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Variations of Initial Shape of Deflection of a Stretched String

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Abstract

In this paper, One-dimensional wave equation is formulated by applying the Newton's second law of motion. Employing the one dimensional wave equation, speeds of decay of vibration deflection distribution in initially triangular shape of a string stretched at both ends are analyzed. Eigenfunction and the corresponding Eigenvalues with the various inclination of initial position of the string are found. The frequencies of the vibrations are analyzed from the Eigenvalues and Eigenfunction with different mass of the string. The initially triangular shape string is varied for analyzing the deflection of the string. The deflection of a string distribution with different initial triangular shape positions of the string is analyzed. Finite difference method is applied to find the deflection distribution numerically. Deflections of the string with the different mass and tension forces in the different time interval are observed.

Keywords: eigenvalues, eigenfunctions, mathematical modelling, tension forces

Introduction

The wave phenomena of a stretched string problem represent a physical problem that examines a solution to a classical problem in wave propagation. The stretched string results have applications in the study of the vibration of vocal chords and stringed musical instruments such as the guitar, violin and others. The mathematical modelling of the stretched string results have also been applied in the study of dynamics of transmission lines and the dynamics of strings in the manufacture of fibbers and textiles. Several research activities are carried out in the past decades. French (1971) developed the concept of vibrations and waves confined to mechanical system. Parmley et al. (1995) presented the theoretical models for random string mass chain and compared with experimental results. Bolwell (1999) observed the solution of Rayleigh's equations for a loaded flexible string. Lagrange examined the non-harmonic vibrations of a string with a variable density. Chen et al. (2004) obtained the closed-form solutions for the amplitude of the vibration. Gough (2000) elaborated the physics of a vibrating string excited by a hard narrow hammer. Li-qun (2004) investigated the steady-state transverse vibration of an axially moving string. There was a strong debate on the possibility of a slope discontinuity for a plucked string. Wang et al. (2005) adopted the Hamiltonian dynamics to solve the Eigenvalue problem for transverse vibrations of axially moving strings. Santos et al.

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(2006) discussed the normal frequencies in the case of uniform elastic function and mass density. A physical or other problem is translated into a mathematical model as differential equations (Kreyszig, 2006; Weaver et al., 1990). These different points of view can be considered as one of the first written evidences on the difficulties to be surmounted to arrive at the mathematical definition of function, a very familiar concept nowadays in science. Helmholtz used a string loaded at the middle point to determine the exact frequency at which the impulses fuse into a continuous tone (Kreyszig, 2006; Tohyama, 2011). Pokhrel and Lamsal (2020) observed that the variation of maximum deflection by applying the different external forces. Pokhrel et al. (2021) observed the vibration of string of mass spring due to the variation of opposition forces. Viet et al. (2020) developed a dynamic model to study the free vibration of a beam.

In this paper, the model equation of vibrating of a string is presented with fixed at both ends. The frequencies of the vibrations are analyzed with different Eigenfunctions and Eigenvalues. Effects of the Eigenfunction and the corresponding Eigenvalues on the deflection of the string with different initially triangular shape of the string are observed. we also analyze the deflection distribution of the string with different initial triangular shape positions of the string are analyzed, it is used finite difference method to find the deflection distribution numerically. The variation of the deflection of the string with the different mass and tension forces in the different time interval are found.

Formulation of Model Equations

Let A tightly elastic string is stretched between two fixed end points O and A of length OA = l. Taking O as the origin, OA as x- axis. When the string vibrates each point of the string makes small vibration on perpendicular to x- axis. Let P(x, u) and Q($x + \Delta x$, $u + \Delta u$) be two neighbouring point on the string. Consider the motion of the element PQ of the string, and T_1 and T_2 be the tensions at the points P and Q, where the tangents makes angles ψ and $\psi + \Delta \psi$ with x-axis as shown in Fig. 1.1.

