of initial shape of deflection of a stretched string. *Humanities and Social Science Journal*.

Pokhrel, P. R., and Lamsal, B. (2020). Modelling and parameter analysis of deflection of a beam. *BEBECHANA*, *18*, 75-82

Pokhrel, P. R., Lamsal, B., Kafle, J., and Kattel, P. (2020). Analysis of displacement of vibrating of mass spring due to opposition force. *Tribhuvan University Journal*, *35*(1), 21-32.

Wazwaz, A. M. (2009). *Partial differential equations and solitary waves theory.* Springer.