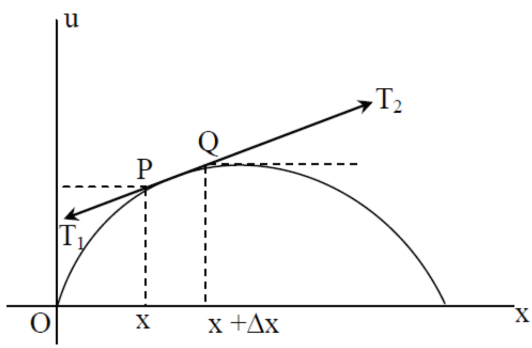
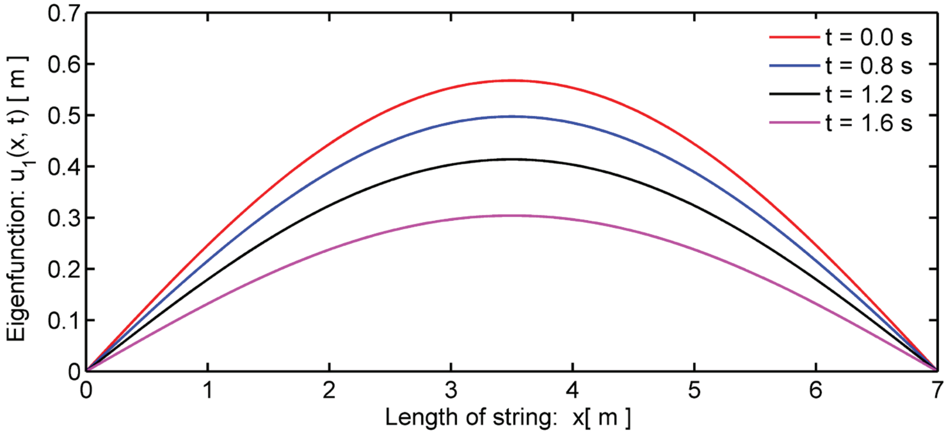


Figure 1.1: Vibrating of string stretched in length l m.



There is no motion in horizontal direction, because the point of the string moves vertically. Thus, the components of tensions in the horizontal must be constant, i.e,

$$T_2 \cos(\psi + \Delta\psi) = T_1 \cos\psi = \text{T (constant)},$$

and the vertical components of the force acting on this element is

$$T_2 \sin(\psi + \Delta\psi) = T_1 \sin\psi.$$

The tangential angles at P and Q is

$$\tan(\psi + \Delta\psi) - \tan\psi = T(u_x(x + \Delta x, t) - u_x(x, t)) \cong T\,\Delta x\, \Delta^2 u / \Delta x^2 .$$

Let w is the weight per unit length, so that $w\,\Delta x$ be the weight of element Δx, then mass of the element is $m = (w\,\Delta x)/g$ per unit length of the string, then Newton's second law of motion states that acceleration of these forces is equal to mass ($\rho\,\Delta x$) of the portion Δ acceleration ($\Delta^2 u/\Delta t^2$) (Chen et al. 2004, G´omez et al. 2007) gives

$$T\,\Delta x\,\Delta^2 u / \Delta x^2 = \left(\frac{w\,\Delta x}{g}\right) \frac{\partial^2 u}{\partial t^2} .$$

$$T_2 \sin(\psi + \Delta\psi) - T_1 \sin\psi = \left(\frac{w\,\Delta x^2}{g}\right). \frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

Dividing both sides of (1) by $T_2 \cos(\psi + \Delta\psi) = T_1 \cos\psi = T$, we get

$$\tan(\psi + \Delta\psi) - \tan\psi = \left(\frac{w\Delta x^2}{t\,g}\right) \frac{\partial^2 u}{\partial t^2} .$$

Tangential angles at Q and P are written as

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \left(\frac{w\,\Delta x}{T g}\right)\frac{\partial^2 u}{\partial t^2}. \qquad ...(2)$$

Dividing both sides of (2) by $\Delta\chi$and taking limit $\Delta\chi \rightarrow 0$ on both sides of (2), we get

$$\lim_{\Delta x \rightarrow 0} \frac{u_x(x+\Delta x,\ t) - u_x(x, t)}{\Delta x} = \left(\frac{w\,\Delta x}{T g}\right)\frac{\partial^2 u}{\partial t^2}.$$

Here, w is weight, T is tension force, and g is gravity of a string of length l. Denoting (T g / w) = c^2, the model equation of vibrating of a string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

This is one-dimensional wave equation. This is a hyperbolic partial differential equation and the solution of this equation is a function of *x* and *t*. So, *u*(*x*, *t*) gives the displacements distribution at *x* at time *t*.

A tightly stretched uniform sting is considered with length l , w as its weight per unit length, T is tension force, and g is the magnitude of the acceleration due to gravity. If the string is subjected to displace by straightening up it at the mid-point (x = l/2), the displacement distribution u(x, t) for x ∈ [0, l] t ∈ [0, t0] is governed by the wave equation (Chen et al., 2004; G´omez et al., 2007):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{T g}{w}, \qquad ...(3)$$

with boundary conditions

u(0, t) = 0, u(l, t) = 0, for all t, ...(4)

and initial condition

$$u(x,\ 0) = f(x) = \begin{cases} k x & \text{if } 0 < x < l/2 \\ k(l - x) & \text{if } l/2 < x < l. \end{cases} \quad = 0 \text{ for all t.} \qquad ...(5)$$

Eigenfunction (Kreyszig, 2006) of this model is obtained as

$$u_n(x, t) = A_n \cos(\lambda_n t) \sin\frac{n \pi x}{l},$$

Corresponding to Eigenvalues $\lambda n = c_n$ p / l of the model (3),

where $A_n = \dfrac{4\, kl \sin(n\pi / 2)}{n^2 \pi^2}$, for n = 1, 2,3, ...

The analytical solution (Pokhrel et al., 2021) for (3) is

$$u(x, t) = \frac{4\,kl}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \cos(\lambda_n t). \qquad ...(6)$$

Results and Discussion

Time evolution of Eigenfunction with corresponding Eigenvalue

The tension forces along the string $T = 0.005$ N, gravity $g = 9.8$ m^2 / s, weight of string $w = 0.025$ kg, length of the string $l = 10$ cm, and the initial constant factor is $k = 1/5$. So, $c^2 = (T\,g/w) = (0.005 \times 9.8)/0.025 = 1.96$ m^4s^2. Hence, the associated Eigenvalue (wave length) $\lambda_1 = 1.4 \times 1 \times \pi = 4.3982$ m / s . The maximum Eigenfunctions are 0.5674 m, 0.4972 m, 0.4136 m and 0.3040 m at time t = 0.0 s, 0.8 s, 1.2 s and 1.6 s . Figure 1.2 shows that Eigenfunction is slowly decreasing from higher to lower during the interval $0 \le x \le 7$ m in different time.

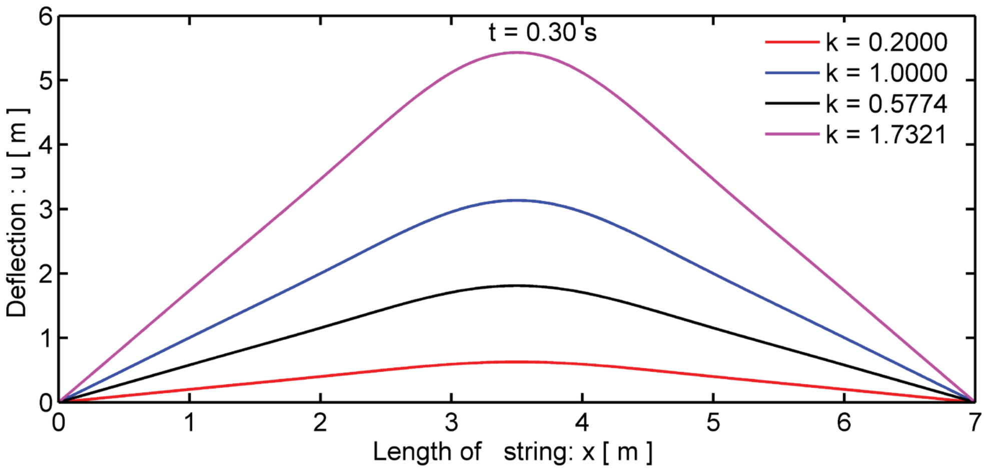


Figure 1.2: The vibration of frequency for wave length $\lambda 1 = 4.3982$ m / s .

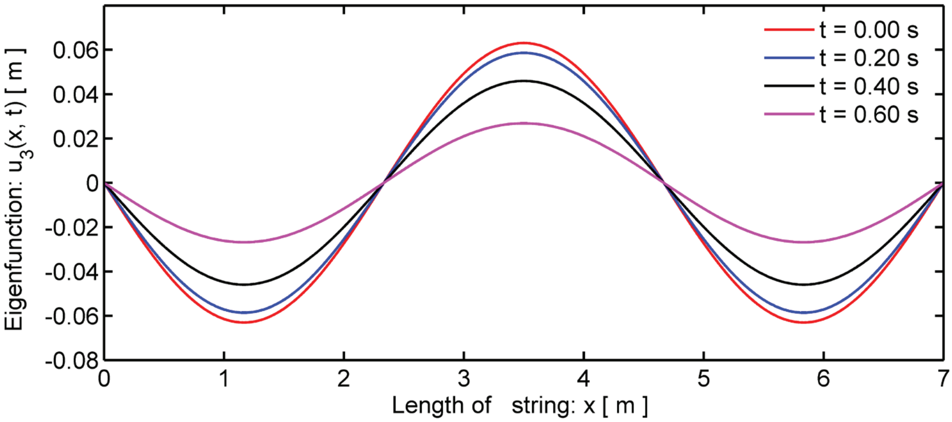


Figure 1.3: The vibration of frequency for wave length λ_3 *= 13.1947 m / s .*

The tension forces along the string $T = 0.005$ N, gravity $g = 9.8$ m^2 / s, weight of string $w = 0.025$ Kg, length of the string $l = 10$ cm, and the initial constant factor is $k = 1/5$. So, $c^2 = (T g / w) = (0.005 \times 9.8) / 0.025 = 1.96$ m^{4}s^2. Hence, the associated Eigenvalues $\lambda_3 = 1.4 \times 3 \times \pi = 13.1947$ m/s.

In this Eigenvalue, the maximum Eigenfunctions 0.0630 m 0.0586 m, 0.0460 m and 0.0268 m at time t = 0.00 s, 0.20 s, 0..40 s and 0.66 s respectively, and the minimum Eigenfunctions are -0.0630 m, - 0.0586 m, - 0.0460 m and - 0.0268 m. Figure 1.3 shows that Eigenfunctions have minimum values at 1.2 m and 5.8 m, and maximum values at 3.5 m in $0 \leq x \leq 7$ m in different time.

The tension forces along the string $T = 0.005$ N, gravity $g = 9.8$ m^2 / s, weight of string $w = 0.025$ Kg, length of the string $l = 7$ m, and the initial constant factor is $k = 1/5$. So, $c^2 = (T g / w) = (0.005 \times 9.8) / 0.025 = 1.96$ m^{4}s^2. Hence, the associated Eigenvalues $\lambda_5 = 1.4 \times 5 \times \pi = 21.9911$ m/s. In this Eigenvalue, the maximum Eigenfunctions are 0.0227m, 0.0199 m, 0.0165 m, 0.0122 m at time t = 0.00 s, 0.16 s, 0.24 s and 0.32 s respectively, and the minimum Eigenfunctions are - 0.0227m, - 0.0199 m, - 0.0165 m and -0.0122 m. Figure 1.4 shows that Eigenfunctions have minimum values at 2.1 m and 4.9 m, and maximum values at 0.8 m, 3.5 m, and 6.4 m in the interval $0 \leq x \leq 7$ m at different time.

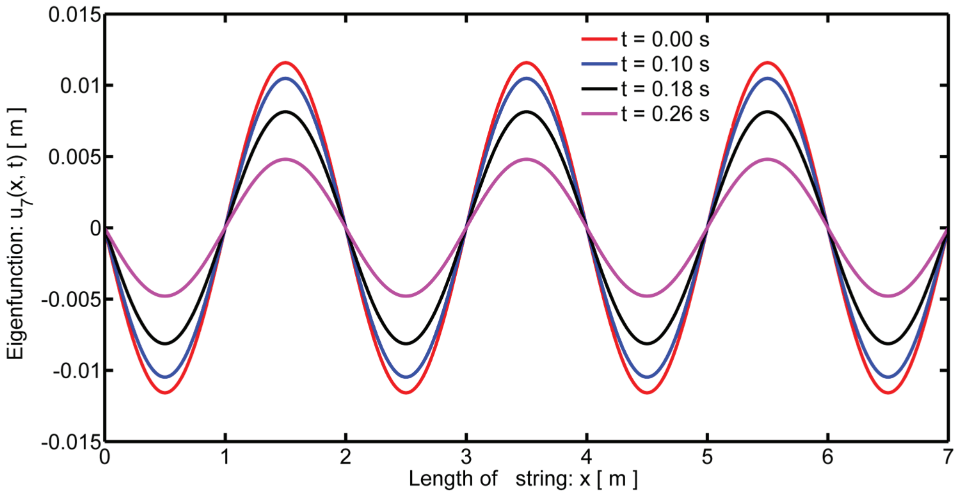


Figure 1.4: The vibration of frequency for wave length λ_5 *= 21.9911 m / s .*

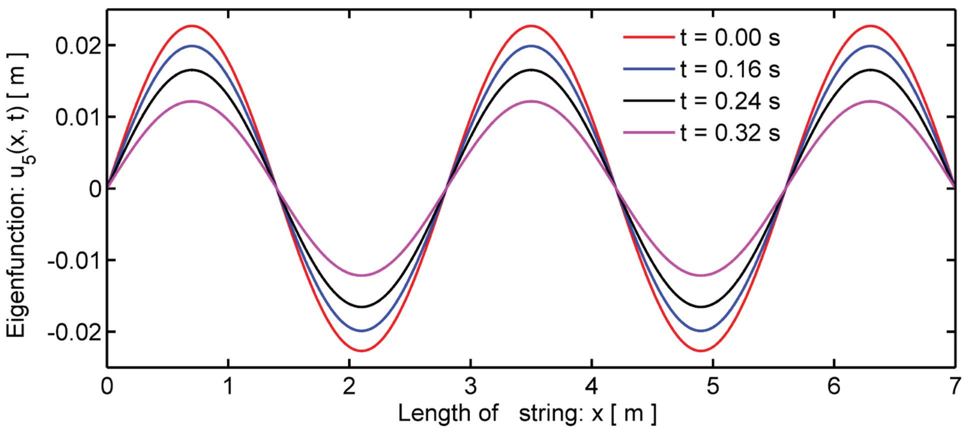


Figure 1.5: The vibration of frequency for wave length λ_7 *= 30.7876 m/s .*

The tension forces along the string T = 0.005 N, gravity g = 9.8 m^2 / s, weight of string w = 0.025 Kg, length of the string l = 10 cm, and the initial constant factor is k = 1/5. So, $c^2 = (T\,g\,/w)$ = (0.005 × 9.8) /0.025 = 1.96 m^4s^2. Hence, the associated Eigenvalues λ_7 =1.4 ×7 ×π= 30.7876 m/s.

In this Eigenvalue, the maximum Eigenfunctions are 0.0116 0.0105 m, 0.0081 m, 0.0048 m at time t = 0.00 s, 0.10 s, 0.18 s and 0.26 s respectively, and the minimum Eigenfunctions are - 0.0116 m, - 0.0105 m, -0.0081 m, and -0.0048 m. Figure 1.5 shows that Eigenfunctions have minimum values at 0.5 m, 2.5 m, 4.5m, 6.5m, and maximum values at 1.5 m, 3.5 m, 5.5 m in the interval $0 \le x \le 7$ m at different time.

Variation of Deflection of rod with different parameter values

The tension forces along the string $T = 0.005$ N, gravity $g = 9.8$ m^2 / s, weight of strings are $w = 0.025$ kg, 0.0125 kg, 0.0063 kg, 0.0031 kg, length of the string $l = 10$ cm, and the initial constant factor is $k = 1/5$. The maximum deflections are 0.6268 m, 0.5933 m, 0.5372 m, 0.4694 m at time t = 0.30 s by varying the weights of the string $w = 0.025$ kg, 0.0125 kg, 0.0063 kg, 0.0031 kg. Figure 1.6 shows that deflection has maximum at 3.5 m with weight $w = 0.0250$kg at time 0.30 s in the interval $0 \leq \times \leq 7$ m.

The tension forces along the string are $T = 0.005$ N, 0.010 N, 0.020 N, 0.040 N, gravity $g = 9.8$ m ^{2}/ s, weight of string w = 0.025 kg, length of the string $l = 7$ m, and the initial constant factor is $k = 1/5$. If the various tension forces are applied in the string at the same period of time t = 0.30 s, the maximum deflections are 0.6268 m, 0.5933m, 0.5372 m 0.4694 m at time t = 0.30 s by varying the tension forces of the string w = 0.005 N, 0.010 N, 0.020 N, 0.040 N. Figure 1.7 shows that deflection has maximum value at 3.5 m with tension force T = 0.005 N at time 0.30 s in the interval $0 \leq x \leq 7$ m.

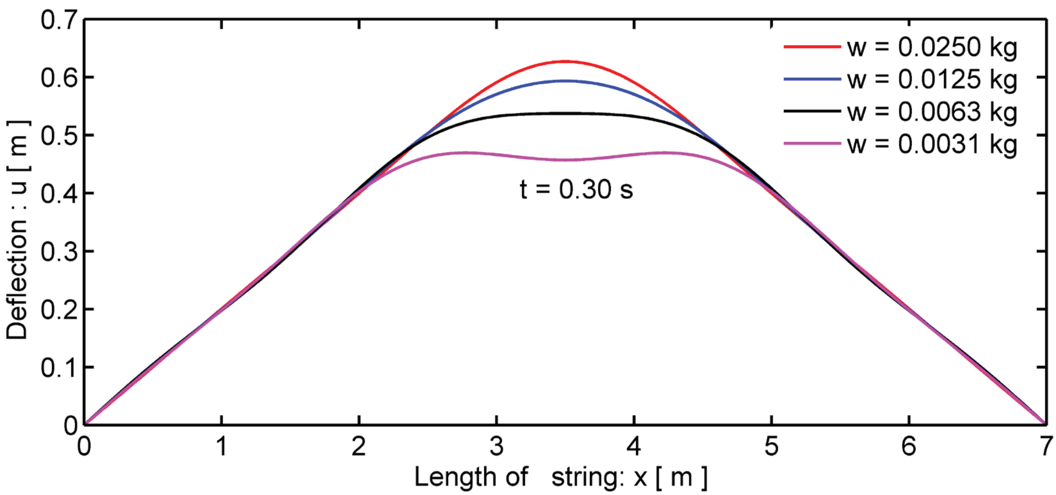


Figure 1.6: Deflection of the string with various weights at $t = 0.3$ s .

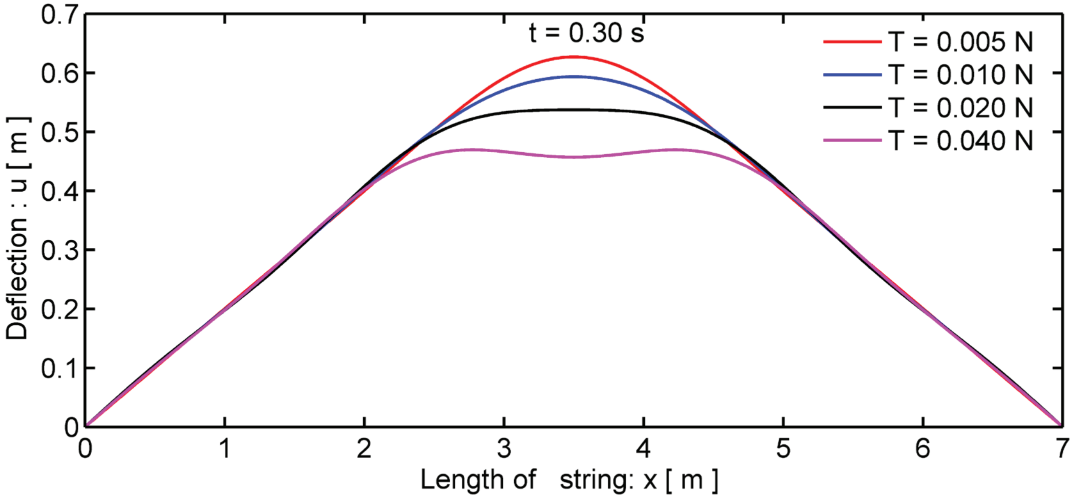


Figure 1.7: The deflection of the string with various tension forces at t = 0.3 s.

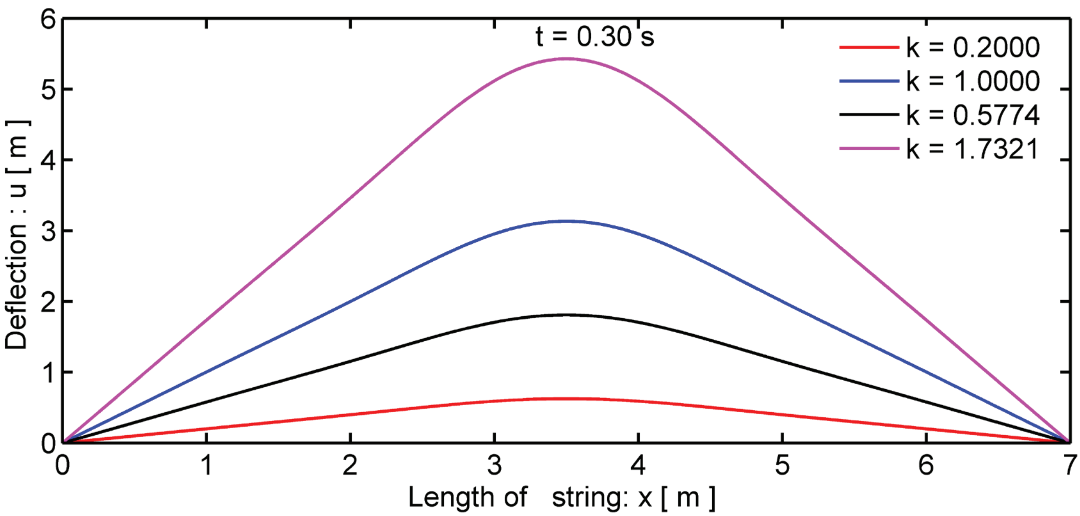


Figure 1.8: The deflection of the string with various initial slopes at t = 0.3 s .

The tension forces along the string are T = 0.005 N, gravity g = 9.8 m ^{2} / s, weight of string w = 0.025 kg, length of the string l = 10 cm, and the inclinations of initial positions of the sting are 11.31 ° and 168.69 °, 30° and 150°, 45 ° and 135 °, and 60 ° and 120°. The constant factors are k = 0.2, 1.0, 0.57774, 1.7321.

In the case of increasing the initial inclination angle of the string as 11.31°, 30 °, 45 ° and 60°, the maximum deflections are 0.6268 m, 1.8096 m, 3.1340 m, and 5.4284 m at the same period of time t = 0.30 s. Figure 1.8 shows that deflection has maximum value at 3.5 m with constant spring k = 1.731 at time t = 0.30 s in the interval $0 \le x \le 7$ m.

Conclusion

I presented the model equation of vibrating of a string with fixed at both ends. I analyzed the frequencies of the vibrations with different Eigenfunctions and Eigenvalues. I observed that the effects of the Eigen function and the corresponding Eigenvalues on the deflection of the string with different initially triangular shape of the string. I also analyzed the deflection distribution of the string with different initial triangular shape positions of the string. I found that the variation of the deflection of the string with the different mass and tension forces in the different time interval.

References

Bolwell, J. (1999). On Rayleigh's equations for the vibrations of a loaded flexible string. *European Journal of Physics*, 2, 305 -312.

Chen, L.Q., Zu, J.W., Wu, J., Yang, X. D. (2004). Transverse vibrations of an axially accelerating viscoelastic string with geometric nonlinearity. *Journal of Engineering Mathematics*, 48,171-182.

French, A. P. (1971). *Vibrations and Waves*. W W Norton & Company, Inc., New York.

G´omez, B. J., Repetto, C. E., Stia, C. R., and Welti, R. (2007). Oscillations of a string with concentrated masses. *European Journal of Physics*, 28, 961-975.

Gough, C. E. (2000). The mass-loaded and nonlinear vibrating string problem revisited. *European Journal of Physics*, 21, L11- L14.

Gunther, L. (2012). *The Vibrating String*. *In: The Physics of Music and Color*. Springer, New York, NY.

Jun, W., Li-qun, C. (2004). Steady-state responses and their stability of nonlinear vibration of an axially accelerating string. *Applied Mathematics and Mechanics(English Edition)*, 25, 1001-1011.

Kreyszig, E. (2006). *Advanced Engineering Mathematics*. John Wiley & Sons, Inc, New York.

Li Sr., Wan, Zq., and Zhang, Jh. (2014). Free vibration of functionally graded beams based on both classical and first-order shear deformation beam theories. *Applied Mathematics and Mechanics* (*English Edition*), 35, 591- 606.

Parmley, S., Zobrist, T., Clough, T., Perez, A. M., Makela, M., and Yu, R. (1995). Vibrational properties of a loaded string. *American Journal of Physics*, 63(6), 547-553.

Pokhrel, P. R., and Lamsal, B. (2020). Modelling and parameter analysis of deflection of a beam. *BEBECHANA*, 18, 75-82.

Pokhrel, P. R., Lamsal, B., Kafle, J., and Kattel, P. (2021). Analysis of displacement of vibrating of mass spring due to opposition force. *Tribhuvan University Journal*, 35(1), 21-32.

Santos, F. C., Coutinho, Y. A., Ribeiro-Pinto, L., and Tort, A. C. (2006). The motion of two masses coupled to a finite mass spring. *European Journal of Physics,* 27, 1037-1051.

Tohyama, M. (2011). *Vibration of String and Wave Propagation*. *In: Sound and Signals. Signals and Communication Technology*. Springer, Berlin, Heidelberg, Germany.

Viet, N. V., Zaki, W., and Wang, Q. (2020). Free vibration characteristics of sectioned unidirectional/bidirectional functionally graded material cantilever beams based on finite element analysis. *Applied Mathematics and Mechanics*, 41, 1787-1804.

Wang, Y., Huang, L., and Liu, X. (2005). Eigenvalue and stability analysis for transverse vibrations of axially moving strings based on Hamiltonian dynamics. *Acta Mechanica Sinica*, 21, 485-494.

Weaver, W., Timoshenko, S. P., and Young, D. H. (1990). *Vibration Problems in Engineering*, Wiley-Interscience, New York.